Modular Invariance of Conformal Field Theory on S^{1}×S^{3} and Circle Fibrations

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Modular invariance of conformal field theory on $S^1 \times S^3$ and circle fibrations

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I conjecture a high-temperature/low-temperature duality for conformal field theories defined on circle fibrations like S^3 and its lens space family. The duality is an exchange between the thermal circle and the fiber circle in the limit where both are small. The conjecture is motivated by the fact that $\pi_1(S^3/\mathbb{Z}_{p\to\infty}) = \mathbb{Z} = \pi_1(S^1 \times S^2)$ and the Gromov-Hausdorff distance between $S^3/\mathbb{Z}_{p\to\infty}$ and $S^1/\mathbb{Z}_{p\to\infty} \times S^2$ vanishes. Several checks of the conjecture are provided: free fields, $\mathcal{N} = 1$ theories in four dimensions (which shows that the Di Pietro-Komargodski supersymmetric Cardy formula and its generalizations are given exactly by a supersymmetric Casimir energy), $\mathcal{N} = 4$ super Yang-Mills at strong coupling, and the six-dimensional $\mathcal{N} = (2, 0)$ theory. For all examples considered, the duality is powerful enough to control the high-temperature asymptotics on the unlensed S^3, relating it to the Casimir energy on a highly lensed S^3. Such large-order quotients are more generally useful for studying quantum field theory on curved spacetimes.

INTRODUCTION

The torus has a nontrivial discrete symmetry group $SL(d, \mathbb{Z})$ which can be used to constrain conformal field theories (CFTs) placed on such a background. As a sample, this symmetry has provided formulas for the asymptotic density of states ($d \geq 2$) [1–3], monotonicity and sign constraints on the torus vacuum energy ($d > 2$) [2, 4], upper bounds on the gaps (above the vacuum) of scaling dimensions and charges of primary operators ($d = 2$) [5–7], and constraints on operator product expansion coefficients ($d = 2$) [8].

The thermal partition function of a CFT on S^{d-1} is computed by placing the theory on $S^1 \times S^{d-1}$. This is a natural object to consider since states of a CFT on S^{d-1} are in one-to-one correspondence with the local operators of the theory. In this case there is no general modular property of the theory on $S^1 \times S^{d-1}$ that is known. The special power of a torus is that it is made of circles, and it is the Hamiltonian interpretation that such a circle admits that allows for distinct-looking but fundamentally equivalent quantizations. In this work we will consider circle fibrations $\mathcal{M}^{d-1}$, which are spaces with a free circle action at every point over some base space $\mathcal{M}^{d-2}$. This is often denoted $S^1 \longrightarrow \mathcal{M}^{d-1} \longrightarrow \mathcal{M}^{d-2}$, and typical examples include odd-dimensional spheres, e.g. S^3 is a circle fibration over S^2. Here we argue that a general circle fibration with a freely acting $U(1)$ can be highly quotiented (we will often say "lensed") into approximating $S^1 \times \mathcal{M}^{d-2}$. There is an emergent circle and the thermal partition function on $S^1/\mathbb{Z}_k \times \mathcal{M}^{d-1}/\mathbb{Z}_p$ inherits an $SL(2, \mathbb{Z})$ invariance when k and p are large with arbitrary ratio. This allows us, for example, to lens the $S^1 \times S^3$ partition function into a $\mathbb{T}^2 \times S^2$ partition function (see figure 1). This then connects the local operator content of the theory – in a precise and quantifiable way – to the $\mathbb{T}^2 \times S^2$ partition function, which admits the aforementioned powerful modular symmetry. Interestingly, in all examples considered in this paper the stronger equality

$$Z[S^1/\mathbb{Z}_k \times \mathcal{M}^{d-1}] \approx Z[S^1 \times \mathcal{M}^{d-1}/\mathbb{Z}_k] \tag{1}$$

is true, which lets us relate e.g. the high-temperature theory on an unlensed S^3 to the low-temperature theory on a highly lensed S^3 (here low temperature means β large compared to the lensed Hopf fiber of the S^3). The rest of this section introduces these ideas in the context of $\mathcal{M}^{d-1} = S^3$.

