Confinement in the Bulk, Deconfinement on the Wall: Infrared Equivalence between Compactified QCD and Quantum Magnets

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Phys. Rev. Lett. **119**, 091601 — Published 1 September 2017

DOI: 10.1103/PhysRevLett.119.091601

Confinement in the bulk, deconfinement on the wall: infrared equivalence between compactified QCD and quantum magnets

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(Dated: August 8, 2017)

In a spontaneously dimerized quantum antiferromagnet, spin-1/2 excitations (spinons) are confined in pairs by strings akin to those confining quarks in non-abelian gauge theories. The system has multiple degenerate ground states (vacua) and domain-walls between regions of different vacua. For two vacua, we demonstrate that spinons on a domain-wall are liberated, in a mechanism strikingly similar to domain-wall deconfinement of quarks in variants of quantum chromodynamics. This observation not only establishes a novel phenomenon in quantum magnetism, but also provides a new direct link between particle physics and condensed-matter physics. The analogy opens doors to improving our understanding of particle confinement and deconfinement by computational and experimental studies in quantum magnetism.

The phenomenon of confinement is well known in quantum chromodynamics (QCD), where quarks are bound by 'strings' and can only be observed within composites; the mesons and baryons. The physics underlying confinement is still poorly understood, e.g., as concerns the nature of the confining strings, because the relevant 3+1 dimensional (D) non-abelian gauge theories are strongly coupled and reliable analytical methods are lacking. Numerical lattice calculations with strings are also challenging, especially in the presence of matter.

Some understanding of confinement has been developed within supersymmetric (SUSY) gauge theories, which generically have multiple vacua. In a conjecture due to Rey and advocated by Witten [1], confining SUSY gauge theories facilitate deconfinement of quarks on domain-walls interpolating between two vacua. Recently [2], by utilization of the special kind of compactification [3, 28], it was demonstrated that this feature transcends SUSY theories and is generic for QCD-like theories. In this work we show that the *liberation on the walls* also transcends QCD-like theories and takes place in quantum magnets as well. We confirm this by large-scale numerical studies of the J-Q model; a quantum spin model amenable to quantum Monte Carlo simulations. All of its QCD-like counterparts suffer from still unsurmountable numerical difficulties, such as the sign problem or the exact chiral limit. Our work establishes domain-wall (DW) deconfinement as a phenomenon in condensed matter physics and provides an unbiased numerical observation from first principles, thus providing valuable insights into the nature of the phenomenon and solidifying its generic nature in QFTs. Finally we also hope that this will further stimulate ideas in overcoming the numerical difficulties in QCD.

In condensed matter physics, certain excitations can be regarded as composites of confined objects, and in some cases deconfinement, or fractionalization, takes place. The most famous example is that of charge $e/3$ excitations in the fractional quantum Hall effect [4, 5]. Another well-established case is the fractionalization of spin waves into spinons carrying spin $S = 1/2$ in spin chains [6–8]. Here we focus on a system with close correspondence with gauge theories: a spontaneously dimerized 2D quantum magnet (a valence-bond-solid, VBS), where spins paired up into localized singlets form columns on the square lattice [9]. Due to lattice symmetries, the pattern can form in multiple ways, corresponding to different vacua. When exciting such a state by breaking a bond, the two unpaired (or triplet-paired) spins are confined by a string of deformed VBS texture. Deconfinement can take place if the VBS is weakened upon approaching a so-called deconfined quantum-critical point [10, 11]. However, the identity of the spinon as a quasi-particle is lost at the critical point, due to the gapless critical host system [12]. Truly deconfined spinons are believed to exist in gapped, topological spin liquid phases [13].

Here we point out that the analogy between quark and spinon confinement is not superficial, but the two phenomena can be described in strikingly similar terms, as illustrated in Fig. 1. We describe a mechanism of spinon liberation on a VBS domain-wall and demonstrate this explicitly by quantum Monte Carlo (QMC) simulations of a spin model hosting Z_4 (four vacua) or Z_2 (two vacua) VBS ground states. Deconfinement on the domain-wall takes place upon reducing the Z_4 symmetry to Z_2. Given the highly non-perturbative nature of confinement, the ability to study this phenomenon in the setting of quantum magnets, in simulations and potentially also in experiments, is a promising avenue for making further progress.

