Anomaly Indicators for Time-Reversal Symmetric Topological Orders

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Anomaly indicators for time-reversal symmetric topological orders

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Some time-reversal symmetric topological orders are anomalous in that they cannot be realized in strictly two-dimensional systems; instead, they can only be realized on the surface of three-dimensional symmetry-protected topological phases. We propose two quantities, which we call *anomaly indicators*, that can detect if a time-reversal symmetric topological order is anomalous in this sense. Both anomaly indicators are expressed in terms of the quantum dimensions, topological spins, and time-reversal properties of the anyons in the given topological order. The first indicator, η_2, applies to bosonic systems while the second indicator, η_f, applies to fermionic systems in the DIII class. We conjecture that η_2, together with a previously known indicator η_1, can detect the two known $\mathbb{Z}_2$ anomalies in the bosonic case, while η_f can detect the $\mathbb{Z}_{16}$ anomaly in the fermionic case.

A useful way to characterize two-dimensional (2D) gapped quantum many-body systems is in terms of the properties of their anyon excitations. For systems with global symmetries, one can study both topological and symmetry properties of anyons. These properties are said to describe the *symmetry-enriched* topological (SET) order in the many-body system[1–6].

An interesting aspect of SET orders is that some of them cannot be realized in strictly 2D systems[7]. Instead, they can only be realized on the surfaces of 3D symmetry-protected topological (SPT) phases — generalizations of the famous topological insulators[8–10]. SET orders of this kind are said to be *anomalous*[11]. More quantitatively, one can define an *anomaly* associated with each SET which takes values in the Abelian group that classifies the corresponding 3D SPT phases (see examples below). This anomaly carries the information of *which* 3D SPT phase can host the SET on its surface[12].

Given that the anomaly associated with each SET tells us which types of physical systems can realize it, it is desirable to have general formulas for determining these anomalies. Such formulas have been found for large classes of SETs with unitary symmetries[5, 13–17]. However, they are generally lacking for SETs with anti-unitary symmetries like time reversal invariance; in the latter case, anomalies have mostly been determined only for specific examples of SETs, and even then their calculation is difficult and involves finding models that realize the SET on the surface of a known SPT phase[18–22].

In this work, we propose general anomaly formulas for the simplest class of time-reversal symmetric SETs — namely those whose *only* symmetry is time-reversal invariance. We consider both bosonic and fermionic systems. In the bosonic case, it is known that there are four time-reversal symmetric 3D SPT phases (including the trivial phase) which are classified by the group $\mathbb{Z}_2 \times \mathbb{Z}_2$ [7, 10, 23, 24]. Hence each time-reversal symmetric SET is associated with a $\mathbb{Z}_2 \times \mathbb{Z}_2$-valued anomaly. Equivalently, each SET is associated with two types of anomalies, each taking values in $\mathbb{Z}_2 = \{\pm 1\}$. One of these two time-reversal anomalies (T anomalies) has been understood previously and is known to be given by the formula

$$\eta_1 = \frac{1}{D} \sum_{a \in \mathcal{C}} d_a^2 e^{i\theta_a}, \tag{1}$$

where $\mathcal{C}$ denotes the set of anyons in the SET, θ_a and d_a are the "topological spin" and "quantum dimension" of the anyon a, and $D = \sqrt{\sum_a d_a^2}$ is the "total quantum dimension" (see Ref. 2 for definitions). As a $\mathbb{Z}_2$ *anomaly indicator*, η_1 has two properties: (1) it only takes the values ± 1 for any time-reversal symmetric SET and (2) if $\eta_1 = -1$, the SET is anomalous[25].

The indicator η_1 is very useful but unfortunately no analogous quantities have been found for other types of T anomalies. In this work we propose two such anomaly indicators: (i) η_2, which detects the second type of $\mathbb{Z}_2$ T anomaly in bosonic systems, and (ii) η_f, which detects the $\mathbb{Z}_{16}$ T anomaly in fermion systems with $\mathcal{T}^2 = -1$. While we are not able to prove that η_2 and η_f are anomaly indicators in the same sense as η_1, we will provide evidence to this effect.

