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Entanglement in non-unitary quantum critical spin chains

Romain Couvreur[1,3], Jesper Lykke Jacobsen[1,2,3] and Hubert Saleur[3,4]
[1] *Laboratoire de Physique Théorique, École Normale Supérieure – PSL Research University, 24 rue Lhomond, F-75231 Paris Cedex 05, France*
[2] *Sorbonne Universités, UPMC Université Paris 6, CNRS UMR 8549, F-75005 Paris, France*
[3] *Institut de Physique Théorique, CEA Saclay, 91191 Gif Sur Yvette, France and*
[4] *Department of Physics and Astronomy, University of Southern California, Los Angeles, CA 90089-0484*
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Entanglement entropy has proven invaluable to our understanding of quantum criticality. It is natural to try to extend the concept to "non-unitary quantum mechanics", which has seen growing interest from areas as diverse as open quantum systems, non-interacting electronic disordered systems, or non-unitary conformal field theory (CFT). We propose and investigate such an extension here, by focussing on the case of one-dimensional quantum group symmetric or supergroup symmetric spin chains. We show that the consideration of left and right eigenstates combined with appropriate definitions of the trace leads to a natural definition of Rényi entropies in a large variety of models. We interpret this definition geometrically in terms of related loop models and calculate the corresponding scaling in the conformal case. This allows us to distinguish the role of the central charge and effective central charge in rational minimal models of CFT, and to define an effective central charge in other, less well understood cases. The example of the $sl(2|1)$ alternating spin chain for percolation is discussed in detail.

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The concept of entanglement entropy has profoundly affected our understanding of quantum systems, especially in the vicinity of critical points [1]. A growing interest in *non-unitary quantum mechanics* (with non-hermitian "Hamiltonians") stems from open quantum systems, where the reservoir coupling can be represented by hermiticity-breaking boundary terms [2]. Another motivation comes from disordered non-interacting electronic systems in $2+1$ dimensions (D) where phase transitions, such as the plateau transition in the integer quantum Hall effect (IQHE), can be investigated—using a supersymmetric formalism and dimensional reduction—via 1D non-hermitian quantum spin chains with supergroup symmetry (SUSY) [3]. SUSY spin chains and quantum field theories with target space SUSY also appear in the AdS/CFT correspondence [4, 5] and in critical geometrical systems such as polymers or percolation [6]. Quantum mechanics with non-hermitian but PT-symmetric "Hamiltonians" also gains increased interest [7].

Can entanglement entropy be meaningfully extended beyond ordinary quantum mechanics? We focus in this Letter on critical 1D spin chains and the associated 2D critical statistical systems and CFTs. This is the area where our understanding of the ordinary case is the deepest, and the one with most immediate applications.

For ordinary critical quantum chains (gapless, with linear dispersion relation), the best known result concerns the entanglement entropy (EE) of a subsystem A of length L with the (infinite) rest B at temperature $T=0$. Let $\rho_A = \mathrm{Tr}_B \rho$ denote the reduced density operator, where $|0\rangle$ is the normalized ground state and $\rho = |0\rangle\langle 0|$. The (von Neumann) EE then reads $S_A = -\mathrm{Tr}_A \rho_A \ln \rho_A$. One has $S \approx \frac{c}{3}\ln(L/a)$ for $L \gg a$, where a is a lattice cutoff and c the central charge of the associated CFT. For the XXZ chain, $c=1$.

Statistical mechanics is ripe with non-hermitian critical spin chains: the Ising chain in an imaginary magnetic field (whose critical point is described by the Yang-Lee singularity), the alternating $sl(2|1)$ chain describing percolation hulls [8], or the alternating $gl(2|2)$ chain describing the IQHE plateau transition [3]. The Ising chain is conceptually the simplest, as it corresponds to a *rational* non-unitary CFT. In this case, abstract arguments [9, 10] suggest replacing the unitary result by

$$S_A \approx \frac{c_{\mathrm{eff}}}{3}\ln(L/a)\,, \tag{1}$$

where c_{eff} is the *effective* central charge. For instance, for the Yang-Lee singularity, $c = -\frac{22}{5}$ but $c_{\mathrm{eff}} = \frac{2}{5}$; in this case (1) was checked numerically [9]. It was also checked analytically for integrable realizations of the non-unitary minimal CFT. The superficial similarity with the result $s \approx \frac{\pi c_{\mathrm{eff}}}{3} T$ for the thermal entropy per unit length of the infinite chain at $T \ll 1$ suggests that (1) is a simple extension of the scaling of the ground-state energy in non-unitary CFT [11]. But the situation is more subtle, as can be seen from the fact that the leading behavior of the EE is independent of the (low-energy) eigenstate in which it is computed [12].

