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Phys. Rev. Lett. **119**, 040403 — Published 27 July 2017

DOI: [10.1103/PhysRevLett.119.040403](https://doi.org/10.1103/PhysRevLett.119.040403)

# Hierarchy construction and non-Abelian families of generic topological orders

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We generalize the hierarchy construction to generic 2+1D topological orders (which can be non-Abelian) by condensing Abelian anyons in one topological order to construct a new one. We show that such construction is reversible and leads to a new equivalence relation between topological orders. We refer to the corresponding equivalent class (the orbit of the hierarchy construction) as “the non-Abelian family”. Each non-Abelian family has one or a few root topological orders with the smallest number of anyon types. All the Abelian topological orders belong to the trivial non-Abelian family whose root is the trivial topological order. We show that Abelian anyons in root topological orders must be bosons or fermions with trivial mutual statistics between them. The classification of topological orders is then greatly simplified, by focusing on the roots of each family: those roots are given by non-Abelian modular extensions of representation categories of Abelian groups.

PACS numbers: 05.30.Pr, 71.10.Pm

**Introduction:** The ultimate dream of classifying objects in nature may be creating a “table” for them. A classic example of such classification result is the “Periodic Table” for chemical elements. As for the topological ordered[1, 2] phases of matter, which draws more and more research interests recently, we are already able to create some “tables” for them[3–8], via the theory of (pre-)modular categories. However, efforts are needed to further understand the tables, for example, to reveal some “periodic” structures in the table.

In the Periodic Table, elements are divided into several “families” (the columns of the table), and those in the same family have similar chemical properties. The underlying reason for this is that elements in the same family have similar outer electron structures, and only differ by “noble gas cores”. The last family consists of noble gas elements, which are chemically “inert”, as they have no outer electrons besides the noble gas cores. Thus the “family” can be considered as the equivalent class up to the “inert” noble gas elements.

When it comes to topological orders, we also have “inert” ones: the Abelian topological orders are “inert”, for example, in the application of topological quantum computation[9, 10]. Abelian anyons can not support non-local topological degeneracy, which is an essential difference from non-Abelian anyons. Is it possible to define equivalent classes for topological orders, which are up to Abelian topological orders? In this letter, we use the hierarchy construction to establish such equivalent classes, which we will call the “non-Abelian families”. The hierarchy construction is well known in the study of Abelian fractional quantum Hall (FQH) states[11–14]. In this letter we generalize it to arbitrary (potentially non-Abelian) topological orders.

We show that the generalized hierarchy construction is reversible. Thus, we can say that two topological orders belong to the same “non-Abelian family” if they are

related by the hierarchy construction. Each non-Abelian family has special “root” topological orders (see Table I), with the following properties:

1. Root states have the smallest rank (number of anyon types) among the non-Abelian family.
2. Abelian anyons in a root state are all bosons or fermions, and have trivial mutual statistics with each other.

Since any topological order in the same non-Abelian family can be reconstructed from a root state, our work simplifies the classification of generic topological orders to the classification of root states.

Our calculation is based on quantitative characterizations of topological orders. One way to do so is to use the  $S, T$  modular matrices obtained from the non-Abelian geometric phases of degenerate ground states on torus [1, 2]. We will show, starting from a topological order described by  $S, T$ , how to obtain another topological order described by new  $S', T'$  via a condensation of Abelian anyons. (For a less general approach based on wave functions, see Ref. 15.) The calculation uses the theory of fusion and braiding of quasiparticles (which will be called *anyons*) in topological order. Such a theory is the so called “unitary modular tensor category (UMTC) theory” (for a review and much more details on UMTC, see Ref. 5).

A UMTC  $\mathcal{C}$  is simply a set of anyons (two anyons connected by a local operator are regarded as the same type), plus data to describe their fusion and braiding. Like the fusion of two spin-1 particles give rise to a “direct sum” of spin-0,1,2 particles:  $1 \otimes 1 = 0 \oplus 1 \oplus 2$ , the fusion of two anyons  $i$  and  $j$  in general gives rise to a “direct sum” of several other anyons:  $i \otimes j = \bigoplus_k N_k^{ij} k$ . So the fusion of anyons is quantitatively described by a rank-3 integer tensor  $N_k^{ij}$ . From  $N_k^{ij}$ , we can determine the internal

degrees of freedom of anyons, which is the so called *quantum dimension*. For example, the quantum dimension of a spin- $S$  particle is  $d = 2S + 1$ . For an anyon  $i$ , its quantum dimension  $d_i$ , which can be non-integer, is the largest eigenvalue of matrix  $N_i$  with  $(N_i)_{kj} = N_k^{ij}$ .

