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Collective modes of a soliton train in a Fermi superfluid

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We characterize the collective modes of a soliton train in a quasi-one-dimensional Fermi superfluid, using a mean-field formalism. In addition to the expected Goldstone and Higgs modes, we find novel long-lived gapped modes associated with oscillations of the soliton cores. The soliton train has an instability that depends strongly on the interaction strength and the spacing of solitons. It can be stabilized by filling each soliton with an unpaired fermion, thus forming a commensurate Fulde-Ferrell-Larkin-Ovchinnikov (FFLO) phase. We find such a state is always dynamically stable, which paves the way for realizing long-lived FFLO states in experiments via phase imprinting.

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A unifying theme of contemporary physics is understanding emergent dynamics of many-particle systems. One motif is the appearance of persistent nonlinear structures such as solitons [1]. Solitons arise naturally in diverse physical systems, including water waves [2], plasmas [3], optical fibers [4], conducting polymers [5–7], superconductors [8–11], Bose-Einstein condensates [12–18], DNA dynamics [19], quantum field theory [20, 21], and early-universe cosmology [22]. They are technologically important, with applications in telecommunications [23], information processing [11, 24], and matter-wave interferometry [25]. Moreover, cold-atom experiments can now engineer matter-wave solitons in atomic superfluids, and directly observe their motion [13–18, 26, 27]. Understanding the behavior of these collective objects is vital to the larger problem of forming a cohesive theory of nonequilibrium dynamics [28]. In particular, the next generation of Fermi gas experiments will be creating clouds with many of these nonlinear defects [27]. While past theoretical studies have shed light on the behavior of individual [29–35] or pairs of solitons [36–38], the behavior and even stability of soliton trains are not understood. Here we study the linearized dynamics of a soliton train in a one-dimensional (1D) Fermi gas, finding a rich set of collective modes. We characterize these modes, finding distinct differences from Bose superfluids, which may generalize to nonlinear excitations of other systems.

We consider a two-component Fermi gas in an elongated trap with tight radial confinement so that the dynamics is effectively 1D [39]. The strong radial confinement suppresses the snake instability by which solitons decay into vortices and sound waves in three dimensions [13, 14, 27, 32–34, 40]. To avoid the idiosyncracies of strictly 1D systems, we envision a weakly-coupled array of such tubes which have long-range superfluid order. Past experiments have studied fermionic superfluids in such geometries [41]. Using phase-imprinting techniques [14, 26, 27, 42], one can generate a train of solitons in the superfluid. The collective modes of the soliton train would show up as pronounced peaks in spectroscopic measurements of the pairing susceptibility, or in the density response of the system [43–47]. Here we extract the collective modes by linearizing the self-consistent Bogoliubov-de Gennes (BdG) equations governing the fermion fields.

The soliton train has two gapless Goldstone modes which originate from the spontaneous breaking of gauge- and translational symmetry: a 'phonon' mode describing phase twists and an 'elastic' mode describing oscillations in the spacing between the domain walls. The elastic mode is only well-defined for wave-vectors smaller than the inverse separation of the solitons, but we find a second gapped branch of oscillations which persists to large wave-vectors [Fig. 2(a)]. This branch is the remnant of the 'Higgs' mode in a uniform superfluid [48, 49].

In addition, we find a twofold degenerate gapped mode which, at small wave-vectors, describes oscillations in the width and grayness of each soliton [Fig. 2(d)-(e)]. To our knowledge, this 'core' mode hasn't appeared before in the literature. It lies outside the particle-hole continua, and should therefore be long-lived, hence easier to detect in experiments than those embedded in a continuum.

However, we also find the soliton train has two kinds of instabilities toward a uniform superfluid state: in one, pairs of neighboring solitons approach and annihilate each other [Fig. 2(f)], whereas in the other, the order parameter moves off into the complex plane [Fig. 2(g)]. Both instabilities grow at the same rate, which depends on the degree of overlap between adjacent solitons. This overlap can be reduced by creating solitons farther apart, or by increasing the attractive interaction strength to produce sharper solitons [Fig. 3(a)-(b)]. Using either approach one can make the instability rate much smaller compared to the frequency of the 'core' modes, thus allowing them to be resolved.