To unambiguously define the partition function of a CFT on a manifold $\mathcal{M}^d$, we often need additional requirements. In the case of a two-dimensional CFT on $\mathbb{T}^2 = S^1_\beta \times S^1_L$ we impose invariance under $SL(2, \mathbb{Z})$, which forces operators to have integer spin and implies a high-temperature/low-temperature duality $Z(\beta/L) = Z(L/\beta)$. For higher-dimensional CFTs on $\mathbb{T}^d$ the analog is invariance under $SL(d, \mathbb{Z})$. The question of whether high-temperature/low-temperature dualities of this nature extend to CFT partition functions on $S^1_\beta \times S^{d-1}_R$ for

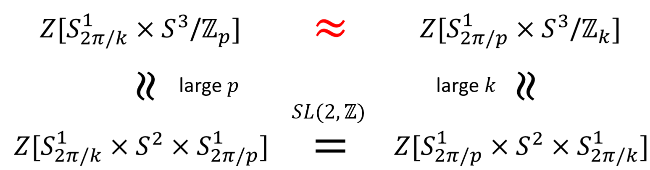

FIG. 1: A series of approximate equivalences motivating the conjecture in the case of a lens space partition function. More generally, we can replace $S^3/\mathbb{Z}_k$ with a smooth manifold $\mathcal{M}^{d-1}/\mathbb{Z}_k$. In all cases considered in this paper, the results can be analytically continued to $p = 1$ and give a correct equality between the high-temperature partition function on the unlensed manifold and the low-temperature partition function on a highly lensed manifold.

$d > 2$ has been discussed [9] since the case of $d = 1$ was understood [1]. While free fields have certain properties under modular transformations [10], there is no general behavior.

When doing thermal physics modular invariance is an invariance under swapping a thermal cycle with a spatial cycle. Odd-dimensional spheres can be understood as circle bundles, so there is an appetizing S^1_L that one cannot help but try to swap with the thermal S^1_β. For $\beta = 2\pi/p$ for some positive integer p, this would mean that the sphere partition function at some temperature would be related to the lens space partition function at some other temperature. This is *not* true in general. However, the conjecture of this paper is that it is true asymptotically as the two cycles shrink to zero size. In other words, we have

$$\lim_{\substack{k\to\infty \\ p\to\infty}} Z\left[S^1_{2\pi/k} \times S^3/\mathbb{Z}_p\right] = \lim_{\substack{k\to\infty \\ p\to\infty}} Z\left[S^1_{2\pi/p} \times S^3/\mathbb{Z}_k\right] \tag{2}$$

for positive integer k, p. The partition function often diverges exponentially in k or p in this limit, so the proper equality is stated in terms of a k- or p-normalized free energy $\log Z$, which in the case of successive limits can be normalized as

$$\lim_{p\to\infty} p \lim_{k\to\infty} \frac{1}{k^3} \log Z\left[S^1_{\frac{2\pi}{k}} \times S^3/\mathbb{Z}_p\right] \tag{3}$$

$$= \lim_{p\to\infty} p \lim_{k\to\infty} \frac{1}{k^3} \log Z\left[S^1_{\frac{2\pi}{p}} \times S^3/\mathbb{Z}_k\right] \tag{4}$$

but we will henceforth leave this implicit. The equality is meant to hold independent of precisely how we take the limit, i.e. for arbitrary ratio p/k, but the normalization in such a double-scaling limit may be theory-dependent.