After knowing the fusion, the braiding of anyons can be fully determined by the fractional part of their angular momentum  $L^z$ :  $s_i = \text{mod}(L_i^z, 1)$ .  $s_i$  is called the *topological spin* (or simply spin) of the anyon  $i$ . The last piece of data to characterize topological orders is the *chiral central charge*  $c$ , which is the number of right-moving edge modes minus the number of left-moving edge modes.

It turns out that two sets of data  $(S, T)$  and  $(N_k^{ij}, s_i)$  can fully determine each other:

$$T_{ij} = e^{2\pi i s_i} \delta_{ij}, \quad S_{ij} = \sum_k e^{2\pi i (s_i + s_j - s_k)} \frac{N_k^{ij} d_k}{D},$$

$$e^{2\pi i s_i} = T_{ii}, \quad N_k^{ij} = \sum_l \frac{S_{li} S_{lj} \overline{S_{lk}}}{S_{l1}}. \quad (1)$$

where  $D = \sqrt{\sum d_i^2}$  is the total quantum dimension.

**Hierarchy construction in generic topological orders:** Let us consider an Abelian anyon condensation in a generic topological order, described by a UMTC  $\mathcal{C}$ . (Such a condensation in an Abelian topological order is discussed in the Supplemental Material [16]) The anyons in  $\mathcal{C}$  are labeled by  $i, j, k, \dots$ . Let  $a_c$  be an Abelian anyon in  $\mathcal{C}$  with spin  $s_c$ . We condense  $a_c$  into the Laughlin state  $\Psi = \prod (z_i - z_j)^{m_c - 2s_c}$ , where  $m_c = \text{even}$  and  $m_c - 2s_c \neq 0$ . [17] The resulting topological order is described by UMTC  $\mathcal{D}$ , determined by  $\mathcal{C}$ ,  $a_c$  and  $m_c$ .

To calculate  $\mathcal{D}$ , we note that the anyons in  $\mathcal{D}$  are the anyons  $i$  in  $\mathcal{C}$  dressed with the vortices of the Laughlin state of  $a_c$ . The vorticity is given by  $m - t_i$ , where  $m$  is an integer, and  $2\pi t_i$  is the mutual statistics angle between anyon  $i$  and the condensing anyon  $a_c$  in the original topological order  $\mathcal{C}$ , which can be extracted from the  $S$  matrix  $e^{-2\pi i t_i} = S_{ia_c}^C / S_{i1}^C$ , or  $-t_i = s_i + s_{a_c} - s_{i \otimes a_c}$ . Thus anyons in  $\mathcal{D}$  are labeled by pairs  $I = (i, m)$ . We like to ask what is the spin and fusion rules of  $I = (i, m)$ ?

The spin of  $(i, m)$  is given by the spin of  $i$  plus the spin of the  $m - t_i$  flux in the Laughlin state:

$$s_{(i,m)} = s_i + \frac{1}{2} \frac{(m - t_i)^2}{m_c - 2s_c} \quad (2)$$

Fusing  $i$  with  $m - t_i$  flux and  $j$  with  $n - t_j$  flux gives us  $i \otimes j$  with  $m - t_i + n - t_j$  flux:

$$(i, m) \otimes (j, n) \sim \oplus_k N_k^{ij} (k, m - t_i + n - t_j + t_k), \quad (3)$$

where  $N_k^{ij}$  is the fusion coefficient in  $\mathcal{C}$ . Since  $a_c$  with  $m_c - t_{a_c} = m_c - 2s_c$  flux is condensed, fusing with  $(a_c, m_c)$  anyon does not change the anyon type in  $\mathcal{D}$ . So, we have an equivalence relation:

$$(i, m) \sim (i \otimes a_c, m - t_i + m_c - 2s_c + t_{i \otimes a_c}), \quad (4)$$

TABLE I. The low-rank root topological orders for bosonic systems. We pick only one root for each non-Abelian family. The rank  $N$  is the number of anyon types and  $c$  the chiral central charge of the edge states.  $s_i$  and  $d_i$  are the topological spin and quantum dimension of type- $i$  anyon. The anyons with red entries have trivial mutual statistics with all Abelian anyons. Here  $\zeta_n^m = \frac{\sin[\pi(m+1)/(n+2)]}{\sin[\pi/(n+2)]}$ . Some roots are the stacking of simpler ones, such as  $4_0 = 2_{14} \boxtimes 2_{26}$ .