Previous work on quark deconfinement on a domain-wall [2] required a non-thermal compactification of the

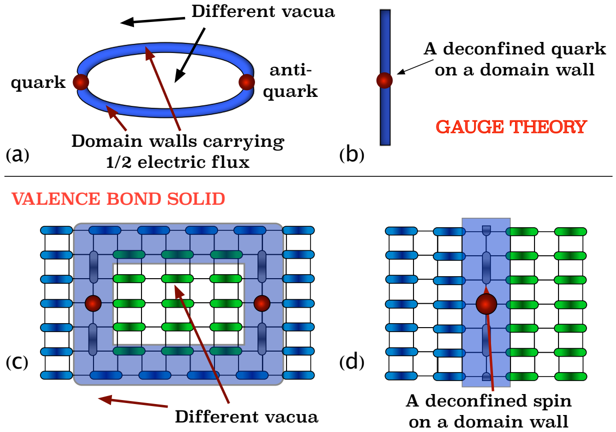

FIG. 1. Deconfinement on domain-walls. A quark-confining string (a) composed out of two strands (domain-walls) separating two vacua [2] allow deconfinement on a domain-wall (b). Confinement in a VBS with Z_2 degeneracy, where two domain-walls form between unpaired spins (c). A domain-wall absorbs the string completely, liberating the spinons (d).

gauge theory, $\mathbb{R}^{3,1} \to \mathbb{R}^{2,1} \times S^1$, where S^1 is the spatial 3-direction (i.e., the long-distance theory is 2+1D) [3, 28]. Such compactifications preserve the center symmetry in the compact direction. We will be interested in two scenarios: QCD with adjoint matter [QCD(adj)] and pure Yang-Mills theory at $\theta = \pi$ [YM_π]. The gauge field in the compact 3-direction is A_3, which upon compactification turns into a compact scalar field in the adjoint representaion. The preservation of the center symmetry effectively causes this scalar field to "condense", Higgsing the gauge group down to the maximally abelian subgroup, which for $SU(2)$ gauge theory is $U(1)$. This is sometimes called the Hosotani mechanism [14]. The effective theory in the IR is then a $U(1)$ 2+1D gauge theory, but it is not free because the underlying non-abelian theory allows for finite action monopoles [15, 16]. By employing the famous dualization of Polyakov [17], the effective $U(1)$ gauge theory with monopoles can be written as a 2+1D theory of a single scalar χ–the dual photon field (see Eq. (1) below).

The mechanism of [2] is operative in both QCD(adj) and YM_π due to the presence of only even-charge monopole-induced non-perturbative potential [18], leading to stable line-like domain-walls carrying half a unit of electric flux. Since quarks carry a whole unit of electric charge, quark–anti-quark pairs are bound by two separate half-flux domain-walls. The fact that fundamental strings can end on domain walls is then a matter of geometry: strings confining the electric charges consist of two domain walls carrying half the electric flux each. The string can then expand and form a domain wall, as illustrated in the top panel of Fig 1.

All of these properties are manifested within the following (Euclidean) effective Lagrangian [2, 3, 19, 20]:

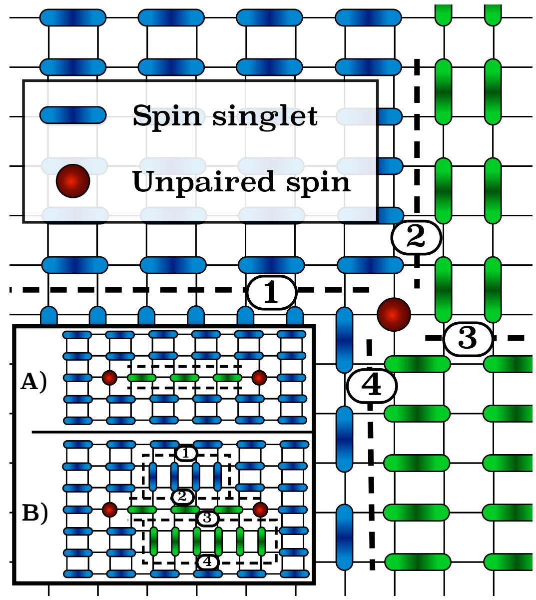

FIG. 2. The vacua of a Z_4 VBS, illustrated with short valence bonds (singlets). Patterns with a relative shift of one lattice spacing are colored with blue and green. The four phases meet at an unpaired spin (red circle). A spinon can be thought of as such a nexus of four different kinds of domain walls (dashed lines labeled by the numbers 1-4) with a spin in the core. Spinons are then not confined by a single string (inlay A), but by *four* string-like domain lines (inlay B).

$$L_{\text{eff}} = M\left[(\partial_\mu \chi)^2 - m^2 \cos(2\chi)\right] \,, \quad (1)$$

where $\chi \sim \chi + 2\pi$ is an angular field referred to as the *dual photon*, $M \propto g(L)/L$ and $m^2 \propto 1/L^2 e^{-8\pi^2/g^2(L)}$, with L the size of the compact direction and $g(L)$ the running gauge coupling at scale L. The dual photon arises upon compactification from $3+1D$ to $2+1D$ because the gauge group abelianizes to $U(1)$ [starting from an $SU(2)$ gauge group]. Finally the $\cos(2\chi)$, rather than $\cos(\chi)$ signifies the lack of charge-1 monopoles. For details of the reasoning leading to Eq. (1) we refer to Supplemental Material [21].

In the path integral, wordlines of charged particles (i.e. quarks) act like 2π-vortices in the field χ, i.e., the winding of the compact field χ around the quark measures its *chromo-electric charge*. The potential $-\cos(2\chi)$, however, forces the χ-field to settle at either $\chi = 0$ or $\chi = \pi$; the two vacua of the $SU(2)$ theory [29]. A quark is then attached to two collimated $1/2$ flux strings, across which the χ-field winds by π; see Fig. 1(b). These half-flux strings are domain walls, which has a remarkable, but simple consequence that the insertion of a confined quark generates a domain wall containing that quark. Consequently, the quark can move freely along the domain wall.

We now discuss an analogous phenomenon in a VBS quantum magnet. The columnar VBS on the uniform square lattice breaks Z_4 symmetry, leading to domain walls when boundary conditions force different patterns (vacua) in different parts of the system. The four vacua (which can be associated with $\chi = 0, \pi/2, \pi, 3\pi/2$) can meet at a single point, in which case the presence of an unpaired spin (spinon) is required at the nexus [30], as illustrated in Fig. 2. A domain wall between, say, the two different horizontal dimer patterns, representing a π winding of χ, will split into two $\pi/2$ domain walls separated by a region with vertical dimers [31].

Confinement of spinons in the VBS is now a matter of topology: an unpaired spin causes misalignment of dimers, forcing interfaces between inequivalent vacua. Two unpaired spins must be connected by two defect lines, which are domain walls separating the two vacua (inlay A of Fig. 2). However, since four domain walls intersect at an unpaired spin, two unpaired spins are also connected by four domain walls. Inlay B of Fig. 2 is therefore a more accurate illustration of the composite nature of strings. We should emphasize however, that the spinons are necessarily dynamical, and the string will break once its energy content becomes comparable to the mass of the $S = 1$ excitation, so that new pairs can be created (in analogy to meson and baryon creation upon separating quarks). This is in contrast to the gauge theories we discussed and to the quantum dimer model, where there is no internal spin structure of the dimers. The strings are then stable and show a domain-wall structure [32]. The dimer model can be thought of as a pure gauge theory without matter fields, while the full quantum magnet inseparably contains matter.

