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In this Letter, we study the effect of topological zero modes on Entanglement Hamiltonians and entropy of free chiral fermions in (1+1)d. We show how Riemann-Hilbert solutions combined with finite rank perturbation theory allow us to obtain exact expressions for Entanglement Hamiltonians. In the absence of the zero mode, the resulting Entanglement Hamiltonians consists of local and bi-local terms. In the periodic sector, the presence of a zero mode leads to an additional non-local contribution to the entanglement Hamiltonian. We derive an exact expression for this term and for the resulting change in the entanglement entropy.

Entanglement Hamiltonians (EH) are the next object to explore in a series of advances in our understanding the quantum structure of many-body states in field theory and condensed matter [1–8]. Indeed, much work has been devoted to understanding entanglement entropy, in a variety of systems, and, subsequently, more detailed information about the spectrum of reduced density matrices is being investigated as well. Entanglement Hamiltonians go one step further in that they contain information about the entanglement spectrum, as well as about the associated eigenvectors, and the possibility to understand the reduced state as a thermal state. Perhaps the most striking of the recent results about Entanglement Hamiltonians is the realization that for a spherical entangling region in a conformal field theory (CFT), Entanglement Hamiltonians have a local form in terms of the original Hamiltonian energy density with a spatially dependent temperature [9–11]. This result is unusual in its simplicity, as, unfortunately, the computation of EHs is, in general, much more involved than entropy and spectrum and only a few results are available. Thus it is a grand challenge to find additional solvable cases.

Here, we present a new method for computing EHs of free fermions in the presence of zero modes. Such modes are usually tied to the appearance of ground state degeneracies and reflect the nature of topological defects through unusual properties such as charge fractionalization [12] and non-abelian braiding [13] and appear in a variety of systems from polyacetylene [14] to topological insulators and superconductors [15–17]. In particular, we compute EHs for chiral fermions, and study in detail the effects of the boundary conditions (BC), periodic/anti-periodic for Majorana and generic for Dirac fermions, and of the zero modes (present in the case of periodic boundary conditions) on entanglement. The edge theory of the $p+ip$ superconductor provides an explicit realization of such a scenario for Majorana fermions and serves as a concrete physical model for our calculation (see, e.g. [18]). Wherever available we make contact with previous results in the literature.

Our approach utilizes the relation between the EH and the resolvent associated with the fermion Green's function adding the contribution of zero modes with an essentially exact re-summation. As we find below, for chiral fermions, the computation of the resolvent can be recast into a Riemann-Hilbert problem (RHP, see below), generalizing the approach of Casini and Huerta [1] who computed the EH for free chiral fermions, albeit without zero modes. Combining the RHP result with exactly summable perturbation theory, we compute the EH for chiral Majorana and Dirac fermions on a finite circle. Our main result is an exact analytic form for the EH, Eq. (15), with a contribution from the topological zero given separately in Eq. (21).

The reduced density matrix ρ_V on a spatial region V is defined to reproduce expectation values of all operators O_V inside V via the relation $\mathrm{Tr_V}(\rho_\mathrm{V} \mathrm{O_V}) = \langle \mathrm{O_V}\rangle$. The EH $\mathcal{H}_V$ is an effective Hamiltonian inside V given by

$$\rho_V = \frac{e^{-\mathcal{H}_V}}{Z_V}. \tag{1}$$

For free fermions, Wick's Theorem implies that the EH is a quadratic operator whose kernel (the single particle EH) H_V is uniquely determined by the equal-time Green's function

$$G(x,y) = \langle \Psi(x)\Psi^\dagger(y)\rangle \tag{2}$$

via the relation $H_V = -\log((P_V G P_V)^{-1} - 1)$ [19]. Here $P_V G P_V$ is the Green's function restricted to the region V by the projectors P_V. As we demonstrate in the supplementary materials, a similar relation holds for Majorana fermions ψ, given $\langle\psi(x)\psi(y)\rangle$, allowing us to treat the problem on a parallel footing.

The relation between H_V and G may be expressed in integral form as:

$$H_V = -\int_{\frac{1}{2}}^{\infty} \left(L(\beta) + L(-\beta)\right) \mathrm{d}\beta, \tag{3}$$

where $L(\beta)$ is the resolvent [1]

$$L(\beta) = (P_V G P_V - 1/2 + \beta)^{-1}. \tag{4}$$

Thus the derivation of the EH reduces to a computation of a projected Green's function resolvent. It turns out that finding the resolvent $L(\beta)$ can be recast as an RHP.