$N_c$	$D^2$	$s_1, s_2, \dots$	$d_1, d_2, \dots$
1 <sub>0</sub>	1	0	1
2 <sub>14</sub> <sup>5</sup>	3.618	0, $\frac{2}{5}, \frac{3}{5}$	1, $\zeta_3^1$
2 <sub>26</sub> <sup>5</sup>	3.618	0, $\frac{2}{5}, \frac{3}{5}$	1, $\zeta_3^1$
3 <sub>5</sub> <sup>7</sup>	4	0, $\frac{1}{2}, \frac{5}{16}$	1, 1, $\zeta_2^1$
3 <sub>8</sub> <sup>7</sup>	9.295	0, $\frac{6}{7}, \frac{12}{7}$	1, $\zeta_5^1, \zeta_5^2$
3 <sub>48</sub> <sup>7</sup>	9.295	0, $\frac{1}{7}, \frac{5}{7}, \frac{12}{7}$	1, $\zeta_5^1, \zeta_5^2$
4 <sub>0</sub>	13.09	0, $\frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, \frac{12}{5}, \frac{13}{5}$	1, $\zeta_3^1, \zeta_3^2, \zeta_8^1, \zeta_8^2$
4 <sub>12</sub> <sup>5</sup>	13.09	0, $\frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, \frac{12}{5}, \frac{13}{5}$	1, $\zeta_3^1, \zeta_3^2, \zeta_8^1, \zeta_8^2$
4 <sub>28</sub> <sup>5</sup>	13.09	0, $\frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, \frac{12}{5}, \frac{13}{5}$	1, $\zeta_3^1, \zeta_3^2, \zeta_8^1, \zeta_8^2$
4 <sub>10</sub> <sup>3</sup>	19.23	0, $\frac{1}{3}, \frac{2}{3}, \frac{4}{3}, \frac{5}{3}, \frac{7}{3}, \frac{8}{3}, \frac{11}{3}$	1, $\zeta_7^1, \zeta_7^2, \zeta_7^3$
4 <sub>14</sub> <sup>3</sup>	19.23	0, $\frac{1}{3}, \frac{2}{3}, \frac{4}{3}, \frac{5}{3}, \frac{7}{3}, \frac{8}{3}, \frac{11}{3}$	1, $\zeta_7^1, \zeta_7^2, \zeta_7^3$
5 <sub>2</sub>	12	0, 0, $\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \frac{7}{2}, \frac{9}{2}, \frac{11}{2}, \frac{13}{2}$	1, 1, $\zeta_4^1, \zeta_4^2, 2$
5 <sub>6</sub>	12	0, 0, $\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \frac{7}{2}, \frac{9}{2}, \frac{11}{2}, \frac{13}{2}$	1, 1, $\zeta_4^1, \zeta_4^2, 2$
5 <sub>16</sub> <sup>11</sup>	34.64	0, $\frac{9}{11}, \frac{12}{11}, \frac{15}{11}, \frac{18}{11}, \frac{21}{11}$	1, $\zeta_9^1, \zeta_9^2, \zeta_9^3, \zeta_9^4$
5 <sub>72</sub> <sup>11</sup>	34.64	0, $\frac{9}{11}, \frac{12}{11}, \frac{15}{11}, \frac{18}{11}, \frac{21}{11}$	1, $\zeta_9^1, \zeta_9^2, \zeta_9^3, \zeta_9^4$
5 <sub>18</sub> <sup>7</sup>	35.34	0, $\frac{6}{7}, \frac{9}{7}, \frac{12}{7}, \frac{15}{7}, \frac{18}{7}, \frac{21}{7}$	1, $\zeta_5^2, \zeta_5^3, \zeta_{12}^1, \zeta_{12}^2$
5 <sub>38</sub> <sup>7</sup>	35.34	0, $\frac{1}{7}, \frac{1}{7}, \frac{6}{7}, \frac{13}{7}, \frac{20}{7}$	1, $\zeta_5^2, \zeta_5^3, \zeta_{12}^1, \zeta_{12}^2$