One can also stabilize the train by filling each soliton with unpaired fermions, i.e., by polarizing the Fermi gas. Such a state constitutes a realization of the long-sought-after Fulde-Ferrell-Larkin-Ovchinnikov (FFLO) phase [50–52], whose experimental evidence in solid state [53] and cold gas [41] systems has so far been indirect. We find the instability rate falls with increasing polarization, vanishing for the 'commensurate FFLO' (C-FFLO) phase with one excess fermion per soliton [Fig. 3(c)]. Thus, a C-FFLO phase is always dynamically stable, even when energetics favor a different state (Fig. 4). This means one can directly engineer stable FFLO states by phase imprinting, as opposed to searching for the one that minimizes the free energy. This enlarged parameter space will facilitate more direct probes of the exotic state. In the Supplemental Material [54] we propose a robust experimental protocol for creating such states, which uses a radio-frequency sweep to selectively transfer atoms in one spin state to a third, noninteracting spin state.

Our results are based on a mean-field BdG formalism. Such a mean-field treatment gives a reasonably accurate description of quasi-1D Fermi gases for moderate to weak interactions, becoming quantitative in the weak-coupling limit [44, 45, 52, 55]. Further, past theoretical work has shown that the 1D BdG equations accurately describe the equilibrium properties of an array of tubes [8, 44].

We start with the many-body Hamiltonian

$$\hat{H} = \int dx \Big[\sum\nolimits_{\sigma=\uparrow,\downarrow} \hat{\Psi}^\dagger_\sigma \hat{H}^{(0)}_\sigma \hat{\Psi}_\sigma + g_{\rm 1D} \hat{\Psi}^\dagger_\uparrow \hat{\Psi}^\dagger_\downarrow \hat{\Psi}_\downarrow \hat{\Psi}_\uparrow \Big], \quad (1)$$

where $\hat{\Psi}_\sigma \equiv \hat{\Psi}_\sigma(x,t)$ denote the fermion field operators in the Heisenberg picture, and $g_{\rm 1D}$ is the 1D coupling constant whose relationship to the 3D scattering length is well-studied [51, 56]. The single-particle Hamiltonian is $\hat{H}^{(0)}_{\uparrow,\downarrow} = -\partial_x^2/2 - \epsilon_{\rm F} \pm h$, where $\epsilon_{\rm F}$ is the Fermi energy, and h is an effective magnetic field which controls the polarization. We have set $\hbar = m = 1$, where m is the mass of each fermion. Attractive interactions ($g_{\rm 1D} < 0$) lead to Cooper pairing, which we encode in the superfluid order parameter $\Delta(x,t) = g_{\rm 1D}\langle \hat{\Psi}_\downarrow(x,t)\hat{\Psi}_\uparrow(x,t)\rangle$. Ignoring quadratic fluctuations about Δ yields mean-field equations of motion for $\hat{\Psi} \equiv (\hat{\Psi}_\uparrow \;\; \hat{\Psi}^\dagger_\downarrow)^T$. The many-body state is formed by occupying fermionic quasiparticle modes $\hat{\gamma}^s_j$, defined by $\hat{\Psi} = \sum_{s,j} e^{isk_{\rm F}x}(U^s_j(x,t) \;\; V^s_j(x,t))^T \hat{\gamma}^s_j$, where $k_{\rm F}$ is the Fermi momentum, (U,V) are coherence factors, and $s = \pm$ breaks modes into right-moving and left-moving. For weak interactions, only the modes near the Fermi points contribute significantly to pairing. Thus we write $(-\partial_x^2/2 - \epsilon_{\rm F})[e^{\pm ik_{\rm F}x}(U^\pm_j, V^\pm_j)] \approx e^{\pm ik_{\rm F}x}[\mp ik_{\rm F}\partial_x(U^\pm_j, V^\pm_j)]$ (the Andreev approximation [57]), obtaining [54]

$$i\partial_t \begin{pmatrix} U^\pm_j \\ V^\pm_j \end{pmatrix} = \begin{pmatrix} \mp ik_{\rm F}\partial_x + h & \Delta(x,t) \\ \Delta^*(x,t) & \pm ik_{\rm F}\partial_x + h \end{pmatrix} \begin{pmatrix} U^\pm_j \\ V^\pm_j \end{pmatrix}, \quad (2)$$

where $$\Delta(x,t) = g_{\rm 1D} \sum\nolimits_{s,j} \langle \hat{\gamma}^{s\dagger}_j \hat{\gamma}^s_j \rangle U^s_j V^{s*}_j. \quad (3)$$

