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# Entanglement Entropy of Eigenstates of Quadratic Fermionic Hamiltonians

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In a seminal paper [Phys. Rev. Lett. **71**, 1291 (1993)], Page proved that the entanglement entropy of typical pure states is  $S_{\text{typ}} \simeq \ln \mathcal{D}_A - (1/2)\mathcal{D}_A^2/\mathcal{D}$ , for  $1 \ll \mathcal{D}_A \leq \sqrt{\mathcal{D}}$ , where  $\mathcal{D}_A$  and  $\mathcal{D}$  are the Hilbert space dimensions of the subsystem and the system, respectively. Typical pure states are hence (nearly) maximally entangled. We develop tools to compute the average entanglement entropy  $\langle S \rangle$  of all eigenstates of quadratic fermionic Hamiltonians. In particular, we derive exact bounds for the most general translationally invariant models,  $\ln \mathcal{D}_A - (\ln \mathcal{D}_A)^2/\ln \mathcal{D} \leq \langle S \rangle \leq \ln \mathcal{D}_A - [1/(2\ln 2)](\ln \mathcal{D}_A)^2/\ln \mathcal{D}$ . Consequently we prove that: (i) if the subsystem size is a finite fraction of the system size then  $\langle S \rangle < \ln \mathcal{D}_A$  in the thermodynamic limit, i.e., the average over eigenstates of the Hamiltonian departs from the result for typical pure states, and (ii) in the limit in which the subsystem size is a vanishing fraction of the system size, the average entanglement entropy is maximal, i.e., typical eigenstates of such Hamiltonians exhibit eigenstate thermalization.

*Introduction.* The concept of entanglement is a cornerstone in modern quantum physics. Different measures of entanglement have been extensively used to probe the structure of pure quantum states [1], and they have started to be measured in experiments with ultracold atoms in optical lattices [2, 3]. Here, we are interested in the bipartite entanglement entropy (referred to as the entanglement entropy) in fermionic lattice systems. In such systems, an upper bound for the entanglement entropy of a subsystem A (smaller than its complement) is  $S_{\text{max}} = \ln \mathcal{D}_A$ , where  $\mathcal{D}$  and  $\mathcal{D}_A$  are the dimensions of the Hilbert space of the system and of the subsystem, with  $\mathcal{D}_A \leq \sqrt{\mathcal{D}}$  (see Fig. 1 for an example for spinless fermions). Note that  $\ln \mathcal{D}_A \propto V_A$ , where  $V_A$  is the number of sites in A, i.e., this upper bound scales with the “volume” of A. (When A is larger than its complement, the Hilbert space of the complement is the one that determines  $S$ .) Almost twenty-four years ago, motivated by the puzzle of information in black hole radiation [4], Page proved [5] that typical (with respect to the Haar measure) pure states nearly saturate that bound (the correction is exponentially small) [6–10]. Their reduced density matrices are thermal at infinite temperature [11–13].

In stark contrast with typical pure states, ground states and low-lying excited states of local Hamiltonians are known to exhibit an *area-law* entanglement [1]. Namely, their entanglement entropy scales with the area of the boundary of the subsystem. On the other hand, most eigenstates of local Hamiltonians at nonzero energy densities above the ground state are expected to have a volume-law entanglement entropy (with the exception of many-body localized systems [14, 15]). Within the eigenstate thermalization hypothesis (ETH) [16–18], one expects volume-law entanglement in all eigenstates (excluding those at the edges of the spectrum) of quantum chaotic Hamiltonians [19–23], with those in the center of the spectrum exhibiting maximal entanglement [23].