This connection between $L(\beta)$ and the RHP arises as follows: for free fermions (in the absence of zero modes), the Green's function $G(x,y)$ computed on the ground state, *un*-restricted to some entangling region V, acts as a single particle operator that projects onto positive energy modes. For chiral fermions in $(1+1)d$, the projection onto positive modes can be interpreted as a projection onto functions *analytic* on the upper/lower half of the complex plane and the computation of the resolvent (4) (associated with the projection of G onto V) can be mapped onto a RHP (see, e.g. [20]). In the case treated here we have a scalar RHP and so exactly solvable. Succinctly put, a RHP is a jump problem along a contour $\mathcal{C}$ in the complex plane, for a piecewise analytic matrix function. Solving it amounts to finding a matrix function $X(z)$ which is analytic on $\mathbb{C}\setminus\mathcal{C}$, and has boundary values on both sides of $\mathcal{C}$, $X_{\pm}(z)$, subject to a jump condition $X_-^{-1}(s)X_+(s) = v(s)$ on $\mathcal{C}$, with a given matrix $v(s)$. The jump function for our case is a scalar determined by the projector onto V, P_V.

Once we have access to the resolvent (4) (either through the RHP or any other method), we can use it to explore various deformations of the system that yield simple perturbations of the Green's function (2). In particular, topologically nontrivial background for the fermion result in the appearance of zero modes [21, 22] manifesting themselves as *finite rank* perturbations of $G(x,y)$. The perturbed resolvent is then computed exactly by summing up the resulting perturbative series.

We carry out this recipe for chiral Dirac/Majorana fermions on a spatial circle with general boundary conditions. Starting with anti-periodic (Neveu-Schwarz, NS) fermions on a circle with no gauge field, inserting a π flux then takes the anti-periodic sector to the periodic (Ramond, R) sector, which has a zero mode whose effect on the EH we derive explicitly. The appearance of ground state degeneracy for periodic boundary conditions has been studied extensively. In particular, Affleck and Ludwig [23], introduced the notion of boundary entropy, to understand the change in degeneracy due to impurities. As noted in [24], boundary entropy can appear as a subleading contribution to the EE of a finite interval. While these calculations are usually carried out by looking at entropy, our EH calculation allows us to study the effect of degeneracy from the point of view of the local density matrix. In Eq. (23), we find the exact change in the entanglement entropy (EE) of free chiral fermions between periodic and anti-periodic BC, for an arbitrary subset of the circle, and the term in the EH associated with it.

Concretely, we compute the EH on a region $V = \cup(a_j, b_j)$ as depicted in Fig. 1. Consider a chiral Dirac fermion in on a ring with periodicity $x \sim x + 2\pi R$. Assuming generic BC, $\Psi(x+2\pi R) = \Psi(x)e^{2\pi\alpha i}$, the mode expansion is $\Psi(x,t) = \frac{1}{\sqrt{2\pi R}}\sum_{k\in\mathbb{Z}+\alpha} b_k e^{-i\frac{k}{R}(t-x)}$, where $\alpha \in [0,1)$. Here $\alpha = \frac{1}{2}$ for the NS sector and $\alpha = 0$ for the R sector. For $\alpha \neq 0$, evaluating the equal time Green's function on the ground state $|\Omega\rangle$, defined by $b_{k>0}|\Omega\rangle = b^\dagger_{k<0}|\Omega\rangle = 0$, gives

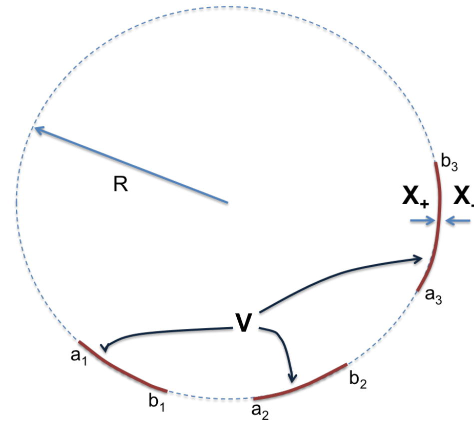

FIG. 1: The entanglement region $V = \cup(a_j, b_j)$ for fermions on a ring of radius R considered in the paper, and an associated Riemann-Hilbert problem.