For a real stationary solution, $\Delta(x,t) = \Delta_0(x)$, the coherence factors are of the form $(U^+_j, V^+_j) = (u_j(x), v_j(x))\, e^{-i(\epsilon_j+h)t}$ and $(U^-_j, V^-_j) = (u^*_j(x), v^*_j(x))\, e^{-i(\epsilon_j+h)t}$, where ϵ_j represents the quasiparticle spectrum.

Prior studies have found [7–9] stationary soliton train solutions of the form $\Delta_0(x) = \Delta_1 k_1 \,\mathrm{sn}(\Delta_1 x/k_{\rm F}, k_1)$ with $\Delta_1 \equiv 2k_{\rm F}k_0K(k_1)/\pi$, where $2\pi/k_0$ is the period of the train, sn is a Jacobi elliptic function [58], K is the complete elliptic integral of the first kind, and $k_1 \in (0,1)$ is a parameter controlling the sharpness of solitons, which is set by imposing self-consistency [Eq. (3)]. The quasiparticle spectrum has a continuum of free states for $|\epsilon| > \epsilon_+$, and a band of midgap states for $|\epsilon| < \epsilon_-$, where $\epsilon_\pm = \Delta_1(1 \pm k_1)/2$ (Fig. 1). The midgap band describes Andreev bound states localized at the soliton cores.

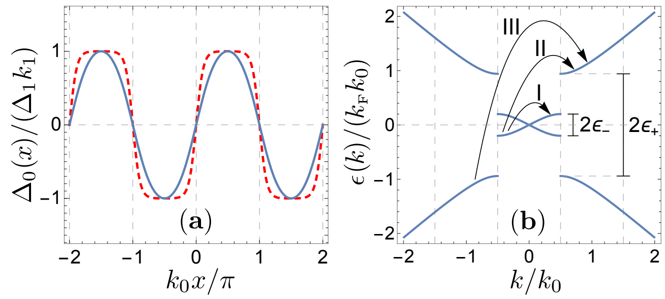

FIG. 1. (Color online.) (**a**) Stationary soliton train profile of the order parameter with wave-vector k_0 for different values of the sharpness parameter k_1. Solid: $k_1 = 0.65$, dashed: $k_1 = 0.999$. The sharpness is set by the soliton spacing, interaction strength, and spin imbalance. (**b**) BdG single-particle spectrum of the soliton train in the extended zone, for $k_1 = 0.65$. The arrows show three types of particle-hole excitations, which give rise to disconnected continua in the collective excitation spectrum (gray regions in Fig. 2(a)).

To find the collective modes, we linearize small fluctuations about the stationary solution. Thus we write $\Delta = \Delta_0(x) + \delta\Delta(x,t)$, $U^+_j = (u_j(x) + \delta u^+_j(x,t))\, e^{-i(\epsilon_j+h)t}$, $U^-_j = (u^*_j(x) + \delta u^-_j(x,t))\, e^{-i(\epsilon_j+h)t}$, and similar expressions for $V^\pm_j$ in Eqs. (2) and (3), yielding a set of coupled equations relating $\delta u^\pm_j$, $\delta v^\pm_j$, and $\delta\Delta$. Next we decompose the fluctuations into frequency components, and use the completeness of the stationary wavefunctions to eliminate $\delta u^\pm_j$ and $\delta v^\pm_j$, thus arriving at an integral equation for $\delta\Delta$. In particular, we write $\delta\Delta = \mathrm{Re}(\delta_a(x)e^{i\Omega t}) + i\,\mathrm{Im}(\delta_p(x)e^{i\Omega t})$ where δ_a and δ_p describe the amplitude and phase fluctuations respectively, and find (full derivation in Supplemental Material [54]),