There are two crucial conditions in the derivation of (1): the left and right ground states $|0_{\mathrm{L}}\rangle, |0_{\mathrm{R}}\rangle$ must be identical, and the full operator content of the theory must be known. These conditions hold for minimal, rational CFT, but in the vast majority of systems the operator content depends on the boundary conditions (so it is unclear what $c_{\rm eff}$ is), and $|0_{\rm L}\rangle \neq |0_{\rm R}\rangle$, begging the question of how exactly ρ, ρ_A and S_A are defined.

In this Letter we explore this vast subject by concentrating on non-Hermitian models with SUSY or quantum group (QG) symmetry. QG symmetric spin chains pervade theoretical physics [13–16] and harbor many applications [8, 17–19]. For the sake of illustration, we focus here on the simplest, prototypical XXZ spin chain, for which we extend the general framework of Coulomb gas and loop model representations to EE calculations. We derive (1) for minimal non-unitary models, and define modified EE involving the true c even in non-unitary cases. We finally introduce a natural, non-trivial EE in SUSY cases, even when the partition function $Z = 1$.

EE and QG symmetry. We first discuss the critical $U_q sl(2)$ QG symmetric XXZ spin chain [13]. Let $\sigma_i^{x,y,z}$ be Pauli matrices acting on space i and define the nearest neighbor interaction

$$e_i = -\frac{1}{2}\left[\sigma_i^x\sigma_{i+1}^x + \sigma_i^y\sigma_{i+1}^y + \frac{q+q^{-1}}{2}(\sigma_i^z\sigma_{i+1}^z - 1) + h_i\right]$$

with $q \in \mathbb{C}$, $|q| = 1$. The Hamiltonian $H = -\sum_{i=1}^{M-1} e_i$ with $h_i = 0$ describes the ordinary critical XXZ chain on M sites, but we add the hermiticity-breaking boundary term $h_i = \frac{q-q^{-1}}{2}(\sigma_i^z - \sigma_{i+1}^z)$ to ensure commutation with the $U_q sl(2)$ QG (whose generators are given in the supplemental material (SM) [20]).

Consider first 2 sites, that is $H = -e_1$. H is not hermitian; its eigenvalues are real [21] but its left and right eigenstates differ. We restrict $\operatorname{Arg} q \in [0, \pi/2]$, so the lowest energy is $E^{(0)} = -(q + q^{-1})$ (the other eigenenergy is $E^{(1)} = 0$). The right ground state, defined as $H|0\rangle = E^{(0)}|0\rangle$, is $|0\rangle = \frac{1}{\sqrt{2}}(q^{-1/2}|\uparrow\downarrow\rangle - q^{1/2}|\downarrow\uparrow\rangle)$. We use the (standard) convention that complex numbers are conjugated when calculating the bra associated with a given ket; therefore $\langle 0|0\rangle = 1$. The density matrix

$$\rho = |0\rangle\langle 0| = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -q^{-1} & 0 \\ 0 & -q & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{2}$$

(in the basis $\uparrow\uparrow, \uparrow\downarrow, \downarrow\uparrow, \downarrow\downarrow$) is normalized, $\operatorname{Tr}\rho = 1$. Taking subsystem A (B) as the left (right) spin, the reduced density operator is $\rho_A = \frac{1}{2}\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, and therefore

$$S_A = \ln 2\,. \tag{3}$$

This coincides with the well-known result for the $sl(2)$ symmetric (hermitian) XXX chain ($q = 1$). But since H is non-hermitian, it is more correct to work with left and right eigenstates defined by $H|E_{\rm R}\rangle = E|E_{\rm R}\rangle$ and $\langle E_{\rm L}|H = E\langle E_{\rm L}|$ (or $H^\dagger|E_{\rm L}\rangle = E|E_{\rm L}\rangle$, since $E \in \mathbb{R}$). Restricting to the sector $S^z = 0$ we have