Second anomaly indicator for bosonic systems.—We propose that the second $\mathbb{Z}_2$ T anomaly for bosonic topological orders can be detected by the following indicator:

$$\eta_2 = \frac{1}{D} \sum_{a \in \mathcal{C}} d_a \mathcal{T}_a^2 e^{i\theta_a} \tag{2}$$

Like η_1, we conjecture that η_2 can only take the values ± 1, and if $\eta_2 = -1$, the SET is anomalous.

In Eq. (2), we have introduced a new quantity, $\mathcal{T}_a^2$. Defining it requires two steps. First, recall that the time reversal operator $\mathcal{T}$ can *permute* different species of anyons. We denote this permutation by $a \to \mathcal{T}(a)$. Next, consider the subset of anyons satisfying $\mathcal{T}(a) = a$, i.e. the anyons that are invariant under the $\mathcal{T}$ permutation. Invariant anyons can be divided into two classes: those that carry a two-fold time-reversal protected Kramers degeneracy, similar to that of a spin-1/2 electron, and those that do not carry such a degeneracy (for a precise definition, see Ref. 26). We will say that an invariant anyon a is a *Kramers doublet* if it belongs to the first class and a *Kramers singlet* otherwise. With this terminology, we define the quantity $\mathcal{T}_a^2$ as follows:

$$\mathcal{T}_a^2 = \begin{cases} 1, & \text{if } \mathcal{T}(a) = a, \text{ and Kramers singlet} \\ -1, & \text{if } \mathcal{T}(a) = a, \text{ and Kramers doublet} \\ 0, & \text{if } \mathcal{T}(a) \neq a \end{cases} \tag{3}$$

It is worth mentioning that there are physical constraints on the $\mathcal{T}$ permutation and $\mathcal{T}^2$ assignments which hold for all SETs whether or not they are anomalous. Here, we list several constraints that will be useful in our later discussion. One example is that the topological spins must satisfy $\theta_{\mathcal{T}(a)} = -\theta_a$ since $\mathcal{T}$ is anti-unitary. Accordingly, all invariant anyons must have $\theta_a = 0$ or π. Another constraint is that $\mathcal{T}$ cannot permute the trivial anyon 1, i.e. $\mathcal{T}(1) = 1$. In addition, the trivial anyon must be a Kramers singlet, that is, $\mathcal{T}_1^2 = 1$. Likewise, permuting an anyon twice should be trivial, so we have $\mathcal{T}[\mathcal{T}(a)] = a$. Lastly, in the case of Abelian topological orders, both the $\mathcal{T}$ permutation and $\mathcal{T}^2$ assignments must respect fusion rules in the sense that

$$\mathcal{T}(a) \times \mathcal{T}(b) = \mathcal{T}(a \times b), \quad \mathcal{T}_a^2 \mathcal{T}_b^2 = \mathcal{T}_{a\times b}^2 \tag{4}$$

where "$\times$" stands for the fusion product, and the second equation holds only for invariant anyons. Note that the above list is not exhaustive; for a more general discussion of constraints, see Ref. 5.

Example.—As an example, let us evaluate η_2 for the well-known toric-code topological order[27]: $\mathcal{C} = \{1, e, m, \epsilon\}$. Here, 1 is the trivial anyon, e and m are bosons, and ϵ is a fermion. All the anyons are Abelian, i.e., $d_a = 1$ for every $a \in \mathcal{C}$. Accordingly, the total quantum dimension is $D = 2$. Consider the case that $\mathcal{T}$ does not permute anyons. Then, there are four possible $\mathcal{T}^2$ assignments: $\mathcal{T}_e^2 = \gamma_e$ and $\mathcal{T}_m^2 = \gamma_m$, with $\gamma_e, \gamma_m = \pm 1$ respectively. The trivial anyon must have $\mathcal{T}_1^2 = 1$, and the fermion ϵ must have $\mathcal{T}_\epsilon^2 = \gamma_e\gamma_m$. The latter follows from the fusion rule $e \times m = \epsilon$ and the constraint (4). Inserting the above information into (2), we obtain

$$\eta_2 = \frac{1}{2}(1 + \gamma_e + \gamma_m - \gamma_e\gamma_m) \tag{5}$$

We observe that $\eta_2 = -1$ if $\gamma_e = \gamma_m = -1$ while $\eta_2 = 1$ otherwise. This agrees with expectations[24]: the first case corresponds to the so-called "eTmT" SET, which is believed to be anomalous, while the other three cases are known to be non-anomalous, i.e. realizable in strictly 2D systems.

