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Phys. Rev. Lett. **119**, 084101 — Published 24 August 2017

DOI: 10.1103/PhysRevLett.119.084101

Stable Chimeras and Independently Synchronizable Clusters

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Cluster synchronization is a phenomenon in which a network self-organizes into a pattern of synchronized sets. It has been shown that diverse patterns of stable cluster synchronization can be captured by symmetries of the network. Here we establish a theoretical basis to divide an arbitrary pattern of symmetry clusters into *independently synchronizable cluster sets*, in which the synchronization stability of the individual clusters in each set is decoupled from that in all the other sets. From this framework, we suggest a new approach to find permanently stable chimera states by capturing two or more symmetry clusters—at least one stable and one unstable—that compose the entire fully symmetric network.

Synchronization is a collective network behavior in which the states of the interacting units evolve in step with each other [1], as observed in animal flocking [2], coordinated firing of neurons [3–5], and synchronous operation of power generators [6]. Beyond the complete synchronization of all units, significant progress has been made on understanding more complex forms of synchronization, including cluster synchronization [7–13] and chimera states [14–27]. In particular, cluster synchronization (CS), in which clusters of nodes exhibit synchronized dynamics, has seen recent breakthoughs: rigorous relations based on group theory have been established between patterns of synchronous clusters and the symmetries of the network structure [11, 12]. Network symmetry can be used to explain various forms of collective behavior, such as remote synchronization, in which two nodes are synchronized despite being connected only through asynchronous ones [28], and isolated desynchronization, in which the synchronization of some clusters is broken without disturbing other clusters [11, 12, 29, 30].

Chimera states, which are characterized by the coexistence of both coherent and incoherent dynamics within the same state, are also intimately related to symmetry. Since the initial discovery [14] and subsequent analysis [15] of such states, numerous studies have found—numerically, analytically, and experimentally—that chimera states can be observed in a wide range of systems (see the review in Ref. [22] and the references therein). However, it was recently found that chimeras in finite-size networks can be long-lived but transient states [17] (i.e., the system will eventually settle onto a simpler state, such as complete synchronization). This raised a fundamental question: are permanently stable chimeras possible with a finite number of oscillators [31]? Evidence for the affirmative answer has so far been limited to numerical simulations [21, 23, 26] (notable exceptions are two case studies with stability analysis: one for "weak" chimeras in bistable populations of phase oscillators [27] and the other for a four-node network of delay-coupled opto-electronic oscillators [25]). Our approach for addressing this problem is to identify symmetry-based "templates" for chimeras: a partition of the network into synchronization clusters including both a stable one and an unstable one.

In this Letter, we develop a general framework that can be used to systematically search for such partitions and, moreover, to characterize any symmetry-based CS patterns in a network. Specifically, we establish that, for any given partition of a network into symmetry clusters, we can uniquely identify groups of clusters in which those in the same group must have the same stability while those from different groups can have different stability (see Fig. 1 for an example). In particular, a single cluster forming a group by itself would be a candidate for the stable cluster in a permanently stable chimera state. We show that these groups can be computed efficiently and provide examples of finding permanent chimeras using this approach. The decoupling of stability between different groups of clusters is derived using a cluster-based coordinate transformation, which is much simpler and is demonstrated to be faster to compute than the one based on the group-theoretical characterization of the network's symmetries [11, 12].

We consider networks of N nodes, each representing an n-dimensional oscillator $\mathbf{x}_i$ governed by

$$\dot{\mathbf{x}}_i(t) = \mathbf{F}(\mathbf{x}_i(t)) + \sigma \sum_{j=1}^{N} A_{ij} \mathbf{H}(\mathbf{x}_j(t)), \qquad (1)$$

where $\mathbf{F}(\mathbf{x})$ determines the uncoupled dynamics of the nodes, σ is the global coupling strength, $A = (A_{ij})_{1\leq i,j\leq N}$ is the adjacency matrix of the network, and $\mathbf{H}(\mathbf{x})$ is the coupling function. For concreteness we consider undirected and unweighted networks, but our theory can be extended naturally to directed and weighted networks [32]. A *CS pattern* for this system is a partition of the network into clusters of oscillators that are in stably synchronized states characterized