$$\delta_{p,a}(x) = -g_{\rm 1D} \int dx' \mathcal{M}^\pm(x,x';\Omega)\, \delta_{p,a}(x')\,, \quad (4)$$

where, at zero temperature,

$$\mathcal{M}^\pm = {\sum_{j,j'}}' \frac{2(\epsilon_j + \epsilon_{j'})}{(\epsilon_j + \epsilon_{j'})^2 - \Omega^2} (u^*_j u_{j'} \pm v^*_j v_{j'})(u'_j u'^*_{j'} \pm v'_j v'^*_{j'}). \quad (5)$$

Here $\Omega \in \mathbb{C}$, the prime on the summation stands for $\epsilon_j > h$, and we have used the notation $(u,v) \equiv (u(x), v(x))$ and $(u',v') \equiv (u(x'), v(x'))$. The collective modes represent non-trivial solutions to Eq. (4).

Periodicity of the soliton train leads to a Brillouin zone structure for the collective modes, i.e., one can write $\delta_{p,a}(x) = e^{iqx}\sum_n C^\pm_n e^{ink_0x}$, where $-k_0/2 < q \le k_0/2$

and $n \in \mathbb{Z}$. However, the stationary solution has an additional symmetry, $\Delta_0(x+\pi/k_0) = -\Delta_0(x)$, which causes the even and odd Fourier modes to decouple in Eq. (4), effectively doubling the Brillouin zone [45]. Thus we consider only odd Fourier components, with $-k_0 < q \le k_0$. Substituting the Fourier expansion into Eq. (4) yields a matrix equation, $C_n^{\pm} = -g_{\rm 1D} \sum_m M_{nm}^{\pm}(q,\Omega)\, C_m^{\pm}$, where

$$M_{nm}^{\pm} = \frac{k_0}{2\pi} \int_{-\pi/k_0}^{\pi/k_0} dx \int dx' e^{-i(q+nk_0)x + i(q+mk_0)x'} \mathcal{M}^{\pm}. \quad (6)$$

We find the collective-mode spectrum by solving $\det(I + g_{\rm 1D} M^{\pm}(q,\Omega)) = 0$. Note that $M^{\pm}(q,\Omega)$ has branch cuts on the real-Ω axis, which originate from particle-hole excitations. Thus while considering real frequencies (ω), we set $\Omega \to \omega + i\,0^+$. We find Ω is either real or imaginary for all collective modes.

The matrices $M^{\pm}$ are related to the pairing susceptibilities $\chi^{\pm}(q,\omega)$, which describe the linear response to a pairing field, as $\chi^{\pm} = -g_{\rm 1D} \mathrm{Tr}\left[(I + g_{\rm 1D} M^{\pm})^{-1} M^{\pm}\right]$ (see Supplemental Material [54] for a derivation). The spectral densities, $\mathrm{Im}\,\chi^{\pm}$, contain isolated poles corresponding to collective modes, and broad particle-hole continua.

The collective excitation spectrum is fully characterized by two dimensionless quantities: n_s, the number of unpaired fermions per soliton, and k_1, which describes the sharpness of the solitons. They are set by the parameters $k_0/k_{\rm F}$, $k_{\rm F} a_{\rm 1D}$, and $h/\epsilon_{\rm F}$, $a_{\rm 1D}$ being the 1D scattering length ($a_{\rm 1D} = -2/g_{\rm 1D}$ [51, 56]). To a good approximation, the dependence on $k_0/k_{\rm F}$ and $k_{\rm F} a_{\rm 1D}$ appears through the combination $w \equiv (k_0/k_{\rm F}) \exp(\pi k_{\rm F} a_{\rm 1D}/2)$, which measures the width of the Andreev bound states. For $h = 0$ and $w \lesssim 2.5$, $k_1 \approx 1 - 8e^{-4\pi/w}$ [54].