6 <sub>3</sub> <sup>10</sup>	14.47	0, $\frac{1}{2}, \frac{11}{16}, \frac{9}{10}, \frac{2}{5}, \frac{7}{80}$	1, 1, $\zeta_2^1, \zeta_3^1, \zeta_3^2, \zeta_2^1 \zeta_3^1$
6 <sub>77</sub> <sup>10</sup>	14.47	0, $\frac{1}{2}, \frac{5}{16}, \frac{1}{10}, \frac{3}{5}, \frac{73}{80}$	1, 1, $\zeta_2^1, \zeta_3^1, \zeta_3^2, \zeta_2^1 \zeta_3^1$
6 <sub>0</sub>	20	0, 0, $\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \frac{7}{2}, \frac{9}{2}, \frac{11}{2}, \frac{13}{2}$	1, 1, 2, 2, $\sqrt{5}, \sqrt{5}$
6 <sub>4</sub>	20	0, 0, $\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \frac{7}{2}, \frac{9}{2}, \frac{11}{2}, \frac{13}{2}$	1, 1, 2, 2, $\sqrt{5}, \sqrt{5}$
6 <sub>58</sub> <sup>35</sup>	33.63	0, $\frac{2}{5}, \frac{1}{7}, \frac{19}{35}, \frac{26}{35}, \frac{4}{35}$	1, $\zeta_3^1, \zeta_5^1, \zeta_5^2, \zeta_3^1 \zeta_5^1, \zeta_5^2 \zeta_3^1$
6 <sub>138</sub> <sup>35</sup>	33.63	0, $\frac{2}{5}, \frac{1}{7}, \frac{19}{35}, \frac{26}{35}, \frac{4}{35}$	1, $\zeta_3^1, \zeta_5^1, \zeta_5^2, \zeta_3^1 \zeta_5^1, \zeta_5^2 \zeta_3^1$
6 <sub>142</sub> <sup>35</sup>	33.63	0, $\frac{3}{5}, \frac{1}{7}, \frac{5}{35}, \frac{26}{35}, \frac{11}{35}$	1, $\zeta_3^1, \zeta_5^1, \zeta_5^2, \zeta_3^1 \zeta_5^1, \zeta_5^2 \zeta_3^1$
6 <sub>222</sub> <sup>35</sup>	33.63	0, $\frac{3}{5}, \frac{1}{7}, \frac{2}{7}, \frac{16}{35}, \frac{31}{35}$	1, $\zeta_3^1, \zeta_5^1, \zeta_5^2, \zeta_3^1 \zeta_5^1, \zeta_5^2 \zeta_3^1$
6 <sub>46</sub> <sup>13</sup>	56.74	0, $\frac{4}{13}, \frac{2}{13}, \frac{7}{13}, \frac{6}{13}, \frac{12}{13}$	1, $\zeta_{11}^1, \zeta_{11}^2, \zeta_{11}^3, \zeta_{11}^4, \zeta_{11}^5$
6 <sub>58</sub> <sup>13</sup>	56.74	0, $\frac{9}{13}, \frac{11}{13}, \frac{6}{13}, \frac{7}{13}, \frac{1}{13}$	1, $\zeta_{11}^1, \zeta_{11}^2, \zeta_{11}^3, \zeta_{11}^4, \zeta_{11}^5$
6 <sub>8</sub> <sup>3</sup>	74.61	0, $\frac{1}{9}, \frac{1}{9}, \frac{1}{9}, \frac{1}{3}, \frac{2}{3}$	1, $\zeta_7^1, \zeta_7^2, \zeta_7^3, \zeta_{16}^4, \zeta_{16}^6$
6 <sub>16</sub> <sup>3</sup>	74.61	0, $\frac{8}{9}, \frac{8}{9}, \frac{8}{9}, \frac{2}{3}, \frac{1}{3}$	1, $\zeta_7^1, \zeta_7^2, \zeta_7^3, \zeta_{16}^4, \zeta_{16}^6$
6 <sub>2</sub>	100.6	0, $\frac{6}{7}, \frac{5}{7}, \frac{3}{7}, 0, \frac{1}{3}$	1, $\frac{3+\sqrt{21}}{2} \times 3, \frac{5+\sqrt{21}}{2}, \frac{7+\sqrt{21}}{2}$
6 <sub>6</sub>	100.6	0, $\frac{1}{7}, \frac{2}{7}, \frac{4}{7}, 0, \frac{2}{3}$	1, $\frac{3+\sqrt{21}}{2} \times 3, \frac{5+\sqrt{21}}{2}, \frac{7+\sqrt{21}}{2}$