$$|0_{\rm R}\rangle = \frac{1}{\sqrt{q+q^{-1}}}\left(q^{-1/2}|\uparrow\downarrow\rangle - q^{1/2}|\downarrow\uparrow\rangle\right) \tag{4}$$

$$|1_{\rm R}\rangle = \frac{1}{\sqrt{q+q^{-1}}}\left(q^{1/2}|\uparrow\downarrow\rangle + q^{-1/2}|\downarrow\uparrow\rangle\right) \tag{5}$$

where $|0_{\rm R}\rangle$, $|1_{\rm R}\rangle$ denote the right eigenstates with energies $E^{(0)}, E^{(1)}$. The left eigenstates $|0_{\rm L}\rangle$, $|1_{\rm L}\rangle$ are obtained from (4)–(5) by $q \to q^{-1}$. Normalizations are such that $\langle i_{\rm L}|i_{\rm R}\rangle = 1$, and $\langle i_{\rm L}|j_{\rm R}\rangle = 0$ for $i \neq j$. Since $\langle 0_{\rm R}|1_{\rm R}\rangle \neq 0$ we need both L and R eigenstates to build a projector onto the ground state. We thus *define*

$$\tilde\rho \equiv |0_{\rm R}\rangle\langle 0_{\rm L}| = \frac{1}{q+q^{-1}}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & q^{-1} & -1 & 0 \\ 0 & -1 & q & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{6}$$

and $\tilde\rho_A = \operatorname{Tr}_B\left(q^{-2\sigma_B^z}\tilde\rho\right) = \frac{1}{q+q^{-1}}\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. We justify the use of a modified trace shortly with both geometrical and QG considerations. Observe that $\tilde\rho_A$ is normalized for the modified trace (note the opposite power of q): $\operatorname{Tr}_A\left(q^{2\sigma_A^z}\tilde\rho_A\right) = 1$. We now define the EE as

$$\tilde S_A = -\operatorname{Tr}\left(q^{2\sigma_A^z}\tilde\rho_A \ln\tilde\rho_A\right) = \ln(q + q^{-1})\,. \tag{7}$$

The result (7) is more appealing that (3): it depends on q through the combination $q + q^{-1}$ which is the quantum dimension of the spin 1/2 representation of $U_q sl(2)$. Note that (7) satisfies $\tilde S_A = \tilde S_B$ (see SM).

Entanglement and loops. Eq. (7) admits an alternative interpretation in terms of loop models. Since e_i obey the Temperley-Lieb (TL) relations,

$$\begin{aligned} e_i^2 &= (q+q^{-1})e_i\,, \\ e_i e_{i\pm1} e_i &= e_i\,, \\ [e_i, e_j] &= 0 \text{ for } |i-j| > 1\,, \end{aligned} \tag{8}$$

their action can be represented in terms of diagrams: $e_i =$ contracts neighboring lines, and multiplication means stacking diagrams vertically, giving weight $n \equiv q + q^{-1}$ to each closed loop. The ground state of $H = -e_1$ is $|0_\ell\rangle = \frac{1}{\sqrt{n}}\cup$ (ℓ stands for loop). We check graphically that $H|0_\ell\rangle = -n|0_\ell\rangle$. With the scalar product ordinarily used in loop models (see SM [20]), $|0_\ell\rangle$ is correctly normalized. The density matrix is $\rho_\ell = \frac{1}{n}|0_\ell\rangle\langle 0_\ell| = \frac{1}{n}$. The partial trace $\rho_{A,\ell} = \operatorname{Tr}_B \rho_\ell$ glues corresponding sites on top and bottom throughout B (here site 2). The resulting reduced density matrix acts only on A (site 1): $\rho_{A,\ell} = \frac{1}{n}\,|$. The gluing of A creates a loop of weight n, so $S_{A,\ell} = -\operatorname{Tr}(\rho_{A,\ell}\log\rho_{A,\ell}) = -n \times \frac{1}{n}\log\frac{1}{n} = \log n$. The agreement with (7) is of course no accident. Indeed, for any spin-1/2 Hamiltonian expressed in the TL algebra (and thus commuting with $U_q sl(2)$), the EE—and in fact, the N-replica Rényi (see below) entropies—obtained with the modified traces and with the loop construction coincide. We shall call these *QG entropies*, and denote them $\tilde S$.