Thanks to the availability of powerful analytical and computational tools to study ground states, many remarkable results have been obtained for the entanglement entropy of such states [24–29]. On the other hand, for ex-

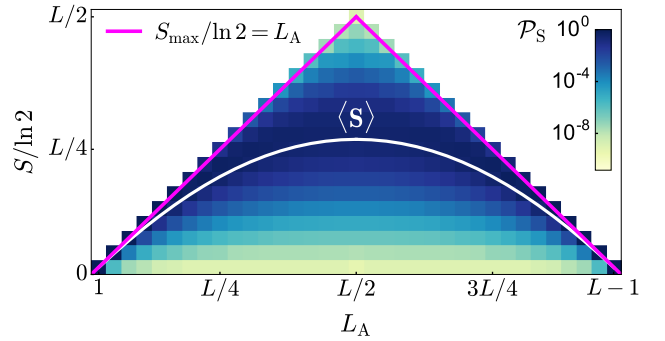


FIG. 1. *Entanglement entropy of eigenstates of noninteracting spinless fermions in a periodic chain with  $L = 36$  sites.* Results are plotted as a function of the linear subsystem size  $L_A$ . (Lower line) Average entanglement entropy  $\langle S \rangle$  of all eigenstates, and (upper line) upper bound  $S_{\text{max}} = \ln \mathcal{D}_A = L_A \ln 2$  for  $L_A \leq L/2$  [ $S_{\text{max}} = (L - L_A) \ln 2$  for  $L_A > L/2$ ]. Each pixel denotes the weight of eigenstates  $|m\rangle$  with target entropy  $S$ , defined as  $\mathcal{P}_S = \mathcal{D}^{-1} \sum_{0 \leq (S - S_m) < \ln 2}$ .

cited states there is a wide gap between what is expected and what has been shown. For interacting Hamiltonians, computational studies are severely limited by finite-size effects so it is difficult to know what happens to the entanglement entropy with increasing the subsystem size. This question was recently addressed for quadratic [30–32] and non-quadratic but integrable [33] Hamiltonians, for which one can study much larger lattices, revealing that randomly generated eigenstates are generally maximally entangled in the limit in which the size of the subsystem is a vanishing fraction of the size of the system (in short, a vanishing subsystem fraction).

In this Letter we prove that, for a nonvanishing subsystem fraction, the average entanglement entropy of all many-body fermionic eigenstates of translationally invariant quadratic Hamiltonians departs from Page’s result. Still, the scaling of that average is volume law. In the limit of vanishing subsystem fraction, we prove that

the overwhelming majority of eigenstates are maximally entangled. Our proof stands on calculations of spectrum averages of eigenstate entanglement entropies, which are based on the insight that such averages can be obtained as traces over even powers of a matrix, without the need of calculating its eigenvalues.

*Entanglement entropy of energy eigenstates.* We study the most general quadratic Hamiltonian of spinless fermions:  $\hat{H} = -\sum_{i,j=1}^V (\Delta_{ij} \hat{f}_i^\dagger \hat{f}_j^\dagger + \Delta_{ij}^* \hat{f}_j \hat{f}_i + t_{ij} \hat{f}_i^\dagger \hat{f}_j)$ , where  $\Delta_{ij} = -\Delta_{ji}$  and  $t_{ij} = t_{ji}^*$ , and  $\hat{f}_i$  is the fermionic annihilation operator at site  $i$ . A Bogoliubov transformation  $\hat{f}_i = \sum_{l=1}^V (\alpha_{il} \hat{c}_l + \beta_{il} \hat{c}_l^\dagger)$  rotates the Hamiltonian so that it commutes with the quasiparticle number operator  $\hat{N}_l = 2\hat{c}_l^\dagger \hat{c}_l - 1$ . Hence, the many-body energy eigenkets  $|m\rangle$  satisfy  $\hat{N}_l|m\rangle = N_l|m\rangle$  with  $N_l = \pm 1$ , and we adopt the binary representation  $m = 1 + \sum_{l=1}^V \frac{1+N_l}{2} 2^{l-1}$  ( $m$  runs from 1 to  $\mathcal{D} = 2^V$ ,  $V$  is the number of lattice sites).

