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Wenlong Wang, M. A. Moore, and Helmut G. Katzgraber

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The Fractal Dimension of Interfaces in Edwards-Anderson and Long-range Ising Spin Glasses: Determining the Applicability of Different Theoretical Descriptions

Wenlong Wang,^{1,*} M. A. Moore,² and Helmut G. Katzgraber^{1,3,4}

¹*Department of Physics and Astronomy, Texas A&M University, College Station, Texas 77843-4242, USA*

²*School of Physics and Astronomy, University of Manchester, Manchester M13 9PL, UK*

³*IQB Information Technologies (IQBit), Vancouver, British Columbia, Canada V6B 4W4*

⁴*Santa Fe Institute, 1399 Hyde Park Road, Santa Fe, New Mexico 87501, USA*

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The fractal dimension of excitations in glassy systems gives information on the critical dimension at which the droplet picture of spin glasses changes to a description based on replica symmetry breaking where the interfaces are space filling. Here, the fractal dimension of domain-wall interfaces is studied using the strong-disorder renormalization group method pioneered by Monthus [Fractals **23**, 1550042 (2015)] both for the Edwards-Anderson spin-glass model in up to eight space dimensions, as well as for the one-dimensional long-ranged Ising spin-glass with power-law interactions. Analyzing the fractal dimension of domain walls, we find that replica symmetry is broken in high-enough space dimensions. Because our results for high-dimensional hypercubic lattices are limited by their small size, we have also studied the behavior of the one-dimensional long-range Ising spin-glass with power-law interactions. For the regime where the power of the decay of the spin-spin interactions with their separation distance corresponds to 6 and higher effective space dimensions, we find again the broken replica symmetry result of space filling excitations. This is not the case for smaller effective space dimensions. These results show that the dimensionality of the spin glass determines which theoretical description is appropriate. Our results will also be of relevance to the Gardner transition of structural glasses.

Spin glasses have been studied for more than half a century but there is still no consensus as to what order parameter describes their low-temperature phase. There are two competing theories: The oldest is the replica symmetry breaking (RSB) theory of Parisi [1–5], which is known to be correct for the Sherrington-Kirkpatrick (SK) model [6], which is the mean-field or infinite-dimensional limit of the short-range Edwards-Anderson (EA) Ising spin-glass model [7], the commonly used model for d -dimensional systems. Within the RSB picture there are a very large number of pure states. In a second theory, known as the “droplet” picture [8–10] there are only two pure states and the low-temperature state is replica symmetric. In the droplet picture the behavior of the low-temperature phase is determined by low-lying excitations or droplets whose (free) energies scale in their linear extent ℓ as ℓ^θ and whose interfaces have a fractal dimension $d_s < d$. In the RSB theory, however, there exist low-lying excitations which cost an energy of $\mathcal{O}(1)$ and which are space filling, that is, $d_s = d$. It has been argued [11] that when $d \leq 6$ the droplet picture applies while for $d > 6$ RSB is the appropriate picture. Note, however, that in finite space dimensions RSB is different to its infinite-dimensional limit; see Newman and Stein [12–14], as well as Read [15] for details. In this paper we study the fractal dimension as a function of the space dimension, $d_s(d)$ [16], to find the space dimension at which the droplets become space-filling, i.e., when $d_s(d) = d$. Our results are consistent with 6 being the critical dimension. It is, of course, difficult to overcome finite-size effects in numerical work near six dimensions. Therefore, our main evidence that 6 is the critical dimension comes from our study of the one-dimensional long-range spin-glass model introduced by Kotliar, Anderson and Stein (KAS) [17]. The calculational technique which we have used is the strong-disorder renor-

malization group (SDRG) introduced by Monthus [18]. This approach produces estimates of d_s , that are in agreement with results on the EA model using other numerical techniques for space dimensions 2 and 3 (also studied by Monthus in Ref. [18]). In this Letter, we extend the results of Ref. [18] up to $d = 8$ space dimensions, and apply the method introduced in the aforementioned reference to the KAS spin-glass model [17].