Coulomb gas calculation of the EE. For the critical QG invariant XXZ chain with $H = -\sum e_i$, the EE $\tilde S$ scales as expected in CFT, but with the *true* central charge $c = 1 - \frac{6}{x(x+1)}$ (instead of $c_{\rm eff} = 1$), where we parametrized $q = {\rm e}^{i\pi/(x+1)}$. The simplest argument for this claim is field theoretical. We follow [22], where the Rényi EE, $S^{(N)} \equiv \frac{1}{1-N} \ln \mathrm{Tr}\, \rho^N$, is computed from N copies of the theory on a Riemann surface with two branch points a distance L apart. As the density operator is obtained by imaginary time evolution, we must project, in the case of non-unitary CFT, onto $|0_{\mathrm{R}}\rangle$ in the "past" and on $|0_{\mathrm{L}}\rangle$ in the "future", to obtain $\tilde{\rho} = |0_{\mathrm{R}}\rangle\langle 0_{\mathrm{L}}|$.

We calculate the QG Rényi EE using the loop model. The geometry of [22] leads to a simple generalization of well-known partition function calculations [23]: an ensemble of dense loops now lives on N sheets (with a cut of length L), and *each loop* has weight n. Let $Z^{(N)}$ denote the partition function. Crucially, there are now *two types* of loops: those which do not intersect the cut close after winding an angle 2π, but those which do close after winding $2N\pi$. To obtain the Rényi EE, we must find the dependence of $Z^{(N)}$ on L.

To this end we use the Coulomb gas (CG) mapping [24, 25]. The TL chain is associated with a model of oriented loops on the square lattice. Assign a phase $\mathrm{e}^{\pm ie_0/4}$ to each left (right) turn. In the plane, the number of left minus the number of right turns is $\Delta N_{\pm} = \pm 4$, so the weight $n = 2\cos e_0$ results from summing over orientations. The oriented loops then provide a vertex model, hence a solid-on-solid model on the dual lattice. Dual height variables are defined by induction, with the (standard) convention that the heights across an oriented loop edge differ by π. In CG theory, the large-distance dynamics of the heights is described by a Gaussian field ϕ with action $A[\phi] = \frac{g}{4\pi}\int \mathrm{d}^2x \left[(\partial_x\phi)^2 + (\partial_y\phi)^2\right]$ and coupling $g = 1 - e_0 = \frac{x}{x+1}$.

With N replicas, we get in this way N bosonic fields $\phi_1, \ldots, \phi_N$. The crux of the matter is the cut: a loop winding N times around one of its ends should still have weight n, whilst, since $\Delta N_{\pm} = \pm 4N$ on the Riemann surface, it gets instead $n' = 2\cos N\pi e_0$. We repair this by placing *electric charges* at the two ends (labelled l, r) of the cut, $e_l = e - e_0$ and $e_r = -e - e_0$, where e will be determined shortly. More precisely, we must insert the vertex operators $\exp[ie_{l,r}(\phi_1 + \ldots + \phi_N)(z_{l,r}, \bar{z}_{l,r})]$ before computing $Z^{(N)}$. This choice leaves unchanged the weight of loops which do not encircle nor intersect the cut. A loop that surround *both* ends (and thus, lives on a single sheet) gathers $\mathrm{e}^{\pm i\pi e_0}$ from the turns, and $\mathrm{e}^{\pm i\pi(e_l+e_r)} = \mathrm{e}^{\mp 2i\pi e_0}$ from the vertex operators (since the loop increases the height of points l and r by $\pm\pi$). The two contributions give in the end $\mathrm{e}^{\mp i\pi e_0}$, summing up to n as required. Finally, for a loop encircling only *one* end we get phases $\mathrm{e}^{\pm ie_{l,r}N\pi}\mathrm{e}^{\pm iN\pi e_0} = \mathrm{e}^{\pm iN\pi e}$, so the correct weight n is obtained setting $e = \frac{e_0}{N}$.