Correlations of a state  $|m\rangle$  are encoded in  $V \times V$  one-body correlation matrices. They form a  $2V \times 2V$  matrix  $J$ , which is a linear complex structure [34]

$$iJ = \begin{pmatrix} \langle m | \hat{f}_i^\dagger \hat{f}_j - \hat{f}_j \hat{f}_i^\dagger | m \rangle & \langle m | \hat{f}_i^\dagger \hat{f}_j^\dagger - \hat{f}_j^\dagger \hat{f}_i | m \rangle \\ \langle m | \hat{f}_i \hat{f}_j - \hat{f}_j \hat{f}_i | m \rangle & \langle m | \hat{f}_i \hat{f}_j^\dagger - \hat{f}_j^\dagger \hat{f}_i | m \rangle \end{pmatrix}. \quad (1)$$

Since the many-body eigenstates  $\{|m\rangle\}$  are Gaussian states, the matrix  $iJ$  fully characterizes them [35–38]. Correlations of a subsystem  $A$  containing  $V_A$  sites are encoded in the restricted complex structure  $[iJ]_A$ , the  $2V_A \times 2V_A$  matrix obtained by restricting the matrix  $iJ$  to the entries with  $i, j \in A$ . The entanglement entropy of subsystem  $A$  in the eigenstate  $|m\rangle$  can be computed as [35, 36]

$$S_m = -\text{Tr} \left\{ \left( \frac{\mathbb{1} + [iJ]_A}{2} \right) \ln \left( \frac{\mathbb{1} + [iJ]_A}{2} \right) \right\}. \quad (2)$$

Expanding Eq. (2) in powers of  $[iJ]_A$ , about  $[iJ]_A = 0$ , allows one to compute the entanglement entropy without calculating the eigenvalues of  $[iJ]_A$

$$S_m = L_A \ln 2 - \sum_{n=1}^{\infty} \frac{\text{Tr}[iJ]_A^{2n}}{4n(2n-1)}, \quad (3)$$

where we use the compressed notation  $\text{Tr}[iJ]_A^{2n} \equiv \text{Tr}\{[iJ]_A^{2n}\}$ . Since the restricted complex structure satisfies  $[iJ]_A^2 \leq \mathbb{1}$  [34], one has

$$0 \leq \text{Tr}[iJ]_A^{2(m+1)} \leq \text{Tr}[iJ]_A^{2m} \leq 2V_A \quad (4)$$

and the series in Eq. (3) is convergent.

Equation (3) allows one to compute the average over the ensemble of all eigenstates  $\{|m\rangle\}$  as

$$\langle S \rangle = V_A \ln 2 - \sum_{n=1}^{\infty} \frac{\langle \text{Tr}[iJ]_A^{2n} \rangle}{4n(2n-1)}, \quad (5)$$

where we define  $\langle O \rangle \equiv \mathcal{D}^{-1} \sum_{m=1}^{\mathcal{D}} O_m$ .

A remarkable property of the series in Eq. (5) is that every higher-order term lowers the average entanglement entropy. Hence, any truncation gives an upper bound. Using the inequality in Eq. (4), one can also produce lower bounds for the average entanglement entropy. To obtain the first-order lower and upper bounds,  $S_1^- \leq \langle S \rangle \leq S_1^+$ , we only need to compute  $\langle \text{Tr}[iJ]_A^2 \rangle$  since: (i) truncating the series in Eq. (5) after the first term results in  $S_1^+$ , and (ii) substituting all averages of higher-order traces by  $\langle \text{Tr}[iJ]_A^2 \rangle$  results in  $S_1^-$ . This gives

$$V_A \ln 2 - \frac{\langle \text{Tr}[iJ]_A^2 \rangle}{2} \ln 2 \leq \langle S \rangle \leq V_A \ln 2 - \frac{\langle \text{Tr}[iJ]_A^2 \rangle}{4}. \quad (6)$$