How can we justify such an equivalence? In the case of lens spaces, for any finite lensing the two manifolds are not related by a large diffeomorphism. So the natural guess is that the conjectured equivalence will not be true at any finite lensing, which we will see is correct in concrete examples. But the conjecture is meant only to hold in the limit of infinite lensing. Notice that something special is happening in the limit: $\pi_1\left(S^3/\mathbb{Z}_{p\to\infty}\right) = \mathbb{Z}_{p\to\infty} = \mathbb{Z} = \pi_1(S^1)$, so it seems that there is an emergent circle justifying the cycle-swapping invariance that would make (2) and its generalizations true. The trivial fibration $S^1/\mathbb{Z}_p \times S^2 = S^1_{2\pi/p} \times S^2$ and the nontrivial fibration $S^3/\mathbb{Z}_p$ become isometric as $p \to \infty$, a notion which we now make precise.

Consider the Gromov-Hausdorff distance d_{GH}, which measures distances between two metric spaces by minimizing over all Hausdorff distances between all possible isometric embeddings of the two metric spaces. In general this is a pseudometric, but since we will be considering compact metric spaces, the distance vanishes if and only if the two spaces are isometric [11]. In particular this means that the two spaces under consideration are diffeomorphic. Both the lens space $\mathcal{M}_1(p) = S^3/\mathbb{Z}_p$ and $\mathcal{M}_2(p) = S^1_{2\pi/p} \times S^2$ converge to $\mathcal{M}_3 = S^2$ as $p \to \infty$ in the Gromov-Hausdorff sense. This is familiar from dimensional reduction in physics, and is an example of collapse with bounded curvature in mathematics. This means that $d_{GH}(\mathcal{M}_1(p\to\infty), \mathcal{M}_3) = 0$ and $d_{GH}(\mathcal{M}_2(p\to\infty), \mathcal{M}_3) = 0$. But by the triangle inequality we therefore have

$$\lim_{p\to\infty} d_{GH}(S^1_{2\pi/p} \times S^2 \;,\;\; S^3/\mathbb{Z}_p) = 0\,, \tag{5}$$

meaning the two spaces become isometric as p becomes large. In particular, for any distance ϵ, we can pick a sufficiently large p such that

$$d_{GH}(S^1_{2\pi/p} \times S^2, S^3/\mathbb{Z}_p) < \epsilon\,. \tag{6}$$

Since the spaces become isometric in the limit of infinite p, this motivates the series of equalities in the limit of large p and large k represented in figure 1. While an exact isometry between two spaces implies they are homeomorphic, two spaces separated by arbitrarily small Gromov-Hausdorff distance can still be very different topological spaces. For example we can replace the lens space in figure 1 with a squashed (Berger) sphere S^3_ν. This one parameter family of solutions also collapses to an S^2 as $\nu \to 0$, motivating the equalities discussed above. However, $\pi_1(S^3_\nu) = 0$ for all ν, meaning it maintains distinct topological properties as ν is varied and there is not as precise a notion of an "emergent circle."[38]

These arguments suggest that the lens space partition function at high temperature and large lensing inherits the mapping class group of the $\mathbb{T}^2 \times S^2$ partition function. This justifies the cycle-swapping equivalence conjectured in (2). (But notice that it does not provide a justification for the stronger equality (1)).

The conjecture of this work can be stated more generally as an invariance under swapping the thermal cycle with the cycle of any manifold $\mathcal{M}^{d-1}$ that can be written as a circle bundle with a (possibly locally) freely acting $U(1)$, which allows lensing the cycle in $\mathcal{M}^{d-1}$ to become arbitrarily small and induce collapse to $\mathcal{M}^{d-2}$. The size of the emergent S^1 is set by $\mathrm{Vol}(S^1/\mathbb{Z}_p \times \mathcal{M}^{d-2}) = \mathrm{Vol}(\mathcal{M}^{d-1}/\mathbb{Z}_p)$. Altogether, the requirement for replacing $\mathcal{M}^{d-1}/\mathbb{Z}_p$ with $S^1/\mathbb{Z}_p \times \mathcal{M}^{d-2}$ will be that the two spaces become isometric and have the same fundamental group as the lensing is taken large.

A subtle point about this equivalence is that it will imply that the high-temperature partition function is equal to a low-temperature partition function. The high-temperature partition function defines a scheme-independent thermal entropy, while the low-temperature partition function depends on the vacuum energy. As is well known, the vacuum energy on curved manifolds is generically scheme-dependent in even-dimensional CFT.