To evaluate the $Z^{(N)}$ we implement the sewing conditions on the surface, $\phi_j(z^+) = \phi_{j+1}(z^-)$ with j mod N, by forming combinations of the fields that obey twisted boundary conditions along the cut. For instance, with $N = 2$, we form $\phi_+ = (\phi_1 + \phi_2)/\sqrt{2}$ and

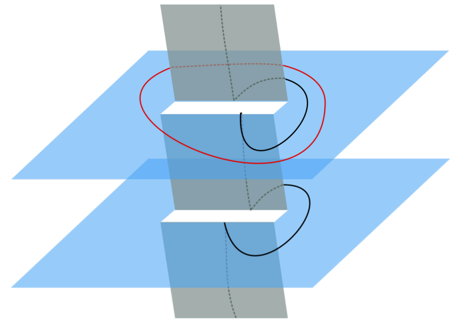

Figure 1. On the Riemann surface used to calculate the Renyi entropy with N replicas (here $N = 2$), the black loop must wind $2\pi N$ times before closing onto itself. The red loop surrounds both ends of the cut.

$\phi_- = (\phi_1 - \phi_2)/\sqrt{2}$. While ϕ_+ does not see the cut, ϕ_- is now twisted: $\phi_-(z^+) = -\phi_-(z^-)$. For arbitrary N, the field $\phi_{\mathrm{sym}} \equiv (\phi_1 + \ldots + \phi_N)/\sqrt{N}$ does not see the cut, while the others are twisted by angles $\mathrm{e}^{2i\pi k/N}$ with $k = 1, \ldots, N-1$. Using that the dimension of the twist fields in a *complex* bosonic theory is [26] $h_{k/N} = k(N-k)/2N^2$ we find that the twisted contribution to the partition function is $Z^{(N)}(\mathrm{twist}) \propto L^{-2x_N}$ with $x_N = \sum_{k=1}^{N-1} h_{k/N} = \frac{1}{12}\left(N - \frac{1}{N}\right)$. Meanwhile, the field ϕ_+, which would not contribute to the EE for a free boson theory (here $e_0 = 0$), now yields a non-trivial term due to the vertex operators with $e_{l,r}$: $Z^{(N)}(\mathrm{charge}) \propto L^{-2x'_N}$ with $x'_N = N\frac{e^2 - e_0^2}{2g} = \frac{e_0^2}{2g}\left(\frac{1}{N} - N\right)$. Assembling everything we get $Z^{(N)} \propto L^{-\frac{1}{6}\left(N - \frac{1}{N}\right)(1 - 6e_0^2/g)}$. Inserting $e_0 = \frac{1}{x+1}$ and $g = \frac{x}{x+1}$ gives the Rényi entropies

$$\tilde{S}_L^{(N)} = \frac{N+1}{6N}\left[1 - \frac{6}{x(x+1)}\right]\ln L \tag{9}$$

($\tilde{S}$ is obtained for $N \to 1$), hence proving our claim.

We emphasize that the $U_q sl(2)$ spin chain differs from the usual one simply by the boundary terms h_i. These are not expected to affect the ordinary EE, and the central charge obtained via the density operator $\rho = |0\rangle\langle 0|$ (with $|0\rangle \propto |0_{\mathrm{R}}\rangle$, but normalized as in our introduction) will be $c_{\mathrm{eff}} = 1$.

Entanglement in non-unitary minimal models. We now discuss the restricted solid-on-solid (RSOS) lattice models, which provide the nicest regularization of non-unitary CFTs. In these models, the variables are "heights" on an A_m Dynkin diagram, with Boltzmann weights that provide yet another representation of the TL algebra (8), with parameter $n = 2\cos\frac{\pi p}{m+1}$ and $p = 1, \ldots, m$. The case $p = 1$ is Hermitian, while $p \neq 1$ leads to negative weights, and hence a non-unitary CFT. One has $c = 1 - 6\frac{p^2}{(m+1)(m+1-p)}$, and, for $p \neq 1$, the effective central charge—determined by the state of lowest conformal weight [11] through $c_{\mathrm{eff}} = c - 24h_{\mathrm{min}}$—is $c_{\mathrm{eff}} = 1 - \frac{6}{(m+1)(m+1-p)}$. The case $(m, p) = (4, 3)$ gives the Yang-Lee singularity universality class discussed in the introduction.