For a given eigenstate  $|m\rangle$  of our Hamiltonian,  $[iJ]_A$  is linear in the quantum numbers  $N_l$ . In fact,  $\langle m | \hat{f}_i^\dagger \hat{f}_j - \hat{f}_j \hat{f}_i^\dagger | m \rangle = \sum_l N_l (\alpha_{il}^* \alpha_{jl} - \beta_{il}^* \beta_{jl})$  and  $\langle m | \hat{f}_i^\dagger \hat{f}_j^\dagger - \hat{f}_j^\dagger \hat{f}_i | m \rangle = \sum_l N_l (\alpha_{il}^* \beta_{jl}^* - \beta_{il}^* \alpha_{jl}^*)$ . The average  $\langle \text{Tr}[iJ]_A^{2n} \rangle$  can therefore be computed from the binomial correlation function  $\langle N_{l_1} \cdots N_{l_{2n}} \rangle$ . In particular, to compute  $\langle \text{Tr}[iJ]_A^2 \rangle$ , we use that  $\langle N_l N_{l'} \rangle = \delta_{ll'}$  to get

$$\langle \text{Tr}[iJ]_A^2 \rangle = 2 \sum_l \sum_{i,j \in A} (|\alpha_{il}^* \alpha_{jl} - \beta_{il}^* \beta_{jl}|^2 + |\alpha_{il} \beta_{jl} - \beta_{il} \alpha_{jl}|^2). \quad (7)$$

Whenever  $\langle \text{Tr}[iJ]_A^2 \rangle / V_A$  does not vanish in the thermodynamic limit,  $\langle S \rangle / V_A < \ln 2$ .

*Bounds for translationally invariant Hamiltonians.* The Bogoliubov coefficients for a translationally invariant system in  $n$  dimensions are:  $\alpha_{il} = e^{i\vec{k}_l \cdot \vec{x}_i} u_{\vec{k}_l} / \sqrt{V}$  and  $\beta_{il} = e^{-i\vec{k}_l \cdot \vec{x}_i} v_{\vec{k}_l} / \sqrt{V}$ , with  $u_{\vec{k}} = u_{-\vec{k}}$ ,  $v_{\vec{k}} = -v_{-\vec{k}}$ , and  $|u_{\vec{k}}|^2 + |v_{\vec{k}}|^2 = 1$ . Substituting these in Eq. (7) leads to

$$\begin{aligned} \langle \text{Tr}[iJ]_A^2 \rangle &= \frac{2V_A^2}{V} - \sum_l \sum_{i,j \in A} \frac{8|v_{\vec{k}_l}|^2 |u_{\vec{k}_l}|^2 \cos 2\vec{k}_l(\vec{x}_i - \vec{x}_j)}{V^2} \\ &= \frac{2V_A^2}{V} - \sum_l \frac{8|v_{\vec{k}_l}|^2 |u_{\vec{k}_l}|^2}{V^2} \prod_\eta \frac{\sin^2(L_\eta[k_l]_\eta)}{\sin^2([k_l]_\eta)}, \end{aligned} \quad (8)$$

where the space sum runs within an  $n$ -dimensional hypercube  $A$  with side lengths  $L_\eta$  containing  $V_A = \prod_{\eta=1}^n L_\eta$  sites. One can bound  $\langle \text{Tr}[iJ]_A^2 \rangle$  from below using  $|u_{\vec{k}_l}|^2 |v_{\vec{k}_l}|^2 \leq 1/4$ . As  $V \rightarrow \infty$ , one can substitute  $\sum_l \rightarrow \frac{V}{(2\pi)^n} \int_{-\pi}^{\pi} d^n k$ . Since  $\int_{-\pi}^{\pi} dk \sin^2(L_\eta k) / \sin^2(k) = 2\pi L_\eta$  then  $\langle \text{Tr}[iJ]_A^2 \rangle \geq 2V_A^2/V - 2V_A/V$ . In the thermodynamic limit, we get that  $\langle \text{Tr}[iJ]_A^2 \rangle = 2V_A^2/V$ . The corresponding *universal* first-order bounds are:

$$\begin{aligned} S_1^+ &= V_A \ln 2 - \frac{1}{2} \frac{V_A^2}{V} = \ln \mathcal{D}_A - \frac{1}{2 \ln 2} \frac{(\ln \mathcal{D}_A)^2}{\ln \mathcal{D}} \\ S_1^- &= V_A \ln 2 \left( 1 - \frac{V_A}{V} \right) = \ln \mathcal{D}_A - \frac{(\ln \mathcal{D}_A)^2}{\ln \mathcal{D}}. \end{aligned} \quad (9)$$