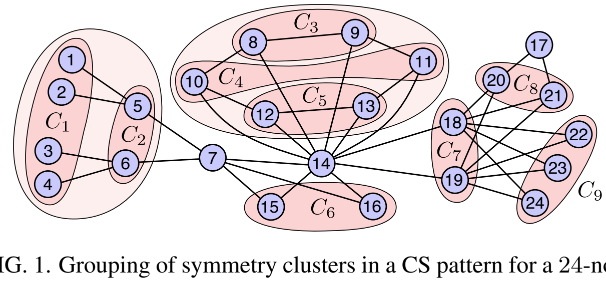

FIG. 1. Grouping of symmetry clusters in a CS pattern for a 24-node network (detailed in Supplemental Material [38], Sec. I).

by $\mathbf{x}_i(t) = \mathbf{x}_j(t)$ for all i and j in the same cluster. The network can be multi-stable and thus allow for multiple CS patterns. For a given network structure, the specific synchronization pattern realized is determined by the initial condition defined by $\mathbf{x}_i(0)$ for all i. A large set of candidate CS patterns—whose stability depend on $\mathbf{F}$, $\mathbf{H}$, σ, and the state of each cluster—can be derived from symmetries in the network [11, 33, 34].

The symmetries of the network (whose structure is represented by A) are determined by the automorphism group Aut(A), defined as the group formed by all node permutations that hold invariant the network topology [and thus the dynamical equation (1)] [11, 35, 36]. Such a network-invariant permutation $i \to \pi(i)$ satisfies $A_{ij} = A_{\pi(i)\pi(j)}$. The orbit $\varphi(G, i)$ of node i under the group $G = \text{Aut}(A)$ is defined as $\varphi(G, i) := \{\pi(i) \,|\, \pi \in G\}$, i.e., the set of all nodes to which node i is mapped under all permutations in G. Since $\varphi(G, i) = \varphi(G, j)$ holds for all nodes $j \in \varphi(G, i)$ by the group property of G, each node in the network belongs to a unique orbit of G. Thus, the set of all orbits of G defines a partition of the network into symmetry clusters, forming a candidate CS pattern. However, and central to this study, there are potentially many other candidate CS patterns determined in the same way by any subgroup G of Aut(A) [12] [where below we use the term subgroup and the notation G to refer either to Aut(A) or any of its proper subgroups]. Figure 1 shows an example of such a pattern.

For a given subgroup G and the associated candidate CS pattern having clusters $C_1, \ldots, C_M$, there are generally multiple synchronous states respecting that pattern. Each such state $\{\mathbf{s}_m(t)\}_{1\le m\le M}$, with $\mathbf{x}_i(t) = \mathbf{s}_m(t)$ for all $i \in C_m$ and for all t, satisfies

$$\dot{\mathbf{s}}_m = \mathbf{F}(\mathbf{s}_m) + \sigma \sum_{m'=1}^{M} \widetilde{A}_{mm'} \mathbf{H}(\mathbf{s}_{m'}), \tag{2}$$

where we have defined $\widetilde{A}_{mm'} := \sum_{j\in C_{m'}} A_{ij}$ with $i \in C_m$. This can be verified by substituting the synchronous state into Eq. (1) and rewriting the summation in Eq. (1) as $\sum_{m'=1}^{M} \sum_{j\in C_{m'}}$. Note that $\widetilde{A}_{mm'}$ is properly defined because the invariance of A under all permutations in G (i.e., $A_{ij} = A_{\pi(i)\pi(j)}$ for all $\pi \in G$) can be used to show that $\widetilde{A}_{mm'}$ as defined does not depend on the choice of node i within C_m. The matrix $\widetilde{A} = (\widetilde{A}_{mm'})_{1\le m,m'\le M}$ can be interpreted as the (possibly directed weighted) adjacency matrix of a coarse-grained version of the original network (called a quotient network [37]), where $\widetilde{A}_{mm'}$ is the number of links from a node in C_m to the cluster $C_{m'}$. Note also that Eq. (2) [thus the set of possible synchronous states for Eq. (1)] is fully determined by the candidate CS pattern and does not directly depend on the associated subgroup G (which may not be unique in general). While similar candidate CS patterns and the corresponding synchronous states can also be formulated using symmetry groupoids [37] and external equitable partitions [13], here we focus on those based on symmetry (sub)groups, as they facilitate our analysis below.