What gives? The resolution to this is that the vacuum energy becomes scheme-independent as the fiber cycle is taken small. In this case, the contribution of any scheme-dependent counterterm goes to zero. This is general and independent of supersymmetry.

Taking the order of a quotient to be infinitely large as a control parameter does not seem to have received much (if any) attention in the literature. This note will show that it provides nontrivial constraints at zeroeth order in perturbation theory around the infinite quotient, with the potential to go to higher orders. For a distinct application of infinite quotients see [12].

Related work includes [13–16].

FOUR DIMENSIONS

Conformally coupled scalar

Consider a conformally coupled scalar at inverse temperature $\beta = 2\pi/k$ on a lens space $S^3_R/\mathbb{Z}_p$ with radius R. This space has volume $V = 2\pi^2 R^3/p$. At high temperature $\beta^3/V \ll 1$ we have[39]

$$\lim_{\substack{k/p\to\infty \\ p\to\infty}} \log Z\left[S^1_{2\pi/k}\times S^3_R/\mathbb{Z}_p\right] = \frac{\pi^4 R^3}{45p(2\pi/k)^3} = \frac{\pi k^3 R^3}{360p} \tag{7}$$

Swapping the thermal cycle with the fiber cycle gives us a low-temperature partition function on a different lens space, for which we have

$$\lim_{\substack{k/p\to\infty \\ p\to\infty}} \log Z\left[S^1_{2\pi R/p}\times S^3_R/\mathbb{Z}_{kR}\right] = -\left(\frac{2\pi R}{p}\right)\left(-\frac{k^3R^3}{720R}\right) = \frac{\pi k^3 R^3}{360p}\,. \tag{8}$$

The above expression illustrates that the low-temperature partition function projects to the vacuum state with energy that becomes scheme-independent as $k\to\infty$:

$$E_{\mathrm{vac},\,S^3_R/\mathbb{Z}_{kR}} = \frac{14-10(kR)^2-(kR)^4}{720R(kR)} \longrightarrow \frac{-k^3R^3}{720R} \tag{9}$$

The expressions above are straightforward to calculate and can be found for example in [17]. We see that the two expressions are equal as predicted by the conjecture. For the rest of the checks, we will not keep the radius of the lens space explicit.

$\mathcal{N}=1$ superconformal theories and Di Pietro-Komargodski formula

In [18], asymptotic Cardy-like formulas for supersymmetric partition functions in four and six dimensions were proposed (see also [19, 20] for early conjectures in this direction). Here we will focus on the case of $\mathcal{N}=1$ superconformal theories in four dimensions. Supersymmetry will be preserved so the "thermal" S^1 will have non-thermal periodicity conditions that match those of the Hopf fiber.

We consider the case of a squashed lens space, a generalization of [18] treated in [21]. The squashed sphere we are lensing is also known as a Berger sphere. The Berger sphere metric can be written as

$$ds^2 = \frac{1}{4}\left(d\theta^2 + \sin^2\theta\, d\phi^2 + \nu^2\left(d\psi + \cos\theta\, d\phi\right)^2\right) \tag{10}$$

where $\psi\sim\psi+4\pi$. The $\nu=1$ point is the unit three-sphere. The high-temperature supersymmetric partition function is given as

$$\lim_{\substack{k/p\to\infty \\ p\to\infty}} \log Z\left[S^1_{2\pi\nu/k}\times S^3_\nu/\mathbb{Z}_p\right] = -\frac{8\pi k}{3p}(a-c)\,. \tag{11}$$