$$G^\alpha(x,y) \equiv \langle\Omega|\Psi(x)\Psi^\dagger(y)|\Omega\rangle = e^{\frac{i\alpha(x-y)}{R}} n(x,y). \qquad (5)$$

where $n(x,y) \equiv \langle x|n|y\rangle$ is the kernel of the single particle projector n onto momentum modes $\langle x|k\rangle = \frac{1}{\sqrt{2\pi R}}e^{\frac{ikx}{R}}$ with k a non-negative integer [29]. Explicitly:

$$n(x,y) \equiv \frac{1}{2\pi R}\sum_{k=0}^{\infty} e^{\frac{ik(x-y+i0^+)}{R}}. \qquad (6)$$

Note that the G^α for $\alpha \neq 0$ are related by spectral flow, implemented by the transformation

$$G^\alpha \;=\; U_\alpha n U_\alpha^{-1}, \quad \alpha \neq 0, \qquad (7)$$

where U_α are unitaries that shift momenta by α: $U_\alpha|k\rangle = |k+\alpha\rangle$. Moreover, U_α commutes with the spatial projection P_V, implying that the projected resolvents (4) and EH kernel (3) for different α are also related by a similarity transform. Thus, we can relate all the $\alpha \neq 0$ resolvents $L^\alpha(\beta) = U_\alpha N(\beta) U_\alpha^{-1}$ to the following one:

$$N(\beta) \equiv \left(P_V n P_V + \beta - \tfrac{1}{2}\right)^{-1} \qquad (8)$$

As an operator acting on functions defined on V, $N(\beta)$ is the resolvent for $P_V n P_V$, which is an *integrable operator* in the sense of [20, 25]. In general, such operators are defined on a curve $\mathcal{C}$ on a manifold conformally related to the Riemann Sphere and have a kernel of the form

$$K(x,y) = \sum_{i=1}^{m} f_i(x) n(x,y) g_i(y), \qquad (9)$$

where $n(x,y)$ is the kernel of a projector onto analytic functions "inside" $\mathcal{C}$, and f_i, g_i are a set of given functions defined on $\mathcal{C}$. It is known that the resolvent, $(1+K)^{-1}$, is related to the solution of an $m\times m$ matrix RHP [20]. In our case, $\mathcal{C}$ is the circle of radius R on the compactified complex plane Fig. 1. Since we only have a single term in K, the matrix RHP reduces to the scalar one and can be stated as follows: Find functions X_+/X_- that are analytic inside/outside the unit disc satisfying

$$X_-^{-1}(x)X_+(x) = 1 + \frac{f(x)g(x)}{\beta - \frac{1}{2}} \equiv X(x); \;\; x \in \mathcal{C}. \quad (10)$$

Equivalently, we can characterize X_+/X_- as containing only k positive /negative momentum modes respectively. This leads to the crucial identities $nX_+^{-1}n = X_+^{-1}n$ and $nX_-^{-1}n = nX_-^{-1}$, and to the resolvent formula [30]:

$$N(\beta) = \frac{1}{\beta - \frac{1}{2}}\left(1 - \frac{1}{\beta-\frac{1}{2}} fX_+^{-1}nX_-g\right) \quad (11)$$

For our application $f = g = \Theta_V$ where Θ_V is the characteristic function of V ($\Theta_V(x) = 1$ if $x \in V$ and is 0 otherwise). Denoting

$$h(\beta) = \frac{1}{2\pi}\ln\frac{\beta+\frac{1}{2}}{\beta-\frac{1}{2}}, \quad (12)$$

our RHP Eq. (10) reduces to $\ln(X_+(x)) - \ln(X_-(x)) = 2\pi h(\beta)\Theta_V(x)$, and has the standard solution

$$\ln X_\pm(x) = ih(\beta)Z^\pm(x); \quad (13)$$
$$Z^\pm(x) = \textstyle\sum_j \ln\left(\frac{e^{\frac{i(x\pm i\epsilon)}{R}} - e^{\frac{ia_j}{R}}}{e^{\frac{i(x\pm i\epsilon)}{R}} - e^{\frac{ib_j}{R}}}\right).$$

Substituting $X_\pm$ from (13) into (11), we find:

$$\langle x|N(\beta)|y\rangle = \tfrac{1}{\beta-1/2}\delta(x-y) - \tfrac{e^{-ih(Z^+(x)-Z^+(y))}}{\beta^2-\frac{1}{4}}n(x,y)$$

This form is the finite radius generalization of the result obtained in [1] for fermions on an infinite line.