Whether a given candidate CS pattern can actually be observed in the system depends on whether the synchronization of the individual clusters is stable. Different clusters are generally interrelated, and in particular can have identical stability, which can be described by the notion of intertwined clusters [11] in the special case where G is the full automorphism group Aut(A). However, direct extension of this notion to an arbitrary subgroup G does not lead to a consistent definition of intertwined clusters (Supplemental Material [38], Sec. II). Below we develop a comprehensive theory that overcomes this difficulty and fully describes the interrelationship between the synchronization stability of the clusters in a given candidate CS pattern. This theory will provide a unique grouping of these clusters and the associated decoupling of the stability equations for clusters belonging to different groups. Specifically, we will define a coordinate system in which the stability equations for each group of clusters (denoted $C_1, ..., C_{M'}$, with sizes $c_1, \ldots, c_{M'}$, respectively) are coupled only within the group. The decoupled equations read:

$$\dot{\boldsymbol{\eta}}_\kappa^{(m)} = D\mathbf{F}(\mathbf{s}_m)\boldsymbol{\eta}_\kappa^{(m)} + \sigma \sum_{m'=1}^{M'} \sum_{\kappa'=2}^{c_{m'}} D\mathbf{H}(\mathbf{s}_{m'}) B_{\kappa\kappa'}^{(mm')} \boldsymbol{\eta}_{\kappa'}^{(m')}, \tag{3}$$

where $\boldsymbol{\eta}_2^{(m)}, \ldots, \boldsymbol{\eta}_{c_m}^{(m)}$ are variables that represent perturbations transverse to the synchronization manifold $\{(\mathbf{x}_1, \ldots, \mathbf{x}_N) \,|\, \mathbf{x}_i = \mathbf{x}_j \text{ for all } i, j \in C_m\}$ of cluster C_m, $D\mathbf{F}$ and $D\mathbf{H}$ are the Jacobian matrices of $\mathbf{F}$ and $\mathbf{H}$, respectively, $\{\mathbf{s}_m(t)\}$ is the synchronous state corresponding to the given CS pattern, and $B_{\kappa\kappa'}^{(mm')}$ is the coupling coefficient between $\boldsymbol{\eta}_\kappa^{(m)}$ and $\boldsymbol{\eta}_{\kappa'}^{(m')}$. This means, in particular, that (a) perturbations applied to a cluster in one group do not propagate to those in other groups, (b) clusters in the same group must have the same stability, and (c) clusters belonging to different groups can have different stability. For a group with $M' = 1$ (i.e., a single cluster), we will show that the coordinate system can be chosen so that Eq. (3) further reduces to

$$\dot{\boldsymbol{\eta}}_\kappa^{(m)} = \left[D\mathbf{F}(\mathbf{s}_m) + \sigma\lambda_\kappa^{(m)} D\mathbf{H}(\mathbf{s}_m)\right]\boldsymbol{\eta}_\kappa^{(m)}, \tag{4}$$

where $\lambda_\kappa^{(m)}$ is an eigenvalue of A. This generalizes the equation defining a master stability function for the stability analysis of complete synchronization (i.e., the special case $M = 1$) in networks of diffusively coupled oscillators [42].

To define the grouping of the clusters $C_1, \ldots, C_M$ in a given candidate CS pattern, we first categorize the nontrivial clusters (i.e., those containing more than one node) into two types: those that are independently synchronizable and those that are not. We say that C_m is an *independently synchronizable cluster* (ISC) if the network has a state in which all nodes in C_m are synchronized while none of the other nodes are required to be synchronized with any other nodes in the network. Mathematically, such a cluster can be completely characterized by the following property of the network structure: there is a subgroup G' of Aut(A) for which $\mathbf{C}^{(G')}$ contains only the cluster C_m, where $\mathbf{C}^{(G')}$ denotes the set of all nontrivial clusters associated with G'. The clusters $C_6, \ldots, C_9$ shown in Fig. 1 are all ISCs.