Evidence.—We now discuss the evidence for our conjecture about η_2.

(1). We have checked that $\eta_2 = 1$ for three large classes of strictly 2D systems: (i) Kitaev's exactly soluble quantum double models with arbitrary finite group G and with $\mathcal{T}$ acting like complex conjugation[27]; (ii) double-layer topological orders $\mathcal{B} \times \bar{\mathcal{B}}$, where $\mathcal{B}$ is an arbitrary bosonic topological order and $\bar{\mathcal{B}}$ is the time reversal partner of $\mathcal{B}$, and the two layers are exchanged under $\mathcal{T}$ permutation;[28] and (iii) Abelian topological orders described by K-matrix theory, discussed in Ref. 26. We discuss details of (i) and (ii) in the Supplementary Material[29], and (iii) can be analyzed straightforwardly using the formula (6) given below.

(2). We have checked that $\eta_2 = -1$ for several systems that are believed to be anomalous. Examples that we considered include (i) the eTmT state discussed above, (ii) the (T-Pfaffian)$_-$ state, and (iii) four copies of the semion-fermion theory. While the latter two examples are fermionic systems — in fact, they correspond to SETs that live at the surface of 3D topological superconductors [30, 31] — they have bosonic counterparts that can be constructed by gauging fermion parity symmetry. Our calculation is for these bosonic counterparts. We present this calculation in the case of the (T-Pfaffian)$_-$ state in the Supplementary Material[29]; the example (iii) can be treated in a similar fashion.

(3). We have checked that η_2 is *multiplicative* under stacking of topological orders. To see this, consider two bosonic topological orders $\mathcal{C}$ and $\mathcal{C}'$, with total quantum dimensions D and D' respectively. In the stacked system $\mathcal{C} \otimes \mathcal{C}'$, anyons are labeled by (a, a') with $a \in \mathcal{C}$ and $a' \in \mathcal{C}'$. One can see that $d_{(a,a')} = d_a d_{a'}$, $\theta_{(a,a')} = \theta_a + \theta_{a'}$, and the total quantum dimension of $\mathcal{C} \otimes \mathcal{C}'$ is DD'. Also, (a, a') is invariant under the $\mathcal{T}$ permutation if and only if both a and a' are invariant, and $\mathcal{T}_{(a,a')}^2 = \mathcal{T}_a^2 \mathcal{T}_{a'}^2$. Putting this all together it follows that η_2 (as well as η_1) is multiplicative under stacking. To see why this result is consistent with expectations, recall that 3D bosonic SPT phases with time reversal symmetry form a $\mathbb{Z}_2 \times \mathbb{Z}_2$ group under stacking. Therefore, we expect that the indicators (η_1, η_2) should also form a $\mathbb{Z}_2 \times \mathbb{Z}_2$ group under stacking. In particular, η_1, η_2 should be multiplicative under stacking, as we just verified.

(4). In the case of Abelian topological orders, we have checked that η_2 does not change under a large class of topological phase transitions, namely those arising from *anyon condensation*[32] (see Supplementary Material[29]). To understand why this property supports our conjecture, note that anomalies can be thought of as properties of *3D bulk* phases whose surfaces support anomalous SETs. On the other hand, topological phase transitions can be thought of as occurring on the *surface*. Since surface phase transitions cannot change bulk properties, anomaly indicators must be invariant under such transitions.

Alternative formula for $\eta_1\eta_2$.— In order to describe some additional evidence for our conjecture, we now discuss an alternative formula for the product indicator $\eta_1\eta_2$. This formula is not as general as (1) and (2) and only applies to the case of Abelian topological orders. It states that $\eta_1\eta_2$ can be computed as[33]

$$\eta_1\eta_2 = e^{i\theta_a} \tag{6}$$

where a is any anyon that obeys

$$e^{i\theta_{a,b}} = \mathcal{T}_b^2 \quad \text{for all } b \in \mathcal{I} \tag{7}$$

Here $\mathcal{I}$ denotes the set of anyons that are invariant under the $\mathcal{T}$ permutation and $\theta_{a,b}$ denotes the mutual statistics between a and b.

Before we derive Eq. (6), let us discuss its implications. First, we can use it to show that η_2 can only take the values $+1$ or -1: to see this, note that Eq. (6) implies that η_2 has unit modulus. The claim then follows from the observation that η_2 is real.