The EH in the $\alpha \neq 0$ sector is obtained by applying (7) to compute the resolvent and then integrating over β as in (3). Adapting the procedure described in [1] to our case, we find:

$$H_V^{\alpha\neq 0} = 4\pi^2 e^{i\alpha\frac{(x-y)}{R}} n_{PV}(x,y)\delta(Z^+(x)-Z^+(y)), \quad (14)$$

where $n_{PV}(x,y) = n(x,y) - \frac{1}{2}\delta(x-y)$ is the principal part of n. Using the distribution kernel (14) to compute explicitly the EH, gives, owing to the δ function, a local contribution

$$\mathcal{H}^{\alpha\neq 0}_{V,\text{loc.}} = -2\pi\int_V dx\Psi^\dagger\left[\frac{i}{|Z'|}\frac{d}{\mathrm{dx}} - \frac{(1-2\alpha)}{2R|Z'|} - \left(\frac{i}{2|Z'|}\right)'\right]\Psi \quad (15)$$

as well as a bi-local contribution with kernel:

$$H^{\alpha\neq 0}_{V,\text{ bi-loc.}} = 2\pi\sum_{l;y_l(x)\neq x}\frac{e^{i\alpha\frac{(x-y)}{R}}|Z'(x)|^{-1}}{R\left(1-e^{\frac{i(x-y)}{R}}\right)}\delta(x-y_l(x)), \quad (16)$$

where $y_l(x)$ are solutions of $Z(x) = Z(y)$.

Let us pause to inspect the single interval result for $\alpha = \frac{1}{2}$. Since the free fermion is a conformal field theory, the CFT results of [10, 11] suggest that the single interval EH should be an integral of the energy density $Q \propto i\Psi^\dagger\partial\Psi$. At first glance, (15) seems at odds with this form, as it contains an additional piece proportional to the density $\rho(x) = \Psi^\dagger(x)\Psi(x)$. However, the CFT expectation and the direct calculation are, in fact, consistent. Indeed, the interpretation of the CFT local energy tensor in [10, 11] as an operator, has to be done with care: in particular, the form $-i\int dx\, c(x)\Psi^\dagger\partial\Psi$, is not Hermitian by itself. However, integration by parts shows that it is made Hermitian with the addition of a term proportional to an integral over the density $(i/2)\int dx\, c'(x)\rho(x)$, which is exactly the extra term in Eq. (15) for $\alpha = 1/2$. In the path integral formalism of [11], this term can be derived by identifying the EH as the generator of conformal rotations fixing the end points of V, which must also generate a phase rotation of the Dirac fermion due to its non-trivial conformal spin. This symmetry generator is the hermitian form of the local energy operator given by $T_{00} = \frac{-i\Psi^\dagger(\partial_x\Psi)+i(\partial_x\Psi^\dagger)\Psi}{2}$. Given this definition, our result (15) takes the form $\mathcal{H}_\mathcal{V} = \int_V dx\beta(x)T_{00}$, with $\beta(x) = 2\pi|Z'(x)|^{-1} = 4\pi R\csc\frac{a-b}{2R}\sin\frac{a-x}{2R}\sin\frac{b-x}{2R}$ as a local entanglement temperature. Turning on a flux then introduces a shift of T_{00} by a conserved charge density $-\mu\rho(x)$ with chemical potential $\mu = \frac{(1-2\alpha)}{2R}$. As shown in [11], this leads to a generalized first law of EE in which small excitations of the vacuum causes a change in EE given by $\delta S_V = \int_V dx\,\beta(x)\delta\langle(T_{00}-\mu\rho(x))\rangle$.