Whether there is RSB or not in dimensions $d \leq 6$ is not only important for spin glasses. In structural glasses there has been much recent interest in the Gardner transition, which is the transition at which replica symmetry breaking is supposed to occur to a glass state of marginal stability (for a review see Ref. [19]). However, recent numerical results have suggested that fluctuation effects about the mean-field solution might destroy the Gardner transition in at least three space dimensions [20]. This result is entirely consistent with our expectation that replica symmetry breaking will be absent for $d \leq 6$.

The Edwards-Anderson model [7] is defined on a d -dimensional cubic lattice of linear extent L by the Hamiltonian

$$\mathcal{H} = - \sum_{\langle ij \rangle} J_{ij} S_i S_j, \quad (1)$$

where the summation is over only nearest-neighbor bonds and the random couplings J_{ij} are chosen from the standard Gaussian distribution of unit variance and zero mean. The Ising spins take the values $S_i \in \{\pm 1\}$ with $i = 1, 2, \dots, L^d$.

We have studied this model in space dimensions $d = 4, \dots, 8$ using the SDRG method [18]. Reference [18] studied the cases of $d = 2$ and 3. The SDRG approach successively traces out the spin whose orientation is most dominated by a single large renormalized bond to another spin; when the

spin is eliminated the couplings of the remaining spins are renormalized accordingly. We refer the reader to Ref. [18] for further details.

The observable we focus on is related to the bond-average of Σ^{DW} , where Σ^{DW} is the number of bonds crossed by the domain wall when the boundary conditions in one direction are changed from periodic to anti-periodic. The SDRG method is essentially a way of constructing a possible ground state of the system. One runs the method twice, first with periodic and next with anti-periodic boundary conditions in one direction, and count the bonds across which the relative spin orientation across the bond has altered because of the change of boundary conditions. Pictures of a domain wall so constructed for dimension $d = 2$ can be found in Ref. [18]. It wanders, indicating that it has a fractal dimension and its length can be described by a fractal exponent d_s , where $\Sigma^{\text{DW}} \sim L^{d_s}$. If the interface were straight across the system, its length would be proportional to L^{d-1} . This means that because of the wandering one expects that $d_s > d - 1$. In the RSB phase the domain walls are space filling, i.e., $d_s = d$. In general, $d - 1 \leq d_s \leq d$.

We first introduce a more formal definition of Σ^{DW} which has a natural extension when we study long-range systems when the definition of an interface is far from obvious. One defines the link overlap [21] via

$$q_\ell = \frac{1}{N_b} \sum_{\langle ij \rangle} S_i^{(\pi)} S_j^{(\pi)} S_i^{(\bar{\pi})} S_j^{(\bar{\pi})} (2\delta_{J_{ij}^\pi, J_{ij}^{\bar{\pi}}} - 1). \quad (2)$$

Here $S_i^{(\pi)}$ and $S_i^{(\bar{\pi})}$ denote the ground states found with periodic (π) and anti-periodic ($\bar{\pi}$) boundary conditions, respectively. One can switch from periodic to anti-periodic boundary conditions by flipping the sign of the bonds crossing a hyperplane of the lattice. N_b is the number of nearest-neighbor bonds in the lattice which for a d -dimensional hyper-cube is given by $N_b = dL^d$. One can then define [21]

$$\Gamma \equiv 1 - q_\ell = \frac{2\Sigma^{\text{DW}}}{dL^d} \sim L^{d_s-d}. \quad (3)$$

In Fig. 1 we show the bond-averaged value of Γ [Eq. 3] vs $\ln L$ which should be a straight line of slope $d_s - d$. In Fig. 2 the value of d_s is plotted for various dimensionalities d . For $d = 1$, $d_s(1) = 0$ (pentagon), while for $d = 2$ we have used the value from Ref. [18], i.e., $d_s(2) = 1.27$ (square), which is in excellent agreement with other numerical estimates [21–27]. For $d = 3$, Ref. [18] quotes $d_s(3) = 2.55$ (square), which is again in good agreement with other estimates [28, 29]. In addition, we estimate $d_s(4) = 3.7358(13)$, which again is in good agreement with Monte Carlo estimates [29]. Note that the largest system in Ref. [29] has $N = 5^4$ spins, which seems to not be in the scaling regime (see Fig. 1). This means that results from small systems tend to overestimate d_s .