Another interesting aspect of the formula (6) is that if we restrict to the case where $\mathcal{T}$ does not permute any anyons, then Eq. (6) agrees with the more specialized time reversal anomaly formula conjectured in Ref. 13.

We now turn to the justification of Eq. (6). We need to establish three points: **(i)** there always exists at least one anyon a satisfying Eq. (7); **(ii)** if there exists multiple a's satisfying Eq. (7), then they all share the same topological spin; and **(iii)** the expression for $\eta_1\eta_2$ in Eq. (6) agrees with Eqs. (1)-(2). We prove the first two points in the Supplementary Material[29]. Here we will focus on the last point. To this end, we multiply Eqs. (1) and (2) together and rewrite the resulting expression:

$$\begin{aligned}\eta_1\eta_2 &= \frac{1}{D^2}\sum_c d_c^2 e^{i\theta_c}\sum_b d_b \mathcal{T}_b^2 e^{-i\theta_b} \\ &= \frac{1}{D^2}\sum_{abc} e^{i\theta_c - i\theta_b} N_{bc}^a d_a d_c \mathcal{T}_b^2 \\ &= \frac{1}{D}\sum_a n_a d_a e^{i\theta_a}, \qquad n_a = \sum_b s_{ab}\mathcal{T}_b^2\end{aligned} \tag{8}$$

Here, the first equality follows from the fact that η_2 is real; the second equality follows from $d_b d_c = \sum_a N_{bc}^a d_a$; the third equality follows from the identity $N_{bc}^a = N_{\bar{b}\bar{c}}^{\bar{a}} = N_{a\bar{b}}^c$ together with the definition of the topological S-matrix[2]: $s_{ab} = \frac{1}{D}\sum_c N_{a\bar{b}}^c e^{i\theta_c - i\theta_a - i\theta_b} d_c$.

So far, our computation of $\eta_1\eta_2$ is completely general. If we specialize now to the Abelian case, then $s_{ab} = e^{-i\theta_{a,b}}/D$. Using the fact that both $\{\mathcal{T}_b^2\}_{b\in\mathcal{I}}$ and $\{e^{i\theta_{a,b}}\}_{b\in\mathcal{I}}$ define one-dimensional representations of the subgroup $\mathcal{I}$, we find that $n_a = |\mathcal{I}|/D$ if a is a solution to (7) and $n_a = 0$ otherwise. Next, substituting n_a into (8) and using property **(ii)** listed above, we deduce that $\eta_1\eta_2 = N|\mathcal{I}|e^{i\theta_a}/D^2$ where a is any solution to (7) and N is the number of such solutions. At the same time, it is not hard to show that $N = D^2/|\mathcal{I}|$. Eq. (6) follows immediately.

Anomaly indicator for fermionic systems.—We now consider time-reversal symmetric SETs in interacting fermionic systems with $\mathcal{T}^2 = -1$ (i.e., DIII class). The T-anomaly for these SETs takes values in $\mathbb{Z}_{16}$, corresponding to the $\mathbb{Z}_{16}$ classification of 3D topological superconductors of DIII class.[30, 31, 34, 35] We propose that this $\mathbb{Z}_{16}$ T anomaly is detected by the following indicator:

$$\eta_f = \frac{1}{\sqrt{2}D}\sum_{a\in\mathcal{C}_f} d_a \tilde{\mathcal{T}}_a^2 e^{i\theta_a} \tag{9}$$

We conjecture that η_f can take 16 different values, $e^{i\pi\nu/8}$ with $\nu = 0, 1, \ldots, 15$, and that the SET is anomalous if $\eta_f \neq 1$.

Let us explain the expression (9). First of all, an essential difference between fermionic and bosonic topological orders is the existence of a *local* fermion f in fermionic topological orders, which has trivial mutual statistics with all anyons and satisfies the fusion rule $f \times f = 1$. We use $\mathcal{C}_f$ to denote the set of all anyons, including f. Anyons in $\mathcal{C}_f$ always come in pairs, $\{a,\ a\times f\}$ where a and $a \times f$ have topological spins that differ by π.