The Ramond sector in which $\alpha = 0$ requires separate consideration, since the zero mode $k = 0$ acts on a doubly degenerate ground state subspace of occupied/unoccupied modes. We choose the state dictated by the zero-temperature limit of the Fermi-Dirac distribution. For a topological zero mode, Fermi-Dirac gives the mixed state $\frac{1}{2}(|\text{occupied}\rangle\langle\text{occupied}| + |\text{empty}\rangle\langle\text{empty}|)$. In this state [31], the Green's function is

$$G^{\alpha=0} = n - \frac{1}{2}|0\rangle\langle 0| \quad (17)$$

and our Resolvent is

$$L^{\alpha=0}(\beta) = (P_V nP_V - \frac{1}{2} + \beta - \frac{1}{2}P_V|0\rangle\langle 0|P_V)^{-1} \quad (18)$$

where $|0\rangle$ is the $k = 0$ momentum mode $\langle x|0\rangle = \frac{1}{\sqrt{2\pi R}}$.

The difference relative to the resolvent (8) is that the zero mode introduces a shift of n by the rank one perturbation $-\frac{1}{2}|0\rangle\langle 0|$. We proceed by treating the zero mode contribution as a Dyson perturbative expansion, which we subsequently re-sum (see supp. material), leading to

$$L^{\alpha=0}(\beta) = N(\beta) + \frac{N(\beta)P_V|0\rangle\langle 0|P_VN(\beta)}{2-\langle 0|P_VN(\beta)P_V|0\rangle}. \quad (19)$$

We note that Eq. (19) may be used for computing the EH for single particle excitations of the vacuum and that the re-summation also works for higher rank perturbations.

While the first term on r.h.s of Eq. (19) will yield a H_V as before, the second term on the r.h.s. is new: it is due to the zero mode and produces a non-local contribution to the EH. Explicitly,

$$\langle x|L^{\mathrm{R}}_{\text{zero-mode}}(\beta)|y\rangle = \frac{2\sinh^2(\pi h)e^{ih(Z(y)-Z(x))}}{\pi R(1+e^{\frac{l_v h}{R}})}, \quad (20)$$

where $l_v = \sum_i(b_i - a_i)$ is the total length of V. Using (3), the contribution from $L^{\mathrm{R}}_{\text{zero-mode}}$ to the EH is:

$$H^{\mathrm{R}}_{V\,\text{zero-mode}} = \frac{-1}{R}\int_{-\infty}^{\infty} dh \frac{1}{1+e^{\frac{l_v h}{R}}} e^{ih(Z(y)-Z(x))} \quad (21)$$
$$= \sum_l \frac{-\pi}{|Z'(x)|R}\delta(x-y_l(x)) + p.v.\frac{\pi i}{2l_v \sinh\left(\frac{\pi R}{l_v}(Z(y)-Z(x))\right)}$$

Thus, the zero mode induced part of the EH has a non-local contribution, even for a single interval. Remarkably, for one interval the $\delta(x-y)$ term exactly cancels the $1/2$ shift in α (the "chemical potential" term) in (15), and we get that, as far as the *strictly local* terms are concerned, $H_{V,\mathrm{loc.}}(\alpha=0) = H_{V,\mathrm{loc.}}(\alpha=1/2)$.

The new non-local contribution to the resolvent due to the zero mode also changes the EE in the R sector relative to the NS. The EE $S_V = -\mathrm{Tr}\rho_V \ln \rho_V$ can be expressed as [1]:

$$S_V \equiv \int_{\frac{1}{2}}^{\infty} \mathrm{d}\beta \mathrm{Tr}[(\beta-1/2)(L(\beta)-L(-\beta)) - \tfrac{2\beta}{\beta+1/2}]. \quad (22)$$

It follows, using (8), that all Dirac fermions with $\alpha \neq 0$ have the same EE (disregarding possible anomaly contributions coming from the UV cutoff).

Using (22) we find the contribution to the EE from the change in BC, $\delta S \equiv S_R - S_{NS}$:

$$\delta S = \frac{l_v}{2R}\int_0^{\infty} dh \tanh\left(\frac{l_v h}{2R}\right)(\coth(h\pi)-1). \quad (23)$$

Expanding in the ratio $\frac{l_v}{R}$ gives:

$$\delta S \sim \sum_{n=1}^{\infty} \frac{(2^{2n}-1)B_{2n}\zeta(2n)l_v^{2n}}{2n(2\pi R)^{2n}} \sim \frac{l_v^2}{48R^2} - \frac{l_v^4}{5760R^4}, \quad (24)$$

where B_{2n} are Bernoulli numbers.