In the gauge theory of the VBS [9–11, 33, 34], spins are represented by vectors on the Bloch sphere coupling to Berry phases. An antiferromagnet is described by a unit vector field $\hat{n}$ on the spatial lattice in continuous time. Haldane [33] showed that the Berry phase in 2+1D has no influence on smooth $\hat{n}$ configurations. However, it couples to singular "hedgehog" configurations in space-time, which can render $\langle \hat{n} \rangle = 0$, e.g., in the VBS phase [9, 34]. The hedgehogs are the analogues of the monopole-instantons, which had a profound influence on the gauge dynamics discussed above. On the square lattice these events appear on the dual lattice (i.e., centers of plaquettes) and couple to Berry phases as $1, e^{i\frac{\pi}{2}}, e^{i\pi}, e^{i3\pi/2}$, depending on which of the four sublattices of the dual lattice they occupy [33]. Further one can write $\hat{n}(x) = u^\dagger(x)\vec{\sigma}u(x)$, where x is a position on the lattice and $u(x) = (u_1(x), u_2(x))$ is a bosonic or fermionic $SU(2)$ doublet, with the constraint $u_1{}^\dagger u_1 + u_2{}^\dagger u_2 = 1$. This parametrization is invariant under the local gauge rotation $u \to e^{i\alpha}u$, and the effective theory with the operators $u_{1,2}$ is therefore a $U(1)$ gauge theory. In the path-integral the hedgehog configurations of $\hat{n}$ appear as monopoles of this $U(1)$ gauge group.

Néel order implies that u condenses, breaking the $U(1)$ gauge symmetry spontaneously. In the absence of Néel order, u can be integrated out and the remaining pure gauge theory can be dualized to a single compact scalar field χ, as before. Here there are four types of monopoles (and their anti-monopoloes), coupling to the χ field and Berry phases as $e^{i\chi + i\frac{k\pi}{2}}$ and $e^{-i\chi - i\frac{k\pi}{2}}$ ($k = 0, 1, 2, 3$). However, only multiple of 4 monopole events are possible [9, 34] due to interference of other charges. A similar effect is responsible for eq. (1) in pure Yang-Mills theory at $\theta = \pi$ (see [28] and Supplementary Materials). The potential $\cos(4\chi)$ then forms and leads to four distinct vacua labeled by $\chi = 0, \pi/2, \pi, 3\pi/2$. Domain walls interpolate between vacua with χ and $\chi + \pi/2$ and carry energy proportional to their length. The insertion of an unpaired spin at x amounts to inserting $u_1^\dagger(x)$ (spin up) or $u_2^\dagger(x)$ (spin down). For definiteness, let $u_{1,2}$ be fermionic, and label the states by the occupation numbers $u_1^\dagger u_1$ and $u_2^\dagger u_2$ as $\Omega = \{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$. The constraint $u_1^\dagger u_1 + u_2^\dagger u_2 = 1$ is obeyed only by the states $|10\rangle$ (up) and $|01\rangle$ (down) at the given site. Insertion of $u_{1,2}^\dagger(x)$ therefore ensures that the spin at x is up or down, because upon projection to physical spin states $u_1^\dagger\Omega, u_2^\dagger\Omega$ contain only $|10\rangle, |01\rangle$. Since $u_{1,2}^\dagger(x)$ are charged under the $U(1)$ gauge group, they represent the fictitious electric charges, which impose winding by 2π on the χ field. An isolated unpaired spin then sources the four domain walls as in Fig. 2.

The Z_4 VBS phase does not allow for spinons to be deconfined on the domain wall, as a single domain wall, say between $\chi = 0$ to $\chi = \pi/2$, would absorb only two out of four domain walls to which an isolated spin is attached. The remaining two cause confinement. If the Lagrangian is deformed, however, to change the ground state degeneracy from 4 to 2, e.g., with horizontal singlets energetically preferred in Fig. 2, the picture changes drastically. Vertical dimers are no longer vacua of the theory, the domain walls 2 and 3 and separately 1 and 2 of Fig. 2 merge, forming π domain walls between the horizontal vacua offset by a Z_2 shift. An isolated spin is then stuck on a domain wall interpolating between these two vacua, as in Fig. 1(d). Such a deformation leads to a $\cos(2\chi)$ term exactly as in Eq. (1) and the parallel with "liberation on the wall" in gauge theories [2] is complete.