The above three relations fully determine the topological order  $\mathcal{D}$ . [5, 6]

It is important to fix a “gauge” for  $t_i$ , say by choosing  $t_i \in [0, 1)$ . The same label  $(i, m)$  may label different anyons under different “gauge” choices of  $t_i$ . Similarly, we have fixed a “gauge” for  $s_c$  that fixed the meaning of  $m_c$ . Note that  $t_{a_c}$  is automatically fixed when  $s_c$  is fixed:  $t_{a_c} = 2s_c$ , while other  $t_i$  can be freely chosen. This ensures that the equivalence relation (4) is compatible with fusion (3), where (4) is generated by fusing with the trivial anyon  $(a_c, m_c)$ . The combinations  $m - t_i, m_c - 2s_c$  determine the final spins and fusion rules; they are gauge-invariant quantities. Thus, if we change the

gauge of  $t_i, s_c$ , i.e., modify them by some integers,  $m, m_c$  should be modified by the same integers to ensure that the construction remains the same.

Below we will study the properties of  $\mathcal{D}$  in detail. Let  $M_c = m_c - 2s_c$ . Applying the equivalence relation (4)  $q$  times, we obtain

$$(i, m) \sim (i \otimes a_c^{\otimes q}, m - t_i + qM_c + t_{i \otimes a_c^{\otimes q}}). \quad (5)$$

Let  $q_c$  be the ‘‘period’’ of  $a_c$ , i.e., the smallest positive integer such that  $a_c^{\otimes q_c} = \mathbf{1}$ . We see that  $(i, m) \sim (i, m + q_c M_c)$ . Thus, we can focus on the reduced range of  $m \in \{0, 1, 2, \dots, q_c |M_c| - 1\}$ . Let  $|\mathcal{C}|, |\mathcal{D}|$  denote the rank of  $\mathcal{C}, \mathcal{D}$  respectively. Now within the reduce range of  $m$ , we have  $q_c |M_c| |\mathcal{C}|$  different labels, and we want to show that the orbit generated by the equivalence relation (5) all have the same length, which is  $q_c$ . To see this, just note that for  $0 < q < q_c$ , either  $i \neq i \otimes a_c^{\otimes q}$ , or if  $i = i \otimes a_c^{\otimes q}$ ,  $m \neq m - t_i + qM_c + t_{i \otimes a_c^{\otimes q}} = m + qM_c$ ; in other words, the labels  $(i, m)$  are all different within  $q_c$  steps. It follows that the rank of  $\mathcal{D}$  is  $|\mathcal{D}| = |M_c| |\mathcal{C}|$ .

Strictly speaking, anyons in  $\mathcal{D}$  should one-to-one correspond to the equivalent classes of  $(i, m)$ . However, as the orbits have the same length, it would be more convenient to use  $(i, m)$  directly (as we will see later, this is the same as working in a pre-modular category  $\tilde{\mathcal{D}}$ ). For example, when we need to sum over anyons in  $\mathcal{D}$ , we can instead do  $\sum_{I \in \mathcal{D}} \rightarrow \frac{1}{q_c} \sum_{i \in \mathcal{C}} \sum_{m=0}^{q_c |M_c| - 1}$ . Now we are ready to calculate other quantities of the new topological order  $\mathcal{D}$ . First, it is easy to see that the quantum dimensions remain the same  $d_{(i,m)} = d_i$ . The total quantum dimension is then

$$D_{\mathcal{D}}^2 = \frac{1}{q_c} \sum_{i \in \mathcal{C}} \sum_{m=0}^{q_c |M_c| - 1} d_{(i,m)}^2 = |M_c| D_{\mathcal{C}}^2. \quad (6)$$