What happens if C_m is not an ISC? In that case, we can still find a set of clusters (containing C_m) that is independently synchronizable as a whole, i.e., the network has a state in which all clusters in the set are synchronized (although the node states can be different for different clusters) while other parts of the network are not required to be synchronized. Formally, we define an *ISC set* to be a minimal such set, i.e., a set $\mathbf{C}$ of nontrivial clusters satisfying: 1) there exists a subgroup G' for which $\mathbf{C} = \mathbf{C}^{(G')}$ (i.e., $\mathbf{C}$ is independently synchronizable), and 2) there is no subgroup G'' for which $\mathbf{C}^{(G'')}$ is a proper subset of $\mathbf{C}$ (i.e., $\mathbf{C}$ is a smallest such set). For example, the clusters C_1 and C_2 in Fig. 1 form an ISC set. Our definition of ISC sets thus provides a higher-order organization of the network nodes into "clusters of clusters"; for any given network structure and any candidate CS pattern [associated with some subgroup of $\mathrm{Aut}(A)$], the (nontrivial) clusters can be uniquely grouped into ISCs and ISC sets [43]. We provide a proof of this unique grouping and also an efficient algorithm [44] for computing the grouping (Supplemental Material [38], Sec. III).

We now construct a cluster-based coordinate system that leads to the decoupling in Eq. (3). For each cluster C_m, we first define a unit vector $\mathbf{u}_1^{(m)}$ parallel to the synchronization manifold for that cluster by setting its ith component to $1/\sqrt{c_m}$ if $i \in C_m$ and zero otherwise. We then choose any set of vectors $\mathbf{u}_2^{(m)}, \ldots, \mathbf{u}_{c_m}^{(m)}$ that, together with $\mathbf{u}_1^{(m)}$, form an orthonormal basis for the c_m-dimensional subspace associated with the cluster C_m (i.e., the subspace of the N-dimensional node coordinate space spanned by $\{\mathbf{e}_i\}_{i \in C_m}$, where $\mathbf{e}_i$ denotes the vector in which the ith component is one and all others are zero). It can be shown using the group-theoretical properties of the ISC sets that the corresponding similarity transformation is guaranteed to block-diagonalize the adjacency matrix A, with one diagonal block for each ISC or ISC set. Applying this transformation to the variational equations for an ISC set $C_1, ..., C_{M'}$, we obtain Eq. (3). In this equation, the diagonal block of the transformed A corresponding to the selected ISC set is further divided into smaller blocks $B^{(mm')} := (B_{\kappa\kappa'}^{(mm')})$ representing the coupling between clusters C_m and $C_{m'}$ within the same ISC set. In contrast, between different ISC sets there is no coupling term in Eq. (3); this indicates that the CS stability of one ISC set can be different from that of another ISC set. We can further show that within the same ISC set, there is no choice of a basis that would decouple the stability of different clusters. For ISCs (i.e., those for which $M' = 1$), we can choose the basis vectors (except for $\mathbf{u}_1^{(m)}$) to be eigenvectors of A, which would diagonalize all the corresponding diagonal blocks of A and decompose the variational equations into individual eigenmodes, which leads to Eq. (4). Similar decoupling of synchronization stability between different ISC sets can also be established for a general class of diffusively coupled oscillators (Supplemental Material [38], Sec. IV for full details).

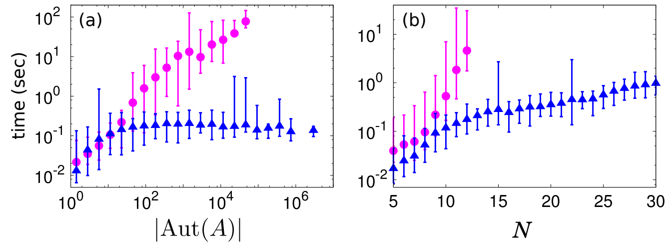

FIG. 2. Computational time for constructing the cluster-based coordinates (▲) and the IRR coordinates (●) for Erdős-Rényi random networks. (a) Average CPU time vs. the number of symmetries $|\mathrm{Aut}(A)|$, estimated from a sample of 10^3 networks with $N = 12$ nodes and L links. We vary L in $N-1 \leq L \leq N(N-1)/2-2$, bin the $|\mathrm{Aut}(A)|$ values, and calculate the average in each bin. The IRR algorithm fails for $|\mathrm{Aut}(A)| > 5 \times 10^4$. (b) Average CPU time vs. the network size N, estimated from a sample of 100 networks with N nodes and $L = N(N-1)/2 - 5$ links. The IRR algorithm fails for $N \geq 13$ because $|\mathrm{Aut}(A)|$ becomes too large for such N. For both (a) and (b), only connected networks are used, and error bars indicate the range of observed values.