Finally, one can see that as the dimensionality d increases, $d_s(d)$ approaches d . However, results from simulations on hypercubic lattices struggle from corrections to scaling. These

TABLE I. Size and number of disorder realizations used in the SDRG approach for the EA and KAS models. d is the space dimension, L is the linear system size and M is the number of disorder realizations used for the average. For the KAS model, we use $M = 3000$ disorder realizations for each $L = 256, 512, 1024, 2048, 4096$, and 8192 at the following σ values: 0.1, 0.25, 0.5, 0.55, 0.6, 0.667, 0.75, 0.896, 1, 1.25, 1.5, 1.75, 2, 2.25, 2.5, 2.75, and 3.

d	L	M
4	{4, 5, 6, 7, 8, 9, 10, 12, 16, 20, 24}	3000
4	28	717
4	32	121
5	{4, 5, 6, 7, 8, 9, 10, 12}	3000
5	14	1342
5	16	581
6	{4, 5, 6, 7, 8}	3000
6	9	1843
6	10	938
7	{4, 5, 6}	3000
7	7	512
8	{4, 5}	3000

make it hard to claim that $d_s = d$ at precisely $d = 6$. To address this point, we turn to the KAS model.

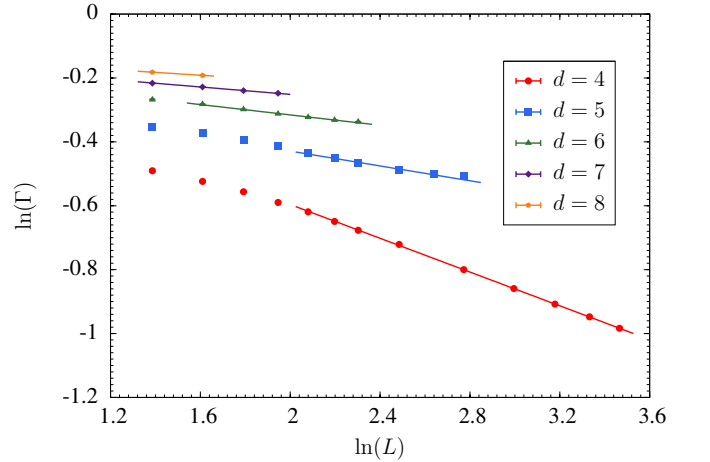


FIG. 1. Γ [see Eq. (3)] for various dimensions d for the EA model as a function of their linear dimension L . Note that $\Gamma \sim L^{d_s-d}$. Our estimate of d_s is determined by the slope of the straight lines drawn through the points at large L values. Error bars are smaller than the symbols.

The one-dimensional KAS model [17] is described by the Hamiltonian in Eq. (1), except that the L spins lie on a ring and the exchange interactions J_{ij} are long-ranged, i.e., $\langle ij \rangle$ denotes a sum over *all* pairs of spins:

$$J_{ij} = c(\sigma, L) \frac{\epsilon_{ij}}{r_{ij}^\sigma}, \quad (4)$$

where r_{ij} is the shortest circular length between sites i and j [30].