Defining the EE for RSOS models is not obvious, since their Hilbert space (we use this term even in the non-unitary case) is not a tensor product like for spin chains. Most recent numerical and analytical work however neglected this fact, and EE was defined using a straightforward partial trace, summing over all heights in B compatible with those in A. In this case, it was argued and checked numerically that $S_A = \frac{c}{3}\ln L$ in the unitary case, and $S_A = \frac{c_{\rm eff}}{3}\ln L$ in the non-unitary case. Note that c matches that of the loop model based on the same TL algebra, with $x+1 \equiv \frac{m+1}{p}$. For details on the QG EE in the RSOS case, see the SM.

The RSOS partition functions can be expressed in terms of loop model ones, Z_ℓ. In the plane, the equivalence [27] replaces equal-height clusters by their surrounding loops, which get the usual weight n through an appropriate choice of weights on A_m. With periodic boundary conditions, the correspondence is more intricate due to non-contractible clusters/loops. On the torus [28], Z_ℓ is defined by giving each loop (contractible or not) weight n, whereas for the RSOS model contractible loops still have weight n, but one sums over sectors where each non-contractible loop gets the weights $n_k = 2\cos\frac{\pi k}{m+1}$ for any $k = 1,\ldots,m$. The same sum occurs (see SM [20] for details) when computing $Z^{(N)}$ of the Riemann surface with N replicas: non-contractible loops are here those winding one end of the cut. Note also that $|0_{\rm L}\rangle = |0_{\rm R}\rangle$ for RSOS models, so the imaginary-time definition of ρ in unambiguous [9, 10].

Crucially, the sum over k is dominated (in the scaling limit) by the sector with the largest n_k, that is $k=1$ and $n_1 = 2\cos\frac{\pi}{m+1}$. In the non-unitary case ($p>1$), $n_1 \neq n$, and the EE is found by extending the above computation. We have still $e_0 = \frac{p}{m+1}$, but now $e = \frac{1}{N(m+1)} = \frac{e_0}{pN}$. To normalize at $N=1$, one must divide by $Z^{(1)}$ to the power N, with the same charges:

$$Z^{(N)}/\left(Z^{(1)}\right)^N \propto L^{-\frac{1}{6}\left(N-\frac{1}{N}\right)\left(1-\frac{6e_0^2}{p^2 g}\right)}, \qquad (10)$$

whence the Rényi entropy $S_A^{(N)} = \frac{N+1}{6N} c_{\rm eff} \ln L$. Hence our construction establishes the claim of [9, 10].

EE in the $sl(2|1)$ SUSY chain. Percolation and other problems with SUSY (see the introduction) have $Z=1$, hence $c=0$, and the EE scales trivially. Having a non-trivial quantity that distinguishes the many $c=0$ universality classes would be very useful. We now show that, by carefully distinguishing left and right eigenstates, and using traces instead of supertraces, one can modify the definition of EE to build such a quantity.

We illustrate this by the $sl(2|1)$ alternating chain [8] which describes percolation hulls. This chain represents the TL algebra (8) with $n=1$, and involves the fundamental (V) and its conjugate ($\bar{V}$) on alternating sites, with $\dim V = 3$. The 2-site Hamiltonian, $H = -e_1$, restricted to the subspace $\{|1\bar{1}\rangle, |2\bar{2}\rangle, |3\bar{3}\rangle\}$ (where $1,2$ are bosonic and 3 is fermionic), reads

$$e_1 = |0_{\rm R}\rangle\langle 0_{\rm L}| = \left(|1\bar{1}\rangle + |2\bar{2}\rangle + |3\bar{3}\rangle\right)\left(\langle 1\bar{1}| + \langle 2\bar{2}| + \langle 3\bar{3}|\right)$$