To numerically study a VBS domain wall we use the J-Q model [35]. Distinguished by the absence of QMC sign problem, it has been used extensively [36] to explore VBS states and deconfined criticality [37–42]. The Hamiltonian $H = -JH_J - Q_xH_x - Q_yH_y$ contains singlet projectors $P_{ij} = 1/4 - \boldsymbol{S}_i \cdot \boldsymbol{S}_j$ as explained in Fig. 3(a). The different Q-interactions for x- and y-oriented singlet projectors allow us to study both Z_4 (for $Q_x = Q_y$) and Z_2 ($Q_x \neq Q_y$) VBSs. We use an unbiased ground-state QMC method [31, 39, 42, 43] and set $Q_x = 1$.

When $Q_x = Q_y = Q$, a deconfined transition takes place at $q = Q/(J + Q) \approx 0.6$; for $q > 0.6$ the ground

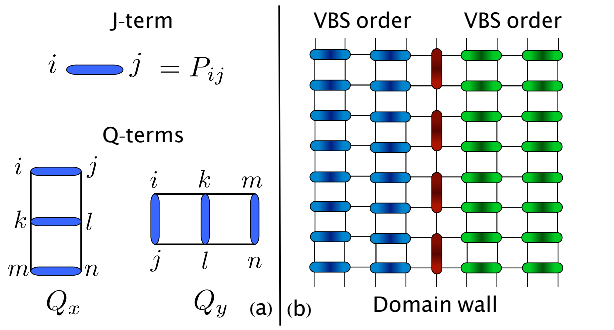

FIG. 3. The J-Q model and domain walls. (a) The singlet projectors P_{ij} of the J and Q terms. (b) Using periodic boundaries in the y-direction and open boundaries in the x-direction, a π domain wall is enforced when L_x is odd and $Q_x > Q_y$. The y-direction is compactified.

state is a Z_4 VBS on a torus of size $L \times L$ with L even [38]. By setting $Q_y < Q_x$, open boundaries in the x direction, and an odd length L_x, the energetics lead to domain wall along the y-direction, as illustrated in Fig. 3(b). The domain wall is broadened by fluctuations and is not fixed at the center of the system.

It is possible to study spinons explicitly using QMC in a basis of valence bonds and unpaired spins [12]. In the present case, it is easier to just confirm that the domain wall hosts a critical mode. With the domain wall along the y-direction, we expect the spin correlations in the x-direction to decay exponentially with distance. This is demonstrated in Fig. 4(a) for two different sets of couplings; one deep in the VBS phase ($J = 0, Q_y/Q_x = 0.6$) and one where fluctuations are more significant ($J/Q_x = 0.5, Q_y/Q_x = 0.6$). The inset panel shows the VBS order parameter [39], demonstrating explicitly the phase change due to the domain wall.

To study correlations along the domain wall we define $C^y(r) = \langle \mathbf{m}(y) \cdot \mathbf{m}(y + r)\rangle$, where $\mathbf{m}$ is total spin on a lattice row. Fig. 4(b) shows that the dependence on $r = L_y/2$ fits the two-point function of the critical Heisenberg chain [44], $C^y(L/2) \sim L^{-1} \ln^{1/2}(L/L_0)$, from which we can infer that spinon excitations, although confined in the bulk, are liberated on the domain wall.

The inset of Fig. 4(b) demonstrates explicitly that deconfinement does not take place in the Z_4 VBS, where the system has two $\pi/2$ domain walls (as in Fig. 2) [31]; the spin correlations decay exponentially, indicating a gap and confined spinons. In the main panel of Fig. 4(b), the data at $Q_y = 0.9$ exhibits a cross-over behavior, where $L \gtrsim 40$ is required to observe the critical Heisenberg behavior. For smaller L the π domain wall is not fully established.