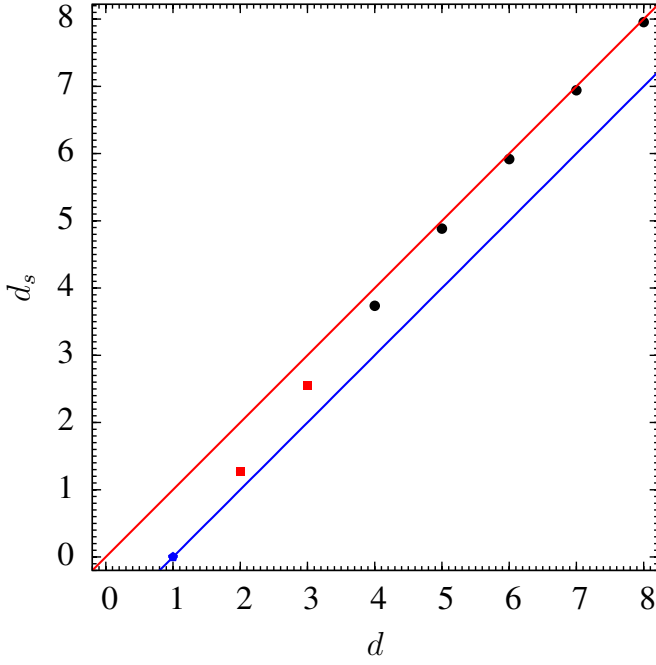


FIG. 2. Values of the fractal dimension d_s as a function of the space dimension d determined using the SDRG method. The top (red) line is the upper bound where $d_s = d$ and the bottom (blue) line is the lower bound where $d_s = d - 1$. The value for $d = 1$ (blue pentagon) can be calculated analytically. The values for $d = 2$ and 3 (red squares) are taken from Ref. [18]. The statistical error bars are smaller than the symbols. No systematic errors have been considered.

The disorder ϵ_{ij} is chosen from a Gaussian distribution of zero mean and standard deviation unity, while the constant $c(\sigma, L)$ in Eq. (4) is fixed to make the mean-field transition temperature $T_c^{\text{MF}} = 1$ and $(T_c^{\text{MF}})^2 = \sum_j [J_{ij}^2]_{\text{av}}$, where $[\cdots]_{\text{av}}$ represents a disorder average and $[J_{ij}^2]_{\text{av}} = c^2(\sigma, L)/r_{ij}^{2\sigma}$ where $1/c(\sigma, L)^2 = \sum_{j=2}^L 1/r_{1j}^{2\sigma}$. Note that in the limit $\sigma \rightarrow 0$ the KAS model reduces to the infinite-range SK model. The advantage of the KAS model is that one can study a large range of linear system sizes.

The KAS model can be taken as an interpolation between the $d = 1$ EA model and the $d = \infty$ SK model as the exponent σ is varied. The phase diagram of this model in the d - σ plane has been deduced from renormalization group arguments in Refs. [10, 31, 32]. For $0 \leq \sigma < 1/2$ it behaves just like the infinite-range SK model. When $1/2 < \sigma < 2/3$ the critical exponents at the spin-glass transition are mean-field like, and this corresponds in the EA model with space dimensions above six. In the interval $2/3 \leq \sigma < 1$ the critical exponents are changed by fluctuations away from their mean-field values. When $\sigma \geq 1$, $T_c(\sigma) = 0$ and when $\sigma > 2$, the long-range zero-temperature fixed point, which controls the value of the exponents d_s and θ , becomes identical to that of the nearest-neighbor one-dimensional EA model, i.e., $d_s = 0$ and $\theta = -1$. There is a convenient mapping between σ and

an effective dimensionality d_{eff} of the short-range EA model [32–36]. For $1/2 < \sigma < 2/3$, it is

$$d_{\text{eff}} = 2/(2\sigma - 1). \quad (5)$$

Thus, right at the value of $\sigma = 2/3$, $d_{\text{eff}} = 6$. The arguments given in Ref. [11] that the critical dimension is 6, below which one sees droplet behavior and above which one sees RSB behavior were directly extended to the KAS model and predicted that only in the interval $\sigma < 2/3$ will one see RSB behavior, so that $\sigma = 2/3$ is the critical value expected for the KAS model.

We have determined d_s for the KAS model from two definitions of d_s . The first definition is via the generalization of the link overlap in Eq. (2) to the long-range KAS model just as done in Ref. [37]:

$$q_\ell = \frac{2}{L(L-1)} \sum_{i < j} w_{ij} S_i^{(\pi)} S_j^{(\pi)} S_i^{(\bar{\pi})} S_j^{(\bar{\pi})} (2\delta_{J_{ij}^\pi, J_{ij}^{\bar{\pi}}} - 1), \quad (6)$$

where $w_{ij} = (L-1)c(\sigma, L)^2/r_{ij}^{2\sigma}$. Note that the sum of w_{ij} over $i < j$ equals $L(L-1)/2$. Anti-periodic boundary conditions can be produced by flipping the sign of the bonds when the shortest paths go through the origin. d_s is then obtained from q_ℓ using Eq. (3) with $d = 1$.