Note that: (i)  $S_1^+$  and  $S_1^-$  fulfill a volume law as they are proportional to  $V_A$ , and (ii) for any nonvanishing subsystem fraction,  $\lim_{V \rightarrow \infty} V_A/V \neq 0$ ,  $S_1^+ < V_A \ln 2$  in

the thermodynamic limit, i.e., the average departs from the result for typical states in the Hilbert space. If the subsystem fraction vanishes in the thermodynamic limit,  $\lim_{V \rightarrow \infty} V_A/V = 0$ , the lower and the upper bounds coincide and  $\lim_{V_A/V \rightarrow 0} S_1^- = \lim_{V_A/V \rightarrow 0} S_1^+ = V_A \ln 2$ . In this limit, the average entanglement entropy is maximal, i.e., typical eigenstates of the Hamiltonian have a typical (*à la* Page [5]) entanglement entropy.

*Entanglement entropy bounds for free fermions.* We now apply our construction to free fermions on a translationally invariant chain with  $L$  lattice sites, described by the Hamiltonian  $\hat{H} = -\sum_{i,j=1}^L t_{j-i} \hat{f}_i^\dagger \hat{f}_j$ . In this case, the Bogoliubov coefficients are  $u_k = 1$  and  $v_k = 0$ , so that the eigenstates are plane waves. This allows us to obtain closed form expressions for finite systems. We denote the linear subsystem size as  $L_A$ . Figure 1 shows the distribution of  $S_m$  for all eigenstates  $|m\rangle$  in a lattice with  $L = 36$  sites, as well as the corresponding average  $\langle S \rangle$ . It is remarkable that when  $L_A$  departs from 1, the entanglement entropy of the eigenstates with the most weight departs from  $S_{\max} = L_A \ln 2$  [30, 32].

Using  $\langle N_k N_{k'} \rangle = \delta_{kk'}$ , we can explicitly compute

$$\langle \text{Tr} [iJ]_A^2 \rangle = \frac{2}{L^2} \sum_{i,j=1}^{L_A} \sum_{k,k'=1}^L \langle N_k N_{k'} \rangle e^{i \frac{2\pi}{L} (k-k')(i-j)} = \frac{2L_A^2}{L} \quad (10)$$

for finite systems. The first-order bounds for  $\langle S \rangle$ , for finite systems, are then given by Eq. (9) and (11) upon replacing  $V \rightarrow L$  and  $V_A \rightarrow L_A$ .

It is straightforward to calculate bounds beyond the first order. A general procedure to compute averages of traces of  $[iJ]_A^{2n}$  is presented in Ref. [34]. The main insight from our analysis is that the term  $\langle \text{Tr} [iJ]_A^{2n} \rangle / L_A$  is a polynomial that, when  $L \rightarrow \infty$ , only contains powers from  $(L_A/L)^n$  to  $(L_A/L)^{2n-1}$ . For the second order upper ( $S_2^+$ ) and lower ( $S_2^-$ ) bounds [34], one gets

$$S_2^+ = L_A \ln 2 - \frac{1}{2} \frac{L_A^2}{L} - \frac{2}{9} \frac{L_A^3}{L^2} + \frac{1}{6} \frac{L_A^4}{L^3}, \quad (11)$$

$$S_2^- = L_A \ln 2 - \frac{1}{2} \frac{L_A^2}{L} - (2 \ln 2 - 1) \left( \frac{4}{3} \frac{L_A^3}{L^2} - \frac{L_A^4}{L^3} \right).$$

In order to obtain Eq. (11), we neglected finite-size corrections of order  $\mathcal{O}(1/L)$  and higher.