The eigenvectors are $|0_{\rm R}\rangle = |1\bar{1}\rangle + |2\bar{2}\rangle + |3\bar{3}\rangle$ and $\langle 0_{\rm L}| = \langle 1\bar{1}| + \langle 2\bar{2}| - \langle 3\bar{3}|$; note that conjugation is supergroup invariant (i.e., $\langle\bar{3}|\bar{3}\rangle = -1$). Hence, despite the misleading expression, H is not unitary. The density operator is $\tilde{\rho} = e_1$ and satisfies ${\rm STr}\,\tilde{\rho} \equiv {\rm Tr}(-1)^F\tilde{\rho} = 1$. The reduced density operator $\tilde{\rho}_A = {\rm STr}_B\,\tilde{\rho} = |1\rangle\langle 1| + |2\rangle\langle 2| + |3\rangle\langle 3|$. If we define the Rényi EE also with the supertrace, we get ${\rm STr}\,\tilde{\rho}_A^N = 1$ for all N. It is more interesting (and natural) to take instead the *normal trace* of $\tilde{\rho}$; this requires a renormalization factor to ensure ${\rm Tr}\,\tilde{\rho}_A = 1$. We obtain then $\tilde{\rho}_A^N = \frac{1}{3^N}\left(|1\rangle\langle 1| + |2\rangle\langle 2| + |3\rangle\langle 3|\right)$ and thus $\tilde{S}_A^{(N)} = \ln 3$. This equals the QG Rényi EE with $n=3$.

This calculation carries over to arbitrary size. One finds that $\tilde{S}_A = \tilde{S}_{A,\ell}$ with weight $n=1$, *provided* non-contractible loops winding around one cut end in the replica calculation get the modified weight $\tilde{n} = 3$ instead of n. We can then use the CG framework developed in the context of the non-unitary minimal models to calculate the scaling behavior. We use (10), with $g = \frac{2}{3}$ for percolation ($n=1$), and $\tilde{n} = 2\cos\pi e_0$. It follows that e_0 is purely imaginary, and that $\tilde{S}^{(N)} \sim \frac{N+1}{6N} c_{\rm eff} \log L$ with $c_{\rm eff} = 1 + \frac{9}{\pi^2}\left(\log\frac{3+\sqrt{5}}{2}\right)^2 \sim 1.84464\ldots$.

Numerical checks. All these results were checked numerically. As an illustration, we discuss only the case $q = e^{2i\pi/5}$, for which the RSOS and loop models have $c = -3/5$, while $c_{\rm eff} = 3/5$ for the RSOS model. In the corresponding $U_q sl(2)$ chain, we measured the (ordinary) EE as in (3), the QG Rényi EE $\tilde{S}^{(2)}$ as in (7), and the QG Rényi EE for the modified loop model where non-contractible loops have fugacity $n_1 = 2\cos\frac{\pi}{5}$ (instead of $n = 2\cos\frac{2\pi}{5}$). This, recall, should coincide asymptotically with the Rényi EE for the RSOS model. Results (see SM) fully agree with our predictions.

Conclusion. While we have mostly discussed the critical case, we stress that the QG EE can be defined also away from criticality. An interesting example is the $sl(2|1)$ alternating chain, for which staggering makes the theory massive (this corresponds to shifting the topological angle away from $\Theta = \pi$ in the sigma-model representation). Properties of the QG Rényi EE along this (and other) RG flows will be reported elsewhere.

Summarizing, this analysis completes our understanding of EE in 1D, providing a natural extension to non-unitary models in their critical or near-critical regimes. In many situations (such as phenomenological "Hamiltonians" for open systems) certain aspects will differ, but our work may provide the first step in the right direction. We also establish a long-awaited "Coulomb gas" handle on the correspondence between lattice models and quantum information quantities. Using this (see SM), we show that, in the case of non-compact theories, the usual $\frac{c}{3}\ln L$ term is corrected by $\ln\ln L$ terms (with, most likely, a non-universal amplitude), in agreement with recent independent work [29].