Because we are unsure of the topological significance of d_s calculated in this way, we use a second approach whose topological significance is clear. Fortunately, it gives very similar results to that of our first definition. Let $\tau_i = S_i^{(\pi)} S_i^{(\bar{\pi})}$, and define an “island” as a sequence in which all the τ_i are of the same sign. For the EA-model-limit of the KAS model, i.e., when $\sigma > 2$, there are only two islands but when the long-range zero-temperature fixed point [31] controls the behavior, there are many islands; we denote by N_I the number of islands produced by the change from periodic to anti-periodic boundary conditions. Formally, N_I can be computed via

$$N_I = \frac{1}{4} \sum_{i=1}^L (\tau_{i+1} - \tau_i)^2, \quad (7)$$

where $\tau_{L+1} = \tau_1$. We define d_s via $N_I \sim L^{d_s}$. The islands have a distribution of sizes with their mean size $L_0 = L/N_I \sim L^{1-d_s}$. In the RSB region where $d_s = d = 1$ L_0 is independent of the size of the system and is of $\mathcal{O}(1)$, a result which we obtained previously from direct studies in the SK limit [38].

We have used these two quite distinct definitions of d_s to compute the fractal dimension as a function of σ using the SDRG method. The details of the system sizes and numbers of disorder realizations can be found in Table I. Our results for N_I and Γ are shown in Fig. 3. From these we have extracted values for d_s which are shown in Fig. 4. The values obtained for d_s from Γ and N_I are reassuringly similar. The most striking feature of our results are, first, $d_s \simeq 1 (= d)$ when $\sigma < 2/3$, and second, d_s decreases from unity as σ increases past $2/3$. Because $\sigma = 2/3$ maps to $d = 6$ according to Eq. (5) we believe that this is strong evidence that 6

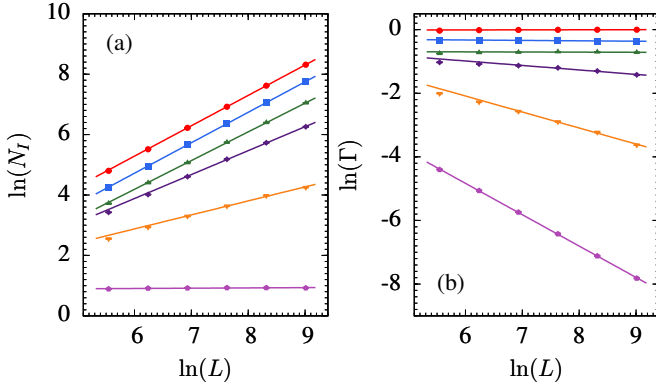


FIG. 3. Dependence of N_I (expected form $\sim L^{d_s}$) and Γ (expected form $\sim L^{d_s-1}$) on L for the KAS model obtained via the SDRG method for a few representative values of the exponent σ : 0.1, 0.5, 0.667, 0.75, 1.0, and 2.0. In both panels (a) and (b) the values of σ increase from top to bottom. Error bars are smaller than the symbols.

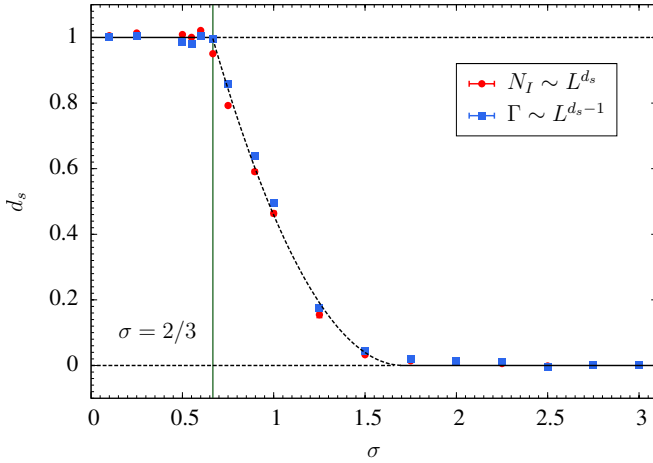


FIG. 4. Exponent d_s calculated from the scaling of N_I (red circles) and Γ (blue squares) using the SDRG method at all the σ values which we have studied (see Table I). The vertical (green) line marks where $\sigma = 2/3$, which is the value of σ at which we expect d_s to decrease below unity. The black or dashed lines are guides to the eye; the dashed line interpolating $d_s = 1$ and $d_s = 0$ is a cubic fit. The values for d_s are obtained by fitting a straight line to the four largest system sizes studied ($L = 1024, 2048, 4096$, and 8192), except for values of $\sigma \geq 1.5$, where all system sizes are used for the fits. Error bars are smaller than the symbols.

is the dimension below which the droplet picture applies and that only in more than 6 space dimensions will one find RSB effects, just as anticipated in Ref. [11].