The  $S$  matrix is

$$\begin{aligned} S_{(i,m),(j,n)}^{\mathcal{D}} &= \sum_k \frac{N_k^{ij}}{D_{\mathcal{D}}} d_k e^{2\pi i [s_{(i,m)} + s_{(j,n)} - s_{(k,m+n+t_k-t_i-t_j)}]} \\ &= \frac{1}{\sqrt{|M_c|}} S_{ij}^{\mathcal{C}} e^{-2\pi i \frac{(m-t_i)(n-t_j)}{M_c}}. \end{aligned} \quad (7)$$

It is straightforward to check that  $S_{(i,m),(j,n)}^{\mathcal{D}}$  is unitary (with respect to equivalent classes of  $(i, m)$ ). Moreover, this formula for  $S$  can recover the equivalence relation (5) and fusion rules (3) via unitarity and Verlinde formula.

The new  $S^{\mathcal{D}}, T^{\mathcal{D}}$  matrices ( $T^{\mathcal{D}}$ -matrix is determined by the spin of anyons  $s_{(i,m)}^{\mathcal{D}}$  in (2)), as well as  $S^{\mathcal{C}}, T^{\mathcal{C}}$ , should both obey the modular relations  $STS = e^{2\pi i \frac{c}{8}} T^{\dagger} S T^{\dagger}$ , from which we can extract the central charge of  $\mathcal{D}$ . The new central charge is found to be (see the Supplementary Material [16])

$$c^{\mathcal{D}} = c^{\mathcal{C}} + \text{sgn}(M_c). \quad (8)$$

Clearly, the one-step hierarchy construction described by (2), (7), and (8) is fully determined by an Abelian anyon  $a_c$  and  $M_c$ , where  $M_c + 2s_c$  is an even integer. In the Supplemental Material [16], we discuss the above hierarchy construction more rigorously at the full categorical level.

As an application, let us explain the ‘‘eight-fold way’’ observed in the table of topological orders[5, 6]: whenever there is a fermionic quasiparticle, the topological order has eight companions with the same rank and quantum dimensions but different spins and central charges. If we apply the one-step condensation with  $a_c$  being a fermion, and  $M_c = \pm 1$ , a new topological order of the same rank is obtained. [18] The spins of the anyons carrying fermion parity flux (having non-trivial mutual statistics with the fermion  $a_c$ ) are shifted by  $\pm 1/8$ , and the central charge is shifted by  $\pm 1$ , while all the quantum dimensions remain the same. If we repeat it eight times, we will go back to the original state (up to an  $E_8$  state), generating the ‘‘eight fold way’’.

**Reverse construction and non-Abelian families:** The one-step condensation from  $\mathcal{C}$  to  $\mathcal{D}$  is always reversible. In  $\mathcal{D}$ , choosing  $a'_c = (\mathbf{1}, 1)$ ,  $s'_c = \frac{1}{2M_c}$ ,  $m'_c = 0$ ,  $M'_c = -1/M_c$ , and repeating the construction, we will go back to  $\mathcal{C}$ . One may first perform the construction for a pre-modular  $\tilde{\mathcal{D}}$  and then reduce the resulting category to a modular category. Taking  $(j, n) = \bar{a}'_c = (\mathbf{1}, -1)$  in (7) we find that the mutual statistics between  $(i, m)$  and  $a'_c = (\mathbf{1}, 1)$  is  $t'_{(i,m)} = \frac{m-t_i}{M_c}$ . Let  $(i, m, p), (j, n, q)$  label the anyons after the above one-step condensation; the new  $S$  matrix is

$$\begin{aligned} S_{(i,m,p),(j,n,q)} &= S_{ij}^{\mathcal{C}} e^{-\frac{2\pi i(m-t_i)(n-t_j)}{M_c}} e^{-\frac{2\pi i(p-t'_{(i,m)})(q-t'_{(j,n)})}{M'_c}} \\ &= S_{ij}^{\mathcal{C}} e^{2\pi i(t_i q + t_j p - t_{a_c} p q)} = S_{i \otimes \bar{a}_c^{\otimes p}, j \otimes \bar{a}_c^{\otimes q}}, \end{aligned} \quad (9)$$