In 3D the physics of the domain wall should be even richer. The membrane-like domain wall may host Néel order, in which case its excitations are spin waves. However, the domain wall could also be a spin liquid with

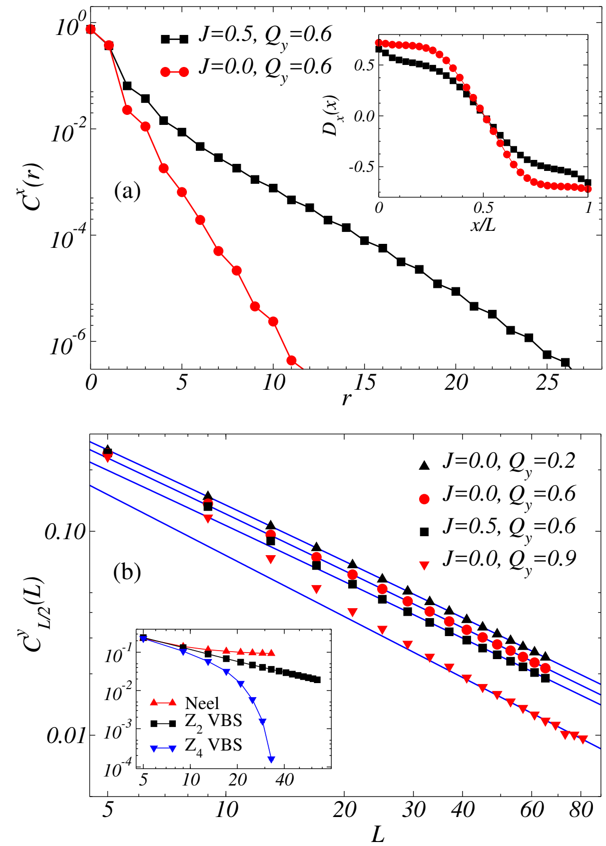

FIG. 4. Correlations in the presence of a domain wall. The x boundaries of the $(L+1) \times L$ J-Q lattice (L even) are open, which forces a domain wall in the y-direction. The coupling $Q_x = 1$. (a) Spin correlations transverse to the domain wall at $Q_y = 0.6$ and $L = 32$. Averaging has been performed over all spin pairs separated by $(\Delta x = r, \Delta y = 0)$. The inset shows the VBS (dimer) order parameter vs the lateral system coordinate. (b) Correlations parallel to the domain wall at $r = L/2$ fitted to the critical Heisenberg form. The inset shows the behavior in three different phases of the model: Néel-ordered ($J = 5.0$, $Q_y = 0.6$), Z_2 VBS ($J = 0.5$, $Q_y = 0.6$), and Z_4 VBS ($J = 0.0$, $Q_y = 1.0$).

deconfined spinons. In addition to possible realizations in magnetic solids, a natural setting to study domain-wall deconfinement experimentally with high tunability would be optical lattices, where there are efforts underway to design quantum spin Hamiltonians [45]. On a more fundamental level, studies of various other aspects of confinement in quantum magnets, e.g., the nature of the confining string and its breaking when matter is created (here spinons, but more generally fermions can be introduced by doping), may provide valuable information relevant also in QCD.

Realistic QCD regimes do not have degenerate vacua, but non-degenerate, so-called k-vacua most likely exist [46]. It is precisely two or more of these vacua that become degenerate in the SUSY limit, or when the topological angle is dialed to $\theta = \pi$ (as we did here). Part of the string tension should be due to the excitation of these k-vacua, even in the regime which is inaccessible to reliable computations. This would in turn imply that the topological charge density fluctuation is sensitive to the presence of the QCD string, which can be tested on the lattice. The QCD string may have nontrivial interactions with the axion—a hypothetical particle which, among other intriguing features, is a candidate for dark matter. An intriguing question is to what extent some of these open issues can also be studied in quantum magnets, or in the richer setting of doped quantum magnets.

Acknowledgments: This research was supported by the DOE under Grants No. DE-FG02-03ER41260 (TS) and DE-SC0013036 (MÜ) and by the NSF under Grant No. DMR-1410126 (AWS). HS would like to thank Boston University's Condensed Matter Theory Visitors program for support. TS and AWS would like to thank the European Centre for Theoretical Studies in Nuclear Physics and Related Areas for support during the workshop *Recent Advances in Monte Carlo Methods 2015*, where the study was conceived. Some of the computations were carried out using Boston University's Shared Computing Cluster.

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