At $\sigma > 2$ the long-range fixed point is unstable and the renormalization group flows go to the short-range fixed point, that of the $d = 1$ EA model [31]. For the EA model in one space dimension, $d_s = 0$ and $\theta = -1$. We were expecting that d_s would go to zero at $\sigma = 2$; it is possible that d_s is just very small in the interval $1.5 < \sigma < 2$.

There are small finite-size corrections when using the

SDRG method. For $\sigma \sim 1$, there is a downward curvature in the data (Fig. 3) so that if we had been able to study larger L values, our estimates of d_s might have decreased. However, the behavior in the crucial region where σ is close to $2/3$ is less affected by finite-size effects. Monthus and Garel [39] have obtained estimates for d_s from exact studies on the KAS model for $L \leq 24$. They found $d_s(\sigma = 0.62) \simeq 1$, $d_s(\sigma = 0.75) \simeq 0.94$, $d_s(0.87) \simeq 0.82$, $d_s(\sigma = 1) \simeq 0.72$, and $d_s(\sigma = 1.25) \simeq 0.4$. These results illustrate clearly that estimates of d_s from small systems tend to be high.

We now discuss the accuracy of the SDRG method. First, we note that SDRG is considerably better than the Migdal-Kadanoff (MK) approximation which gives $d_s^{\text{MK}} = d - 1$ [18], which coincides with the lower bound on d_s and so never gives $d_s = d$. The SDRG method can be used to determine θ as well as d_s . In two dimensions; it gives $\theta \simeq 0$ [18]; its established value is close to -0.28 [40]. The SDRG method is only exact for special cases. Like the MK approximation, it is exact in one space dimension for the EA model but its performance for the energy per spin and the exponent θ then steadily deteriorates with increasing space dimension d . Monthus [18] suggested that it does a good job for the exponent d_s because that exponent is dominated by short length-scale optimization which is well captured by the early steps of the SDRG method, but that it does badly for the interface free-energy exponent θ which also requires optimization on the longest length scales. We also suspect that its success in determining d_s might be connected with the fact that the domain wall is a self-similar fractal. That means it has the same fractal dimension d_s whether that fractal dimension is studied on short or long length scales. In $d = 2$ and $d = 3$ Monthus [18] showed that the SDRG worked on short length scales but fails on long length scales. We believe the consequence of this might just be that in determining the length of the domain wall $\Sigma^{\text{DW}} = AL^{d_s}$, the exponent d_s is correctly determined from the short length-scale behavior, but to obtain the coefficient A correctly one would need a treatment also valid on long length scales. In the KAS model at $\sigma = 0.1$ the SDRG fails on short length scales but works on long length scales. Again, we believe that the exponent $d_s = d = 1$ is correct, but that the coefficient A is only approximate.

One worrisome issue is that numerical work around six space dimensions could suffer from poor precision, so how can one be confident that $d = 6$ is a special space dimension below which RSB does not occur (aside from a rigorous proof). There is another numerical procedure, the greedy algorithm [41–44] in which one satisfies the bonds in the order of the couplings $|J_{ij}|$ unless a closed loop appears, where one skips to the next largest bond. We have found that as $d \rightarrow 6$ from below the values of d_s obtained from the GA approach those from the SDRG, which is not surprising when one examines how the SDRG works. For $d = 2$, however, the GA is certainly poorer than the SDRG, because it predicts $d_s = 1.216(1)$ [44]. Jackson and Read [43], however, have an analytical argument that 6 is a special space dimension for the GA algorithm. This gives us confidence that 6 is the space

dimension above which interfaces are space filling.