In Eq. (9), we have introduced a new quantity $\tilde{\mathcal{T}}_a^2$. To define it, we first introduce a related quantity:

$$\mathcal{T}_a^2 = \begin{cases} 1, & \text{if } \mathcal{T}(a) = a, \text{ and Kramers singlet} \\ -1, & \text{if } \mathcal{T}(a) = a, \text{ and Kramers doublet} \\ \pm i, & \text{if } \mathcal{T}(a) = a \times f \\ 0, & \text{otherwise} \end{cases} \tag{10}$$

(We will explain how to determine the signs in the $\pm i$'s below). With this definition, $\tilde{\mathcal{T}}_a^2$ is given by:

$$\tilde{\mathcal{T}}_a^2 = \begin{cases} -i\mathcal{T}_a^2, & \text{if } \mathcal{T}(a) = a \times f \\ \mathcal{T}_a^2, & \text{otherwise} \end{cases} \tag{11}$$

Here, the minus sign in the $-i$ in (11) is simply a matter of convention. In this convention, the surface of a DIII-class topological superconductor with index ν carries an anomaly $\eta_f = e^{i\nu\pi/8}$. If instead we used $+i$ in (11), the indicator defined through (9) would be the complex conjugate of η_f in the current convention.

We now explain how the $\pm i$'s in (10) are assigned. This is subtle because when $\mathcal{T}(a) = a \times f$, time reversal symmetry guarantees that a and $a \times f$ are degenerate in energy. Thus, a and $a \times f$ *always* form a doublet. Nevertheless, previous work has shown that the anyons obeying $\mathcal{T}(a) = a \times f$, can be divided into two classes which can be assigned the values $\mathcal{T}_a^2 = i$ and $\mathcal{T}_a^2 = -i$ respectively.[30, 31] Unlike Kramers doublets/singlets, the physical distinction between anyons with $\mathcal{T}_a^2 = \pm i$ is subtle, and the assignments depend on a sign convention; however, once a convention has been fixed, the $\mathcal{T}^2$ assignments are unambiguous.[31]

As in the bosonic case, the $\mathcal{T}$ permutation and $\mathcal{T}^2$ assignments must satisfy certain constraints. In particular, the relation $\theta_{\mathcal{T}(a)} = -\theta_a$ implies that the invariant anyons must have topological spin $\theta_a = 0$ or π while the anyons with $\mathcal{T}(a) = a \times f$ must have $\theta_a = \pm\pi/2$. Also, the trivial anyon 1 and the local fermion f must be invariant under the $\mathcal{T}$ permutation, and must have $\mathcal{T}_1^2 = 1$ and $\mathcal{T}_f^2 = -1$. Lastly, in the case of Abelian topological orders, there are constraints similar to Eq. (4). However, instead of $\mathcal{T}_a^2$, it is $\tilde{\mathcal{T}}_a^2$ that satisfies the relation $\tilde{\mathcal{T}}_a^2\tilde{\mathcal{T}}_b^2 = \tilde{\mathcal{T}}_{a\times b}^2$, for all nonzero $\tilde{\mathcal{T}}_a^2$'s.[30, 31]

Examples.—Let us evaluate η_f for two examples. Our first example is the so-called semion-fermion (SF) topological order. This system contains four Abelian anyons $\{1, f, s, \bar{s}\}$, where s is a semion with $\theta_s = \pi/2$, and $\bar{s} = s \times f$ is an anti-semion with $\theta_{\bar{s}} = -\pi/2$. The $\mathcal{T}$ permutation takes $\mathcal{T}(s) = \bar{s}$ and $\mathcal{T}(\bar{s}) = s$. As for the $\mathcal{T}^2$ assignments, we have $\mathcal{T}_f^2 = -1$, and $\mathcal{T}_1^2 = 1$ while there are two possibilities for $\mathcal{T}_s^2$ and $\mathcal{T}_{\bar{s}}^2$, namely $\mathcal{T}_s^2 = -\mathcal{T}_{\bar{s}}^2 = i\sigma$, with $\sigma = \pm 1$. These two possibilities correspond to two types of semion-fermion topological orders known as SF_+ and SF_-. Inserting this information into (9) and using the definition (11) gives

$$\eta_f\big|_{\mathrm{SF}_\sigma} = e^{i\sigma\pi/4} \tag{12}$$

This agrees with previous work which has argued that the SF_+ and SF_- topological orders are anomalous and live on the surfaces of $\nu = 2$ and $\nu = 14$ topological superconductors, respectively[30, 31].