which means that we can identify  $(i, m, p)$  with  $i \otimes \bar{a}_c^{\otimes p}$  ( $\bar{a}_c$  denotes the anti-particle of  $a_c$ ). It is easy to check that they have the same spin  $s_{(i,m,p)} = s_{i \otimes \bar{a}_c^{\otimes p}}$ . Therefore,  $i \sim (i \otimes a_c^{\otimes p}, m, p)$ ,  $\forall m, p$ , we have come back to the original state  $\mathcal{C}$ . Therefore, generic hierarchy constructions are reversible, which defines an equivalence relation between topological orders. We call the corresponding equivalent classes the ‘‘non-Abelian families’’.

Now we examine the important quantity  $M_c = m_c - 2s_c$  which relates the ranks before and after the one-step condensation,  $|\mathcal{D}| = |M_c| |\mathcal{C}|$ . Since  $m_c$  is a freely chosen even integer, when  $a_c$  is not a boson or fermion ( $s_c \neq 0$  or  $1/2 \pmod{1}$ ), we can always make  $0 < |M_c| < 1$ , which means that the rank is reduced after one-step condensation. We then have the first important conclusion: *Each non-Abelian family have ‘‘root’’ topological orders with the smallest rank; the Abelian anyons in the ‘‘root’’ states are all bosons or fermions.*

We can further show that the Abelian bosons or fermions in the ‘‘root’’ states have trivial mutual statistics among them. To see this, assuming that  $a, b$  are

Abelian anyons in a root state. Since the mutual statistics is given by  $DS_{ab} = \exp[2\pi i(s_a + s_b - s_{a \otimes b})]$ , and  $a, b, a \otimes b$  are all bosons or fermions, non-trivial mutual statistics can only be  $DS_{ab} = -1$ . Now consider two cases: (1) one of  $a, b$ , say  $a$ , is a fermion, then by condensing  $a$  (choosing  $a_c = a, m_c = 2, s_c = 1/2, t_b = 1/2$ ), in the new topological order, the rank remains the same but  $s_{(b,0)} = s_b + \frac{t_b^2}{2M_c} = s_b + 1/8$ , which means  $(b, 0)$  is an Abelian anyon but neither a boson nor a fermion. By condensing  $(b, 0)$  again we can reduce the rank, which conflicts with the “root” state assumption. (2)  $a, b$  are all bosons. Still we condense  $a$  with  $m_c = 2, s_c = 0, t_b = 1/2$ . In the new topological order the rank is doubled but  $s_{(b,0)} = s_b + \frac{t_b^2}{2M_c} = 1/16$ , which means further condensing  $(b, 0)$  with  $m'_c = 0$  the rank is reduced to  $1/8$ , which is again, smaller than the rank of the beginning root state, thus contradictory.

Therefore, *in the root states, Abelian anyons are bosons or fermions with trivial mutual statistics*. We also have a straightforward corollary: *all Abelian topological orders have the same unique root state, which is the trivial topological order*. In other words, all Abelian topological orders are in the same trivial non-Abelian family, which resembles the noble gas family in the Periodic Table. Thus, the non-Abelian families are indeed equivalent classes up to Abelian topological orders.

To easily determine if two states belong to the same non-Abelian family, it is very helpful to introduce some *non-Abelian invariants*. One is the fractional part of the central charge. Since the one-step condensation changes the central charge by  $\text{sgn}(M_c)$  (see (8)), we know that central charges in the same non-Abelian family can only differ by integers. Another invariant is the quantum dimension. It is not hard to check that, in the one-step condensation, the number of anyons with the same quantum dimension is also multiplied by  $|M_c|$ . The third invariant is a bit involved. Note that in the one-step condensation, if  $i$  has trivial mutual statistics with  $a_c, t_i = 0$ , then  $(i, 0)$  in  $\mathcal{D}$  have the same spin as  $i$  in  $\mathcal{C}$  and the same mutual statistics with  $(j, m), \forall m$  as that between  $i$  and  $j$  in  $\mathcal{C}$ . Therefore, the centralizer of Abelian anyons, namely, the subset of anyons that have trivial mutual statistics with all Abelian anyons (the anyons in black in Table I), is the same within one non-Abelian family. These facts enable us to quickly tell that two states are not in the same non-Abelian family.