Figure 2(a) shows the collective-mode spectrum for $n_s = 0$, $k_0/k_{\rm F} = 0.05$, and $k_{\rm F} a_{\rm 1D} = 2.6$ in the extended-zone scheme. Its structure is representative of the $n_s = 0$ case. The two-particle continuum has three separate regions, corresponding to particle-hole excitation between different bands of the quasiparticle spectrum [Fig. 1(b)].

We find two gapless Goldstone modes. The Goldstone phase mode is described by $\delta_p(x) \propto \Delta_0(x) e^{iqx}$ and $\omega = k_{\rm F} q$. It is the analog of the Anderson-Bogoliubov phonon mode in a uniform Fermi superfluid [59]. The Goldstone amplitude mode represents elastic deformations of Δ, and has a second gapped branch extending to large wave-vectors, which forms the analog of the 'Higgs' mode in a uniform Fermi superfluid [48, 49]. Both branches are expressed by $\delta_a(x) \propto u_{\frac{q}{2}}(x)\, v_{\frac{q}{2}}(x)$ and $\omega = 2\,\epsilon(\frac{q}{2})$, where $\epsilon(k)$ is the single-particle dispersion. Like the 'Higgs' mode, both branches sit on the threshold for particle-hole excitations, and will therefore be damped [48, 60]. In contrast, the excitation spectrum of a soliton train in a Bose superfluid, modeled by the Gross-Pitaevskii equation, is comprised only of two undamped gapless modes [Fig. 2(c)]. They have a similar dispersion to the fermionic case for small q, but each mode contains both amplitude and phase variations [54].

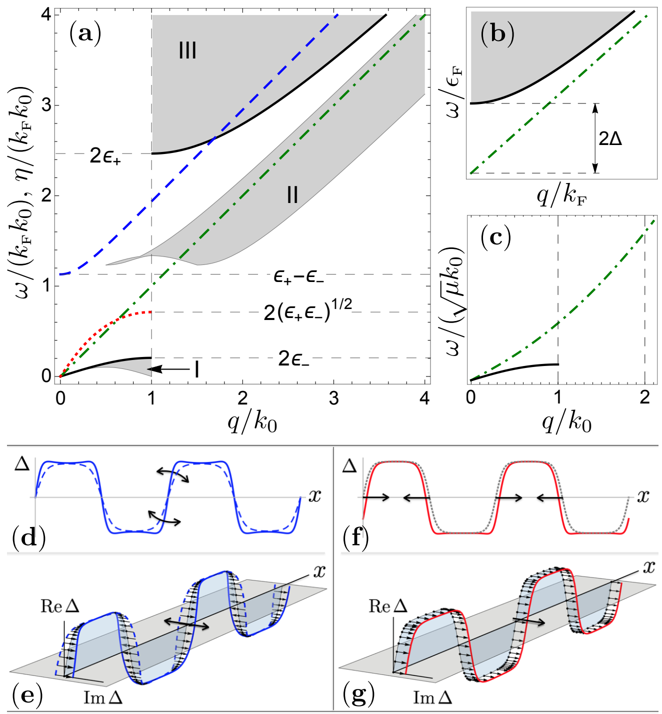

FIG. 2. (Color online.) Collective-mode spectrum of (**a**) soliton train in a Fermi superfluid with no spin imbalance, for $k_0/k_{\rm F} = 0.05$ and $k_{\rm F} a_{\rm 1D} = 2.6$, (**b**) a uniform Fermi superfluid, (**c**) soliton train in a Bose-Einstein condensate, modeled by the Gross-Pitaevskii equation. There are two gapless Goldstone modes in (**a**): a 'phonon' mode (dot-dashed, green) and an 'elastic' mode (solid, black) which describe phase twists and elastic deformations of the order parameter respectively. The 'phonon' mode is the analog of the Anderson-Bogoliubov mode of a uniform superfluid in (**b**). A second gapped branch of amplitude oscillations (solid, black) forms the remnant of the 'Higgs' mode in (**b**). Both 'elastic and 'Higgs' modes in (**a**) reside on an edge of the two-particle continua, shaded in gray, which originate from three types of particle-hole excitations, as shown in Fig. 1(b). Additionally, we find novel, twofold degenerate gapped modes in (**a**) (dashed, blue) which, for small q, describe width and grayness oscillations of each soliton, as illustrated in (**d**) and (**e**). The soliton train also has instabilities toward a uniform superfluid state, which show up as twofold degenerate unstable modes. The dotted (red) curve in (**a**) gives the growth rate η of these modes. The most unstable mode consists of pairs of solitons annihilating one another (**f**) or the order parameter moving off in the complex plane (**g**). In contrast, the spectrum of a bosonic soliton train in (**c**) only contains two gapless Goldstone modes.