In Fig. 2(a), we compare the first- and second-order bounds with the average  $\langle S \rangle$  computed on a lattice with  $L = 36$  sites. The bounds can be seen to be very close to the numerically computed average. At  $L_A/L = 1/2$ , where the relative deviation is largest, we get that  $0.52 < \langle S \rangle / [(L/2) \ln 2] < 0.59$ . In Fig. S1 of Ref. [34], we extrapolate numerical results for  $\langle S \rangle$  to the limit  $L \rightarrow \infty$ . Finite-size effects are found to be exponentially small in  $L$  [39], and we obtain  $\lim_{L \rightarrow \infty} \langle S \rangle / [(L/2) \ln 2] = 0.5378(1)$ .

*Entanglement entropy variance for free fermions.* In order to understand whether the average of the entanglement entropy over all eigenstates is representative of

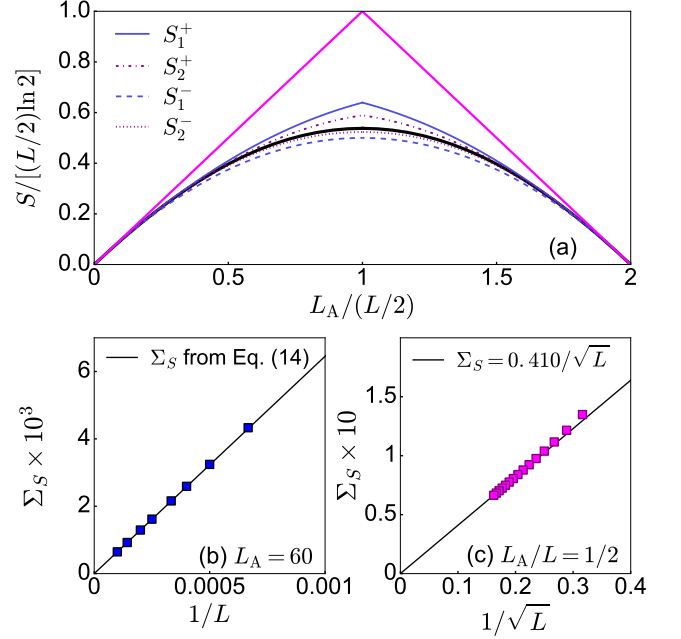


FIG. 2. Entanglement entropy mean, bounds, and variance for free fermions on a periodic chain. (a) Upper bounds ( $S_1^+$ ,  $S_2^+$ ) and lower bounds ( $S_1^-$ ,  $S_2^-$ ), given by Eqs. (9) and (11), for the average entanglement entropy. The upper (magenta) line is the maximal entanglement entropy  $S_{\max}$  and the thick black line is the average entanglement entropy  $\langle S \rangle$  on a lattice with  $L = 36$  sites (same results as in Fig. 1). (b)  $\Sigma_S$  for  $L_A = 60$  in ensembles of  $10^6$  randomly sampled eigenstates and, solid line, the prediction from Eq. (14). (c)  $\Sigma_S$  for  $L_A/L = 1/2$  calculated using all eigenstates in lattices with  $L \leq 38$ . The solid line is a single-parameter fit to  $\Sigma_S = a/\sqrt{L}$  for  $L \geq 30$ , with  $a = 0.410$ .

the entanglement entropy of typical eigenstates, we calculate the variance

$$\Sigma_S^2 = \frac{\langle S^2 \rangle - \langle S \rangle^2}{(L_A \ln 2)^2} = \frac{1}{(L_A \ln 2)^2} \sum_{m,n=1}^{\infty} F_{m,n}, \quad (12)$$

where

$$F_{m,n} = \frac{\langle \text{Tr} [iJ]_A^{2m} \text{Tr} [iJ]_A^{2n} \rangle - \langle \text{Tr} [iJ]_A^{2m} \rangle \langle \text{Tr} [iJ]_A^{2n} \rangle}{4m(2m-1)4n(2n-1)}. \quad (13)$$