**Examples:** Realizations of non-Abelian FQH states were first proposed in Ref. 19 and 20. One of them is [19, 21]  $\Psi_{\nu=1}(\{z_i\}) = [\chi_2(\{z_i\})]^2$ , where  $\chi_k(\{z_i\})$  is the many-fermion wave function with  $k$  filled Landau levels. The bulk effective theory is the  $SU(2)_{-2}^f$  Chern-Simons (CS) theory with 3 types of anyons and the edge has  $c = 5/2$  (see the Supplemental Material [16]). So the state is one of the root state  $N_c = 3\frac{1}{2}$  in Table I. Another bosonic non-abelian FQH liquid at  $\nu = 1$  is [20]

$\Psi_{\nu=1} = \mathcal{A}(\frac{1}{z_1-z_2}, \frac{1}{z_2-z_3}, \dots) \prod (z_i - z_j)$ , whose edge has  $c = 3/2$ . It is the state described by  $N_c = 3\frac{1}{3}$  which belong to the same non-Abelian family as the  $3\frac{1}{5}$  state above. The experimentally realized  $\nu = 5/2$  FQH state is likely to belong to this non-Abelian family [22–24].

A more interesting non-Abelian state (which can perform universal topological quantum computation [25]) is  $\Psi_{\nu=\frac{3}{2}}(\{z_i\}) = [\chi_3(\{z_i\})]^2$ , whose edge has  $c = \frac{21}{5}$ . The bulk effective theory is the  $SU(2)_{-3}^f$  CS theory with 4 types of anyons [19, 21]. So the state is  $N_c = 4\frac{21}{5}$ , which belongs to the same non-Abelian family as the state  $2\frac{26}{5}$  in Table I (see the Supplemental Material [16], which contains more examples of non-Abelian states and non-Abelian families).

We like to remark that the topological orders studied in this paper do not require and do not have any symmetry. However, some  $c = 0$  topological orders with a  $Z_2$  automorphism  $i \rightarrow i'$  that changes the sign of spins  $s_i = -s_{i'}$  can be realized by time-reversal symmetric states [26].

**Conclusion and Outlook:** In this letter we introduced the hierarchy construction in generic topological orders, which established a new equivalence relation: Two topological orders related by the hierarchy construction belong to the same “non-Abelian family”. This reveals intriguing new structures in the classification of topological orders. Non-Abelian families are equivalent classes up to Abelian topological orders. Topological orders in the same non-Abelian family share some properties, such as quantum dimensions and the fractional part of central charges.

In particular we studied the “root” states, the states in a non-Abelian family with the smallest rank. Other states can be constructed from the root states via the hierarchy construction. Thus, classifying all topological orders is the same as classifying all root states, namely, all states such that their Abelian anyons have trivial mutual statistics. In other words, we can try to generate all possible topological orders by constructing all the root states, which can be obtained by *starting with an Abelian group  $G$ , extending its representation category  $\text{Rep}(G)$  or  $\text{sRep}(G^f)$  to a modular category [7, 8] while requiring all the extra anyons being non-Abelian (which is referred to as a non-Abelian modular extension)*. This is a promising future problem and may be an efficient way to produce tables of topological orders.

Although in this letter we focused on bosonic topological orders with no symmetry (described by modular categories), the construction also applies to bosonic/fermionic topological orders with any symmetry (described by certain pre-modular categories)[7, 8]. The same argument goes for non-Abelian families and root states with symmetries.

TL thanks Zhenghan Wang for helpful discussions. This research was supported by NSF Grant No. DMR-1506475 and NSFC 11274192.

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- [17] This is different from the so called “anyon condensation” categorical approach where only bosons condensing into the trivial state is considered.
- [18] Physically this amounts to condensing the fermionic quasiparticle into an integer quantum Hall state. If we instead condense the fermionic quasiparticle into  $p \pm ip$  states we are able to obtain the “sixteen fold way”. However, such condensation is beyond the construction of this work.
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