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[1] P. Calabrese, J. Cardy and B. Doyon (eds.), *Entanglement entropy in extended quantum systems*, J. Phys. A: Math. Theor. **42** (2009), and references therein.
[2] T. Prozen, J. Phys. A: Math. Theor. **48**, 373001 (2015).
[3] M. Zirnbauer, J. Math. Phys. **38**, 2007 (1997), and references therein.
[4] N. Beisert et al., Lett. Math. Phys. **99**, 3 (2012).
[5] T. Quella and V. Schomerus, J. Phys. A: Math. Theor. **46**, 494010 (2013).
[6] G. Parisi and N. Sourlas, J. Physique Lett. **41**, 403 (1980).
[7] C.M. Bender, J. Phys. Conf. Ser. **631**, 012002 (2015).
[8] N. Read and H. Saleur, Nucl. Phys. B **613**, 409 (2001).
[9] D. Bianchini, O. Castro-Alvaredo, B. Doyon, E. Levi and F. Ravanini, J. Phys. A: Math. Gen. **48**, 04FT01 (2015).
[10] D. Bianchini, O. Castro-Alvaredo and B. Doyon, Nucl. Phys. B **896**, 835 (2015).
[11] C. Itzykson, H. Saleur and J.B. Zuber, Euro. Phys. Lett. **2**, 91 (1986).
[12] F. Alcaraz, M. Berganza and G. Sierra, Phys. Rev. Lett. **106**, 201601 (2011).
[13] V. Pasquier and H. Saleur, Nucl. Phys. B **330**, 523 (1990).
[14] V. G. Drinfeld, J Math Sci **41**, 898-915 (1988).
[15] D. Bernard and A. LeClair, Commun.Math. Phys. **142**, 99-138 (1991).
[16] C. Gomez, M. Altaba and G. Sierra, *Quantum Groups in Two-dimensional Physics*, Cambridge Univ. Press (1996).
[17] A. Feiguin, S. Trebst, A. W. W. Ludwig, M. Troyer, A. Kitaev, Z. Wang and M. H. Freedman, Phys. Rev. Lett. **98**, 160409 (2007).
[18] J. L. Jacobsen, N. Read and H. Saleur, Phys. Rev. Lett. **90**, 090601 (2003).
[19] I. Affleck, T. Kennedy, E. H. Lieb and H. Tasaki, Commun.Math.Phys. **115**, 477 (1988).
[20] See Supplemental material, which includes Refs. [30–34], for discussions of the quantum group symmetry, properties of the QG entropy, calculations of the entropy in different representations, the RSOS case, the non compact case and numerics.
[21] A. Morin-Duchesne, J. Rasmussen, P. Ruelle and Y. Saint Aubin, J. Stat. Mech. (2016) 053105.
[22] J. Cardy and P. Calabrese, J. Phys. A: Math. Theor. **42**, 504005 (2009).
[23] P. Di Francesco, H. Saleur and J.B. Zuber, J. Stat. Phys. **49**, 57 (1987).
[24] B. Nienhuis, J. Stat. Phys. **34**, 731 (1984).
[25] J.L. Jacobsen, *Conformal field theory applied to loop models*, in A.J. Guttmann (ed.), *Polygons, polyominoes and polycubes*, Lecture Notes in Physics **775**, 347–424 (Springer, 2009).
[26] D. Friedan, E. Martinec and S. Shenker, Nucl. Phys. B **282**, 13 (1987).
[27] V. Pasquier, J. Phys. A: Math. Gen. **20**, L1229 (1987).
[28] P. Di Francesco, H. Saleur and J.B. Zuber, Nucl. Phys. B **300**, 393 (1988).
[29] D. Bianchini and O. Castro-Alavaredo, *Branch point twist field correlators in the massive free boson theory*, arXiv:1607.05656; O. Blondeau Fournier and B. Doyon, *Expectation values of twist fields and universal entanglement saturation of the massive free boson*, in preparation.
[30] H. Saleur and M. Bauer, Nucl. Phys. B **320**, 591 (1989).
[31] P. Calabrese and A. Lefèvre, Phys. Rev. A **78**, 032329 (2008).
[32] S. Fredenhagen, M.R. Gaberdiel and C. Keller, J. Phys. A: Math. Theor. **42**, 495403 (2009).
[33] I. Affleck and A.W.W. Ludwig, Phys. Rev. Lett. **67**, 161 (1991).
[34] I. Runkel and G.M.T. Watts, JHEP 09 (2001) 006.