Our second example is the $\mathrm{SO}(3)_6$ topological order[30]. This theory also contains four anyons $\{1, f, s, \bar{s}\}$, with $\theta_s = \pi/2$ and $\theta_{\bar{s}} = -\pi/2$. The anyons s and $\bar{s}$ are non-Abelian with $d_s = d_{\bar{s}} = 1 + \sqrt{2}$. The $\mathcal{T}$ permutation is the same as in the semion-fermion topological order, and like that case there are two variants of $\mathrm{SO}(3)_6$ with $\mathcal{T}_s^2 = -\mathcal{T}_{\bar{s}}^2 = \pm i$. We will refer to these two possibilities as $\mathrm{SO}(3)_{6,+}$ and $\mathrm{SO}(3)_{6,-}$. Substituting this data into (9), we obtain

$$\eta_f\big|_{\mathrm{SO}(3)_{6,\sigma}} = e^{i\sigma 3\pi/8} \tag{13}$$

where $\sigma = \pm 1$. Previous work has argued that the $\mathrm{SO}(3)_{6,\pm}$ topological orders are anomalous and live on the surfaces of topological superconductors with odd index ν, but the values of ν have not been determined[30]. Our conjecture reveals these values: it implies that the $\mathrm{SO}(3)_{6,+}$ topological order lives on the surface of a $\nu = 3$ topological superconductor, while $\mathrm{SO}(3)_{6,-}$ lives on the surface of a $\nu = 13$ topological superconductor.

Evidence.—We now turn to the evidence for our conjecture about η_f.

(1). We have checked that $\eta_f = 1$ for three large classes of strictly 2D fermionic topological orders. The first two classes are obtained by taking the 2D *bosonic* systems that we discussed earlier — namely (i) Kitaev's quantum double models and (ii) double layer bosonic topological orders of the form $\mathcal{B} \times \bar{\mathcal{B}}$ — and stacking them with a fermionic atomic insulator. The third class consists of (iii) all fermionic Abelian topological orders described by K-matrix theory[26]. Actually, the fact that $\eta_f = 1$ for classes (i) and (ii) follows immediately from our previous result that $\eta_2 = 1$ for the corresponding bosonic systems, since it is easy to show that $\eta_f = \eta_2$ for any fermionic system obtained by stacking a bosonic system with an atomic insulator. As for class (iii), these systems can be analyzed via an alternative formula for η_f, similar to (6). This alternative formula is discussed in the Supplementary Material [29].

(2). We have checked that $\eta_f \neq 1$ for several systems that are believed to be anomalous, including the $(\text{T-Pfaffian})_-$ state, N copies of the semion-fermion state ($N \notin 8\mathbb{Z}$), and N' copies of the $\mathrm{SO}(3)_6$ state ($N' \notin 16\mathbb{Z}$)[19–22, 30, 31]. On the other hand, we have checked that $\eta_f = 1$ for the Moore-Read$\times U(1)_{-2}$ state, T_{96} state, and $(\text{T-Pfaffian})_+$ state from Refs. 19–22, 30, and 31. This agrees with expectations since the latter topological orders are believed to be realizable in strictly 2D.

(3). We have checked that η_f is *multiplicative* under stacking of topological orders.

(4). For the case of Abelian topological orders, we have checked that η_f does not change under any topological phase transition arising from anyon condensation (see Supplementary Material[29]).

Discussion.—To sum up, we propose two quantities, η_2 (2) and η_f (9), for detecting anomalies in time-reversal symmetric bosonic SETs and DIII-class fermionic SETs, respectively. Our proposal remains a conjecture. One possible approach to prove our conjecture would be to construct, for each SET, a corresponding 3+1D topological field theory that supports the SET on its 2+1D boundary. If the SET is *not* anomalous then the partition function of this 3+1D theory should equal 1 for every closed spacetime manifold. Thus, if one could show that the partition function on some (non-orientable) closed manifold is equal to η_2 or η_f, then our conjecture would follow[36, 37]. Another possible approach would be to investigate η_2 and η_f in the context of 1+1D conformal field theory (CFT). Indeed, the relation $\eta_1 = e^{i2\pi c_-/8}$, which underlies the η_1 anomaly, was first proven in CFT[2, 38]. Hence, it seems plausible that relations analogous to $\eta_2 = 1$ or $\eta_f = 1$ can also be derived in the context of time-reversal symmetric CFT.

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