The computation of  $F_{m,n}$  is, in general, a daunting task. However, by using a summation technique to compute higher-order traces [34], we are able to extract key properties of  $\Sigma_S$ . In particular, we are able to prove that  $\Sigma_S$  vanishes with increasing the system size as  $\Sigma_S \sim 1/\sqrt{L}$  or faster [34]. Furthermore, in the limit of vanishing subsystem fraction (fixed  $L_A$  for  $L \rightarrow \infty$ ), we obtain the lowest order term in  $L$  to be

$$\Sigma_S^2 = \frac{1}{L^2} \frac{1}{(\ln 2)^2} \left( \frac{L_A}{3} + \frac{1}{6L_A} \right). \quad (14)$$

Numerical results for  $\Sigma_S$  in this limit, reported in Fig. 2(b), confirm the accuracy of this prediction. Nu-

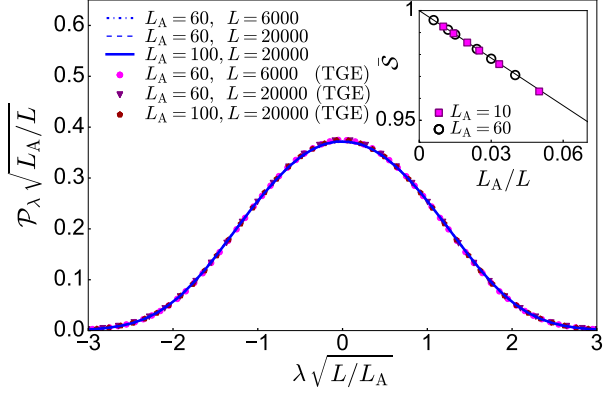


FIG. 3. *Distribution of eigenvalues of the restricted complex structure for vanishing subsystem fraction.* The overlapping solid lines depict  $\mathcal{P}_\lambda$ , which are averages of the discrete distribution  $p_\lambda = \sum_{|\lambda_j - \lambda| < \delta\lambda/2}$  over  $10^6$  random eigenstates, where  $\lambda_j$  are eigenvalues of  $[iJ]_A$  and we take  $\delta\lambda = 10^{-2}$ . The symbols depict  $\mathcal{P}_\lambda^{(\text{TGE})}$ , which are averages of the discrete distribution  $p_\lambda^{(\text{TGE})} = \sum_{|\lambda_j - \lambda| < \delta\lambda/2}$  over  $10^6$  realizations of the Toeplitz Gaussian Ensemble (TGE), where  $\lambda_j$  are the eigenvalues of the TGE and we take  $\delta\lambda = 10^{-2}$ . Both axes are renormalized to show data collapse. (Inset) Scaling of  $\bar{S}$  [see Eq. (16)]. The symbols show numerical results of an average over  $10^6$  random eigenstates. The solid line shows the results of Eq. (16).

merical results for  $L_A/L = 1/2$ , reported in Fig. 2(c), confirm that  $\Sigma_S \sim 1/\sqrt{L}$  for a nonvanishing subsystem fraction. The vanishing of the variance proves that the average and typical entanglement entropies are identical.

*Eigenvalue distribution for free fermions.* Our results for the average entanglement entropy allow us to unveil some remarkable properties of the eigenvalues  $\lambda_j$  of  $[iJ]_A$  in energy eigenstates. Eigenvalues  $\lambda_i$  satisfy  $|\lambda_j| \leq 1$  [34]. It is also straightforward to prove that the average of the sum of eigenvalues vanishes:  $\langle \sum_j \lambda_j \rangle = \langle \text{Tr}[iJ]_A \rangle = L_A(1 - 2\langle N \rangle/L) = 0$ , where  $\langle N \rangle = L/2$  is the average number of particles. On the other hand, the average of the variance of the eigenvalues of  $[iJ]_A$  can be calculated using Eq. (10), yielding

$$\sigma^2 = \frac{1}{2L_A} \left\langle \sum_j \lambda_j^2 \right\rangle = \frac{1}{2L_A} \langle \text{Tr}[iJ]_A^2 \rangle = \frac{L_A}{L}. \quad (15)$$