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* wenlongcmp@gmail.com

- [1] G. Parisi, “Infinite number of order parameters for spin-glasses,” *Phys. Rev. Lett.* **43**, 1754 (1979).
- [2] G. Parisi, “Order parameter for spin-glasses,” *Phys. Rev. Lett.* **50**, 1946 (1983).
- [3] R. Rammal, G. Toulouse, and M. A. Virasoro, “Ultrametricity for physicists,” *Rev. Mod. Phys.* **58**, 765 (1986).
- [4] M. Mézard, G. Parisi, and M. A. Virasoro, *Spin Glass Theory and Beyond* (World Scientific, Singapore, 1987).
- [5] G. Parisi, “Some considerations of finite dimensional spin glasses,” *J. Phys. A* **41**, 324002 (2008).
- [6] D. Sherrington and S. Kirkpatrick, “Solvable model of a spin glass,” *Phys. Rev. Lett.* **35**, 1792 (1975).
- [7] S. F. Edwards and P. W. Anderson, “Theory of spin glasses,” *J. Phys. F: Met. Phys.* **5**, 965 (1975).
- [8] W. L. McMillan, “Domain-wall renormalization-group study of the three-dimensional random Ising model,” *Phys. Rev. B* **30**, R476 (1984).
- [9] A. J. Bray and M. A. Moore, “Scaling theory of the ordered phase of spin glasses,” in *Heidelberg Colloquium on Glassy Dynamics and Optimization*, edited by L. Van Hemmen and I. Morgenstern (Springer, New York, 1986) p. 121.
- [10] D. S. Fisher and D. A. Huse, “Equilibrium behavior of the spin-glass ordered phase,” *Phys. Rev. B* **38**, 386 (1988).
- [11] M. A. Moore and A. J. Bray, “Disappearance of the de Almeida-Thouless line in six dimensions,” *Phys. Rev. B* **83**, 224408 (2011).
- [12] C. M. Newman and D. L. Stein, “Simplicity of state and overlap structure in finite-volume realistic spin glasses,” *Phys. Rev. E* **57**, 1356 (1998).
- [13] C. M. Newman and D. L. Stein, “Interfaces and the Question of Regional Congruence in Spin Glasses,” *Phys. Rev. Lett.* **87**, 077201 (2001).
- [14] C. M. Newman and D. L. Stein, “The State(s) of Replica Symmetry Breaking: Mean Field Theories vs Short-Ranged Spin Glasses,” *J. Stat. Phys.* **106**, 213 (2002).
- [15] N. Read, “Short-range Ising spin glasses: the metastate interpretation of replica symmetry breaking,” *Phys. Rev. E* **90**, 032142 (2014).
- [16] We note that the behavior of the stiffness exponent θ was discussed in detail in Ref. [38].
- [17] G. Kotliar, P. W. Anderson, and D. L. Stein, “One-dimensional spin-glass model with long-range random interactions,” *Phys. Rev. B* **27**, 602 (1983).
- [18] C. Monthus, “Fractal dimension of spin-glasses interfaces in dimension $d = 2$ and $d = 3$ via strong disorder renormalization at zero temperature,” *Fractals* **23**, 1550042 (2015).
- [19] P. Charbonneau, J. Kurchan, G. Parisi, P. Urbani, and F. Zamponi, “Fractal free energy landscapes in structural glasses,” *Nat. Comm.* **5**, 3725 (2014).
- [20] C. Scalliet, L. Berthier, and F. Zamponi, “Absence of marginal stability in structural glasses,” (2017), (arXiv:1706.04112).
- [21] A. K. Hartmann and A. P. Young, “Large-scale low-energy excitations in the two-dimensional Ising spin glass,” *Phys. Rev. B* **66**, 094419 (2002).
- [22] A. J. Bray and M. A. Moore, “Chaotic Nature of the Spin-Glass Phase,” *Phys. Rev. Lett.* **58**, 57 (1987).
- [23] A. A. Middleton, “Energetics and geometry of excitations in random systems,” *Phys. Rev. B* **63**, 060202(R) (2001).
- [24] O. Melchert and A. K. Hartmann, “Fractal dimension of domain walls in two-dimensional Ising spin glasses,” *Phys. Rev. B* **76**, 174411 (2007).
- [25] C. Amoruso, A. K. Hartmann, M. B. Hastings, and M. A. Moore, “Conformal Invariance and SLE in Two-Dimensional Ising Spin Glasses,” *Phys. Rev. Lett.* **97**, 267202 (2006).
- [26] D. Bernard, P. Le Doussal, and A. A. Middleton, “Possible description of domain walls in two-dimensional spin glasses by stochastic Loewner evolutions,” *Phys. Rev. B* **76**, 020403 (2007).
- [27] S. Risau-Gusman and F. Romá, “Fractal dimension of domain walls in the Edwards-Anderson spin glass model,” *Phys. Rev. B* **77**, 134435 (2008).
- [28] M. Palassini and A. P. Young, “Nature of the spin glass state,” *Phys. Rev. Lett.* **85**, 3017 (2000).
- [29] H. G. Katzgraber, M. Palassini, and A. P. Young, “Monte Carlo simulations of spin glasses at low temperatures,” *Phys. Rev. B* **63**, 184422 (2001).
- [30] C. Monthus, “One-dimensional Ising spin-glass with power-law interaction: real-space renormalization at zero temperature,” *J. Stat. Mech.* P06015 (2014).
- [31] A. J. Bray, M. A. Moore, and A. P. Young, “Lower critical dimension of metallic vector spin-glasses,” *Phys. Rev. Lett.* **56**, 2641 (1986).
- [32] H. G. Katzgraber and A. P. Young, “Monte Carlo studies of the one-dimensional Ising spin glass with power-law interactions,” *Phys. Rev. B* **67**, 134410 (2003).
- [33] H. G. Katzgraber, D. Larson, and A. P. Young, “Study of the de Almeida-Thouless line using power-law diluted one-dimensional Ising spin glasses,” *Phys. Rev. Lett.* **102**, 177205 (2009).
- [34] L. Leuzzi, G. Parisi, F. Ricci-Tersenghi, and J. J. Ruiz-Lorenzo, “Ising Spin-Glass Transition in a Magnetic Field Outside the Limit of Validity of Mean-Field Theory,” *Phys. Rev. Lett.* **103**, 267201 (2009).
- [35] R. A. Baños, L. A. Fernandez, V. Martin-Mayor, and A. P. Young, “Correspondence between long-range and short-range spin glasses,” *Phys. Rev. B* **86**, 134416 (2012).
- [36] T. Aspelmeier, Helmut G. Katzgraber, Derek Larson, M. A. Moore, Matthew Wittmann, and Joonhyun Yeo, “Finite-size critical scaling in Ising spin glasses in the mean-field regime,” *Phys. Rev. E* **93**, 032123 (2016).
- [37] H. G. Katzgraber and A. P. Young, “Geometry of large-scale low-energy excitations in the one-dimensional Ising spin glass with power-law interactions,” *Phys. Rev. B* **68**, 224408 (2003).
- [38] T. Aspelmeier, Wenlong Wang, M. A. Moore, and Hel-