The thermal circle is normalized to match the size of the emergent circle in the lens space when $k=p$. The size of the emergent circle is fixed by the volume condition discussed in the introduction, i.e. equating $\mathrm{Vol}[S^3_\nu/\mathbb{Z}_p] = \mathrm{Vol}[S^1\times S^2]$ for S^2 with radius $1/2$ sets the size of the emergent S^1 to be $2\pi\nu/p$. The supersymmetric vacuum energy on a lensed Berger sphere $S^3_\nu/\mathbb{Z}_k$ is given as [22]

$$E_{\mathrm{susy},\ S^3_\nu/\mathbb{Z}_k} = \frac{16}{27k\nu}(3c-2a) + \frac{4k}{3\nu}(a-c)\,. \tag{12}$$

Unlike the previous section, this vacuum energy is manifestly scheme-independent for arbitrary fiber size, thanks to supersymmetry. The low-temperature partition function is given by the supersymmetric vacuum energy through a projection to the ground state:

$$\lim_{\substack{k/p\to\infty \\ p\to\infty}} \log Z\left[S^1_{2\pi\nu/p}\times S^3_\nu/\mathbb{Z}_k\right] = -\left(\frac{2\pi\nu}{p}\right)\left(\frac{4k}{3\nu}\right)(a-c)\,. \tag{13}$$

As advertised, the high-temperature partition function on a given squashed lens space (11) equals the low-temperature partition function on a different squashed lens space (13). Notice that it was crucial that the limit $k\to\infty$ picked out the second etrm in (12).

It is somewhat surprising that the equivalence holds at $\mathcal{O}(k)$ for supersymmetric partition functions, since for non-supersymmetric partition functions – which have an extensive leading term $\mathcal{O}(k^3)$ – we will see in upcoming sections that the equivalence fails at $\mathcal{O}(k)$ (we did not calculate the conformally coupled scalar to this order to see disagreement). Thus, agreement at $\mathcal{O}(k)$ seems special to supersymmetric partition functions.

We also see from the above results that the large squashing limit $\nu \to 0$ (where we do not put a factor of ν in the size of the thermal circle now), which also collapses the spatial manifold to S^2, does not give an appropriate modular equivalence since it does not isolate the second term in (12). In particular, $Z[S^1_{2\pi/k} \times S^3_\nu] \neq Z[S^1_{2\pi\nu} \times S^3_{k^{-1}}]$ at large k and small ν, except for the trivial case $\nu = k^{-1}$. Presumably the approach to equivalence of fundamental groups as the manifold collapses is important for supersymmetric free energies but not as important for extensive free energies.

A similar check, not included here, can be performed for the theory on an ellipsoidal lens space, where the ellipsoid preserves a $U(1)\times U(1)$ isometry of S^3, as opposed to the $SU(2) \times U(1)$ preserved by the Berger sphere.

Notice that the duality is powerful enough to control the high-temperature partition function on an unlensed S^3_ν (which in the case of a round S^3 is counting local operators that sit in short representations of the superconformal group), equating it with the low-temperature partition function on a highly lensed S^3_ν. Also notice that the superconformal index, which is shifted from the supersymmetric partition function by a vacuum energy factor $e^{\beta E_{\rm susy}}$, would not exhibit nice modular properties. This precisely mimics the case of modular invariance in two-dimensional CFTs, where the shift of operator dimensions by $-c/12$ is necessary to exhibit modular invariance.

$\mathcal{N} = 4$ super Yang-Mills at weak coupling

Now consider $\mathcal{N} = 4$ $SU(N)$ super Yang-Mills theory at weak 't Hooft coupling. The relevant partition functions and vacuum energies we will use in this section have been calculated in [23].

For the NS-NS partition function, which corresponds to antiperiodic periodicity conditions for the fermions along the thermal cycle and the fiber cycle (requiring k to be even), we have

$$\lim_{\substack{k/p\to\infty \\ p\to\infty}} \log Z\left[S^1_{2\pi/k} \times S^3/\mathbb{Z}_p\right] = \frac{\pi N^2 k^3}{24p}\,, \tag{14}$$

$$\lim_{\substack{k/p\to\infty \\ p\to\infty}} \log Z\left[S^1_{2\pi/p} \times S^3/\mathbb{Z}_k\right] = -\frac{2\pi}{p}\, E^{(\rm NS)}_{{\rm vac},\, S^3/\mathbb{Z}_k} = \frac{\pi N^2 k^3}{24p}\,. \tag{15}$$