Our construction of the cluster-based coordinates (see Supplemental Material [38], Sec. III C for an algorithm [44]) has significant computational advantage over the existing method of constructing block-diagonalizing coordinates based on irreducible representation (IRR) of subgroup G [11] (detailed in Supplemental Material [38], Sec. III A). Figure 2 shows the time it takes to compute the two coordinate systems for $G = \mathrm{Aut}(A)$ as a function of $|\mathrm{Aut}(A)|$ and N using a single processor core on a workstation. We observe that, as the network size and the number of symmetries grow, the computational time grows very quickly for the IRR coordinates (and the particular implementation fails to compute for $N > 12$), while it grows much slower for the cluster-based coordinates and appears to saturate as a function of $|\mathrm{Aut}(A)|$.

The theory we established above can be used to identify chimera states that are permanently stable, when applied to a fully symmetric network [i.e., one in which $\mathrm{Aut}(A)$ has only one symmetry cluster]. To do this, we need to consider a proper subgroup of $\mathrm{Aut}(A)$ that has at least one ISC strictly smaller than the network (since a chimera state requires at least one stable cluster and one unstable cluster). This suggests the following two-step procedure for finding stable chimera states. First, we choose a fully symmetric network structure that has an ISC C_1 that is strictly smaller than the network itself. Then, we find system parameters satisfying the following conditions: (i) the complete synchronization of the network [i.e., any state with $\mathbf{x}_i(t) = \mathbf{s}(t)$, $\forall i$ in Eq. (1)] is unstable; (ii) for each candidate CS pattern in which C_1 appears, the synchronization of C_1 is stable but the synchronization of the other clusters is unstable according to Eqs. (3) and/or (4). These conditions are necessary for a symmetry cluster-based chimera state to exist and persist indefinitely.

As an example of applying this approach, we consider the electro-optic system studied in Ref. [11], whose dynamics is

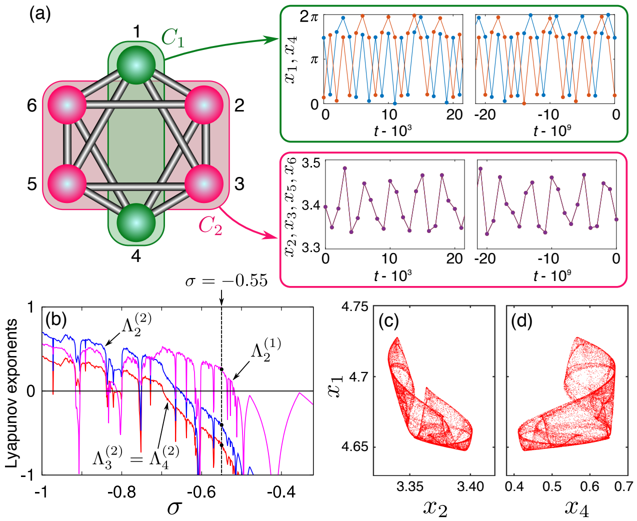

FIG. 3. Permanently chimera state in the electro-optic system described by Eq. (5). We use $\beta = \frac{2\pi}{3} - 4\sigma$ and $\delta = \frac{\pi}{6}$ for a given coupling strength σ. (a) Six-node ring structure of the network. Shown in the two boxes on the right are time series of the nodes in clusters C_1 and C_2 obtained from directly iterating Eq. (5) for 10^9 iterations with $\sigma = -0.55$. (b) Transverse Lyapunov exponents for C_1 [$\Lambda_2^{(1)}$ (magenta)] and C_2 [$\Lambda_2^{(2)}$ (blue), $\Lambda_3^{(2)} = \Lambda_4^{(2)}$ (red)], estimated over $10^7 \leq t \leq 2 \times 10^7$. (c–d) Last 10^5 iterations of the time series, where we plot x_1 against x_2 (c) and x_4 (d), showing only every 4th iterate (since the state is approximately 4-periodic). We thus see that this is a chimera state in which C_2 is stably synchronized, while the dynamics within C_1 is chaotic.