- mut G. Katzgraber, “Interface free-energy exponent in the one-dimensional Ising spin glass with long-range interactions in both the droplet and broken replica symmetry regions,” *Phys. Rev. E* **94**, 022116 (2016).
- [39] C. Monthus and T. Garel, “Chaos properties of the one-dimensional long-range Ising spin-glass,” *J. Stat. Mech.* P03020 (2014).
- [40] A. K. Hartmann, A. J. Bray, A. C. Carter, M. A. Moore, and A. P. Young, “The stiffness exponent of two-dimensional Ising spin glasses for non-periodic boundary conditions using aspect-ratio scaling,” *Phys. Rev. B* **66**, 224401 (2002).
- [41] M. Cieplak, A. Maritan, and J. R. Banavar, “Optimal paths and domain walls in the strong disorder limit,” *Phys. Rev. Lett.* **72**, 2320 (1994).
- [42] C. M. Newman and D. L. Stein, “Spin-glass model with dimension-dependent ground state multiplicity,” *Phys. Rev. Lett.* **72**, 2286 (1994).
- [43] T. S. Jackson and N. Read, “Theory of minimum spanning trees. I. Mean-field theory and strongly disordered spin-glass model,” *Phys. Rev. E* **81**, 021130 (2010).
- [44] S. M. Sweeney and A. A. Middleton, “Minimal spanning trees at the percolation threshold: A numerical calculation,” *Phys. Rev. E* **88**, 032129 (2013).