Remarkably, using the replica trick, Herzog and Nishioka have previously found the form (24) for δS, noting that the series is not convergent [26]. The exact expression Eq. (23) for the entropy allows us to identify the replica trick result of [26] as an asymptotic expansion of the convergent integral (23). As a check we take the limiting case where V is the entire circle. Plugging $l_v \to 2\pi R$ into (23), we find $\delta S = \log 2$, as expected from the degeneracy [32].

Finally, we show that a similar story holds for Majorana fermions. Using [27] we find that the relation between the Majorana Green's function

$$G_{\mathcal{M}}(x,y) = \langle\psi(x)\psi(y)\rangle \quad (25)$$

and EH is $H_V^{\mathcal{M}} = \frac{1}{2}\ln((P_V G_{\mathcal{M}} P_V)^{-1} - 1)$ [33]. This relation differs from the Dirac one by the important factor of $-1/2$, while $G_{\mathcal{M}}(x,y)$ agrees with the Dirac case in the NS and R sector. In addition, since $\psi(x)^2$ is a constant for Majoranas, we can ignore the $\delta(x-y)$ terms in the EH kernel. Thus we find that in the NS sector, the Majorana EH has the local term

$$\mathcal{H}^{\mathrm{NS}}_{V,\mathrm{loc.}} = \pi\int_V dx \frac{i}{Z'(x)}\psi(x)\partial_x\psi(x) \quad (26)$$

while the bilocal kernel (16) remains the same. In the R sector the situation is more subtle, as the Majorana zero mode b_0 has no natural partner to create a complex fermion. To get the minimal, non trivial Hilbert space representation of the Clifford algebra containing b_0, we have to introduce an additional Majorana. A physical realization of such a situation is the edge of a $p+ip$ superconductor, in the presence of a (π flux) vortex in the bulk (see, e.g. [18]). Such a vortex acts to change the BC on the edge into R type and binds a Majorana zero energy mode to its core. Combining the Majorana operator b_0' for the bulk-core zero-mode with our edge zero mode b_0, forms a complex/Dirac fermion operator $a_0 = \frac{1}{\sqrt{2}}(b_0 + ib_0')$ which switches between it's associated "$|\text{occupied}\rangle$" and "$|\text{empty}\rangle$" ground states. Since $b_0^2 = \frac{1}{2}$ by the Majorana anti-commutation relations, the Green's function is same as the Dirac one in (17) for any choice of ground state. Up to the $-\frac{1}{2}$ factor, the resulting R sector single particle EH is equal to the Dirac one, while the EE changes by a factor of $\frac{1}{2}$.

To summarize, in this letter we provide a framework for the computation of EH in the presence of topological defects by combining RHP and finite rank perturbation theory. We present the first exact calculation of an EH on a compact space, with and without zero modes. We find that the Hamiltonian consists of a local term in agreement with the CFT, as well as non-local terms associated with the zero mode and with several entangling intervals. We note that our method also applies to non-topological zero modes (however, such modes are unstable, and will typically break the ground state degeneracy). We also find an expression, Eq. (23), representing the *local* entropic signature of the presence of a boundary changing operator that reproduces boundary entropies computed previously. Immediate possible applications for the method are the EH and EE of excited states, and low temperatures [28]. While we concentrated here on the simple case of a scalar RHP, a full matrix problem appears immediately in more involved situations, in particular, for non-relativistic fermions with a finite Fermi sea [34].

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[29] For our right-moving sector chiral Lagrangian, the single particle momentum states $|k\rangle$ are states of energy $E_k = k$. Therefore a projector onto $k \geq 0$ momentum modes is also a projector onto states with $E_k \geq 0$.
[30] See supplementary material for more details.
[31] The following also applies to the pure state $\frac{1}{2}(|\text{occupied}\rangle + |\text{empty}\rangle$.
[32] The requirement of Wick's theorem to hold in the presence of zero modes, forces the state on the full circle to be a mixed state, hence the nonzero entropy for the full circle in the R sector.
[33] Supplementary Online Material.
[34] G. Wong, I. Klich and D. Vaman, *in preparation.*