$\mathcal{N} = 4$ super Yang-Mills at strong coupling

We can also check the conjecture at strong coupling using the AdS/CFT correspondence. In fact, for this case we will be able to illustrate modular S-invariance at intermediate temperatures since we will have access to $\log Z$ as a function of arbitrary k and p. This requires using the proposed phase structure on $S^1_{2\pi/k} \times S^3/\mathbb{Z}_p$, with antiperiodic fermions along both cycles, for which there is strong evidence but no proof. The bulk phase structure has two saddles, the thermal Eguchi-Hanson-AdS (EH) metric representing the confined phase and the AdS-Schwarzschild/$\mathbb{Z}_p$ black hole (BH) representing the deconfined phase. We can test the proposed modular invariance at medium temperatures due to the exact expressions available for the on-shell actions. At large k and p, independent of how we take the limit (i.e. independent of exactly what ratio of powers of k to p we keep fixed), we have

$$\log Z_{\rm BH} = \frac{\pi^2 k^3}{64pG} - \frac{3\pi^2 k}{16pG} + \mathcal{O}\left(\frac{1}{kp}\right)\,, \tag{16}$$

$$\log Z_{\rm EH} = \frac{\pi^2 p^3}{64kG} - \frac{\pi^2 p}{8kG} + \mathcal{O}\left(\frac{1}{kp}\right)\,. \tag{17}$$

As advertised, the leading order answer is invariant under swapping $k \leftrightarrow p$, since this maps us from the confined (deconfined) to the deconfined (confined) phase, leaving the partition function invariant. Notice that the first subleading correction ruins this invariance, illustrating the necessity of the limit. As required by modularity, the phase transition at large p occurs at $k = p$. This is just like the BTZ/thermal AdS$_3$ transition, which by modularity necessarily occurs when the thermal cycle size equals the spatial cycle size.

SIX DIMENSIONS

$\mathcal{N} = (2,0)$ theory on circle fibration over $S^2 \times S^2$

We can test the conjecture for the strongly coupled $\mathcal{N} = (2,0)$ theory. Here we consider supergravity in AdS$_7$ with asymptotic topology $S^1 \times \mathcal{M}^5/\mathbb{Z}_p$. We will consider $\mathcal{M}^5$ to be a circle fibration over $S^2 \times S^2$. We again have two competing saddles, a black hole solution representing the deconfined phase and a solitonic solution representing the confined phase. The relevant solutions and the transition between the two saddles were developed in [24–26]. Taking the large p and large k limit, again in any order, gives us

$$\log Z_{\rm BH} = \frac{\pi^3 k^5}{1296p} - \frac{5\pi^3 k^3}{324p} + \mathcal{O}\left(\frac{k}{p}\right))\,, \tag{18}$$

$$\log Z_{\rm EH} = \frac{\pi^3 p^5}{1296k} - \frac{\pi^3 p^3}{108k} + \mathcal{O}\left(\frac{p}{k}\right)\,. \tag{19}$$

This has the proposed $k \leftrightarrow p$ invariance at leading order, and we again see that it is violated at first subleading order. The phase transition between saddles occurs at

$p = k$ as required by modularity. This context and that of $\mathcal{N} = 4$ in the last section have phase structures that can be predicted (at large k and p) using modular invariance and center symmetry arguments as explained in [27].

OUTLOOK

In the context of odd-dimensional spheres, the conjecture of this work relates sphere partition functions to torus partition functions. This allows the use of modular invariance while maintaining an interpretation in terms of the operator content of the theory.