This allows us to conclude that  $\sigma^2$  vanishes if  $\lim_{L \rightarrow \infty} L_A/L = 0$  (implying  $\langle S \rangle$  is maximal), and cannot vanish if  $\lim_{L \rightarrow \infty} L_A/L \neq 0$ . In Fig. 3, we report results of numerical calculations of the distribution of eigenvalues of  $[iJ]_A$  (for small values of  $L_A/L$ ) in a large ensemble of randomly chosen eigenstates. This distribution can be seen to have a universal form that only depends on the ratio  $L_A/L$ , and whose width is  $\sqrt{L_A/L}$ .

The variance of the distribution of eigenvalues of  $[iJ]_A$  ( $\sigma^2 = L_A/L$ ) is important as it determines how the av-

erage entanglement entropy reaches the maximal value in the thermodynamic as  $L_A/L \rightarrow 0$ . The lowest order correction to  $\langle S \rangle = L_A \ln 2$  in terms of  $L_A/L$  can be read from Eqs. (11), in which the upper and lower bounds coincide up to  $\mathcal{O}[(L_A/L)^2]$ ,

$$\bar{S} \equiv \frac{\langle S \rangle}{L_A \ln 2} = 1 - \frac{1}{2 \ln 2} \frac{L_A}{L} + \mathcal{O}\left[\left(\frac{L_A}{L}\right)^2\right]. \quad (16)$$

A comparison of the latter expression to numerical results, reported in the inset in Fig. 3, yields an almost perfect agreement for  $L_A/L \lesssim 0.05$ .

For a vanishingly small subsystem fraction, the fact that: (i) the average entanglement entropy is maximal, and (ii) the distribution of eigenvalues of  $[iJ]_A$  is universal (see Fig. 3), hints that a random ensemble may explain those results. We construct such an ensemble, the Toeplitz Gaussian Ensemble (TGE). In the TGE, the entries of  $[iJ]_A$  are replaced by random complex numbers whose absolute value is that of a normally distributed variable with zero mean and variance  $1/L$ , and whose phase is uniformly distributed between 0 and  $2\pi$ . As shown in Fig. 3, the corresponding eigenvalue distribution is nearly indistinguishable of the numerical calculation over  $10^6$  random eigenstates. (See also Fig. S2 of Ref. [34], which shows that taking the limit  $L \rightarrow \infty$  first, followed by  $L_A \rightarrow \infty$ , results in two distributions that are identical.) This shows that, in the limit of vanishing subsystem fraction, no specific information beyond the symmetries of the model appears to be encoded in the restricted complex structure  $[iJ]_A$  of typical eigenstates.

*Discussion.* Our work introduces a novel methodology that enables the rigorous study of the entanglement entropy of excited eigenstates of quadratic Hamiltonians. The derivation of exact bounds for the average entanglement entropy of translationally invariant quadratic Hamiltonians reveals a fundamental difference between the results for vanishing and nonvanishing subsystem fractions, which is not captured by the analysis of typical states in the Hilbert space [5]. This highlights the difference in information content between typical eigenstates of physical Hamiltonians, such as those considered here, and typical states in the Hilbert space. The fact that, for vanishing subsystem fraction, typical eigenstates are maximally entangled constitutes a proof that typical eigenstates satisfy ETH for local observables.

We note that Eqs. (1)–(7) also apply to quadratic models of much current interest such as those appearing in disordered [40] and periodically driven (Floquet) [41] systems. While our study focuses on the von Neumann entanglement entropy, the upper bounds derived remain valid for higher-order Renyi entropies, which are bounded from above by the von Neumann entanglement entropy. This is of particular relevance for current experiments with ultracold atoms on optical lattices [2, 3], which can now measure the second Renyi entropy.

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