governed by a discrete-time analog of Eq. (1):

$$x_i(t+1) = \left[\beta \mathcal{I}(x_i(t)) + \sigma \sum_{j=1}^{N} A_{ij} \mathcal{I}(x_j(t)) + \delta\right] \bmod 2\pi, \tag{5}$$

where $x_i(t)$ is the state variable for node i, the function $\mathcal{I}(x) = [1 - \cos(x)]/2$ represents light intensity, and the parameter $\delta > 0$ suppresses the trivial solution $x_i(t) \equiv 0$. We use the six-node ring network shown in Fig. 3(a), which can be partitioned into two ISCs, $C_1 = \{1, 4\}$ and $C_2 = \{2, 3, 5, 6\}$. For this system, condition (i) can be verified to hold true if $\sigma < \frac{\pi}{9} - \frac{2}{3}$, $\beta = \frac{2\pi}{3} - 4\sigma$, and $\delta = \frac{\pi}{6}$. For condition (ii), we only need to study the stability of states in which both C_1 and C_2 are synchronized, since no smaller nontrivial cluster within C_1 is possible. Computing the transverse Lyapunov exponents for these clusters using a discrete-time analog of Eq. (4), we find ranges of σ for which the system satisfies condition (ii) [Fig. 3(b)]: $\Lambda_2^{(1)} > 0$, $\Lambda_2^{(2)} < 0$, and $\Lambda_3^{(2)} = \Lambda_4^{(2)} < 0$, where $\Lambda_2^{(1)}$ denotes the exponent for C_1, while $\Lambda_2^{(2)}$, $\Lambda_3^{(2)}$, and $\Lambda_4^{(2)}$ denote the exponents for C_2. Note that we always have $\Lambda_3^{(2)} = \Lambda_4^{(2)}$ for this system due to the degeneracy of the eigenvalues associated with these exponents. Having verified both conditions (i) and (ii), the system is likely to exhibit a permanent chimera state. Indeed, Fig. 3 shows the trajectory of an example chimera state that emerges from a completely random initial condition and persists even after iterating Eq. (5) for 10^9 time steps. In this state, the dynamics of node 1 appears chaotic with respect to that of both node 2 and node 4 [Fig. 3(c–d)]. This provides further evidence that the system is in a permanent stable chimera state. We also identify permanent chimera states in larger networks using the same approach (see Supplemental Material [38], Sec. V for details and for a larger network example).

The theory of "clusters of clusters" we developed here answers the fundamental question about how the stability of synchronous clusters are interrelated. Moreover, it provides a mechanism for a fully symmetric network to be in a permanently stable state that exhibits coherence and incoherence simultaneously. Our formulation of network-structural conditions under which such chimera states are possible explain why some networks are more likely to exhibit chimeras than others; for example, the observed prevalence of chimera states in star networks [24] is due to the property that any partition of the end nodes yields a CS pattern in which all clusters are ISCs. Our approach for finding chimera states in a fully symmetric network can also be applied to a given symmetry cluster in an arbitrary (not necessarily fully symmetric) network to identify sub-cluster chimeras: symmetry breaking that leads to the coexistence of coherent and incoherent dynamics within that cluster. We hope our work will lead to the discovery of new patterns of synchronization that have not been anticipated before, and also stimulate further studies on cluster synchronization.

This work was supported by ARO (Award No. W911NF-15-1-0272) and by Chonbuk National University.

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[43] It can be shown that ISC sets, defined for arbitrary G, generalize the notion of intertwined clusters. Moreover, being an ISC is a necessary condition for a cluster to undergo the isolated desynchronization defined in Ref. [11].

[44] Our implementation of the algorithms are included as Supplemental Material (`grouping_clusters.zip`) and are also downloadable from `https://github.com/tnishi0/grouping-clusters`