Only modular S-invariance has been checked in this paper, for a handful of conformal field theories, on a few different circle bundles (ellipsoids and Berger spheres in four dimensions and circle bundles over $S^2 \times S^2$ in six dimensions). The most general form of the conjecture is that for any circle fibration with a locally free $U(1)$ action that admits arbitrarily large lensing of the fiber (e.g. Seifert manifolds in three dimensions), there will emerge an $SL(2, \mathbb{Z})$ invariance of the theory in terms of the complex structure τ of the emergent $\mathbb{T}^2$. Phrasing the invariance in terms of the complex structure requires rigid rescalings of the base manifold. It would be interesting to test this general $SL(2, \mathbb{Z})$ invariance and study any modular covariance properties of correlators under the symmetry.

Another important extension to this work is to build a "$1/k$ expansion" for large lensing. As we saw, at subleading order in $1/k$ the modular equality breaks down as expected. Large lensing may generally provide a new perturbative framework to study quantum field theory on certain curved backgrounds.

Unlensed $S^1 \times S^3$ partition function

In every case considered in this work, the conjectured modular invariance turns out to control the high-temperature limit $\beta = 2\pi/k \to 0$ on an unlensed S^3, relating it to the low-temperature partition function on $S^3/\mathbb{Z}_{k\to\infty}$. For thermal partition functions of local CFTs this is not too surprising, as the high-temperature partition function is rather universal and depends only on the volume of the spatial manifold. To be consistent with the formulas of [2, 3], this implies that the coefficient controlling the vacuum energy on $S^3/\mathbb{Z}_{k\to\infty}$ is the same as the coefficient controlling the vacuum energy on $S^1 \times \mathbb{R}^2$. This is indeed the case in the examples considered. Interestingly, though, supersymmetric partition functions on S^3 (whose leading piece is not extensive) also have their high-temperature limit controlled by the vacuum energy on a lens space.

Further applications of large lensing in quantum field theory

The primary assumption used throughout this work is an approximate equivalence between a nontrivial circle fibration and a trivial circle fibration. This can be used for applications unrelated to modular invariance. For example, we can consider CFT$_3$ on S^3. If the approximate equivalence is correct, then $Z[S^3/\mathbb{Z}_{p\to\infty}] \approx Z[S^1_{2\pi/p\to 0} \times S^2]$. Again, the equality should be between the leading singularity in the two path integrals. This limit gives credence to the idea of large-N "topological volume-independence" of center-symmetric gauge theories introduced in [27] by relating it to ordinary large-N volume independence [28–31], which is firmly established.

Another potential application is to counting problems. We know that the finite, cutoff-independent part of $F = -\log|Z_{S^3}|$ [32] is a good monotonic quantity [33]. This free energy admits a counting interpretation in CFT since it can be conformally mapped to the entanglement entropy across an S^1 [34, 35]. It is also known that the thermal coefficient c_{therm} which enters the entropy density on $S^1 \times \mathbb{R}^{d-1}$ as $s \sim c_{\text{therm}} T^{d-1}$ (which is the same as the coefficient ε_{vac} which enters into the vacuum energy on a spatial $S^1 \times \mathbb{R}^{d-2}$) is *not* a good monotonic quantity unless $d = 2$. These two quantities can now be connected by the one-parameter family of lens spaces. The free energy on $S^3/\mathbb{Z}_p$ with thermal boundary conditions on the Hopf fiber as $p \to \infty$ becomes the thermal free energy on $S^1_{2\pi/p} \times S^2$. But this gives precisely c_{therm} up to geometric factors. Understanding this one-parameter family may give intuition into the mechanics of the F-theorem and connect it to the breakdown of the c_{therm}-theorem. One conclusion we can immediately draw is that $F = -\log|Z_{S^3/\mathbb{Z}_p}|$, the lens space free energy, is not a good monotonic quantity for arbitrary p.

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[38] In many examples, however, like CFT_3 on S^3_ν, it is still nevertheless true that $Z[S^3_{\nu\to 0}] \approx Z[S^1_{2\pi\nu} \times S^2]$.

[39] I will almost always write expressions that are formally divergent. For example, for (7) the precise thing to do is to define a density by taking the limit of $pk^{-3}\log Z$, which would give a finite answer.