In Fig. 2(a) we also show a gapped mode that is not present in either a Bose superfluid or a uniform Fermi superfluid (dashed, blue curve). This mode is twofold degenerate, with a phase- and an amplitude sector. For small q, they describe oscillations in the grayness and width of each soliton [Fig. 2(d)-(e)]. In particular, at $q = 0$, these sectors are expressed by $\delta_p(x) \propto \mathrm{cn}(\Delta_1 x/k_{\rm F}, k_1)$, $\delta_a(x) \propto \mathrm{sn}(\Delta_1 x/k_{\rm F}, k_1)\,\mathrm{dn}(\Delta_1 x/k_{\rm F}, k_1)$, and have an energy $\omega = \epsilon_+ - \epsilon_-$. Surprisingly, we find $\delta_a(x) \propto \delta_p'(x)\ \forall q$.

Being outside the continua, these 'core' modes should be long-lived and hence suitable for experimental detection. One can excite the amplitude 'core' mode by a fast ramp to a different interaction strength [see Fig. 1(a)].

The balanced soliton train ($n_s = 0$) has dynamical instabilities toward a uniform superfluid state, which show up as two degenerate solutions to Eq. (4) with an imaginary frequency. The unstable amplitude mode is associated with pairs of solitons approaching one another and annihilating, whereas the unstable phase mode involves the order parameter moving off into the complex plane [Fig. 2(f)-(g)]. The maximum instability occurs at $q = k_0$, where $\delta_a(x) \propto \mathrm{dn}^2(\Delta_1 x/k_{\mathrm{F}}, k_1)$, $\delta_p(x) = $ constant, and the fluctuations grow at a rate $\eta_{\max} = 2(\epsilon_+\epsilon_-)^{1/2}$. For a given soliton spacing, $\eta_{\max}$ is highest at weak interactions, approaching $k_{\mathrm{F}}k_0$. One can lower $\eta_{\max}$ by creating solitons farther apart or increasing the interaction strength [Fig. 3(a)-(b)]. We have verified the instability by direct simulations of the BdG equations without the Andreev approximation. We find a lower bound on the soliton lifetime $\tau_{\min} \sim 8/k_{\mathrm{F}}k_0$, which is saturated at weak interactions. For ^{6}Li atoms with $\epsilon_{\mathrm{F}} = 1.2\ \mu$K (as in [41]) and $k_0/k_{\mathrm{F}} = 0.05$, $\tau_{\min} \approx 0.5$ ms. The instability becomes unnoticeable for $k_{\mathrm{F}}a_{1\mathrm{D}} \lesssim 2$, where adjacent solitons collide elastically, in agreement with previous findings on two-soliton collisions [36, 37]. We present the simulations in the Supplemental Material [54], along with collective-mode spectra at different interactions.

An alternate way to stabilize the soliton train is by filling solitons with unpaired fermions [34]. As we increase n_s from 0, the instability rate falls, becoming zero at $n_s = 1$ for the C-FFLO phase [Fig. 3(c)]. The stability of the C-FFLO phase originates from the absence of zero-energy particle-hole excitations, as the chemical potentials lie within gaps in the single-particle spectrum. For $n_s > 1$, one again has instabilities (see Supplemental Material [54] for more details).

Past studies on FFLO have focused on the phase that minimizes free energy, which occurs at specific values of k_0 within a limited region of the phase diagram [51, 52]. Low-energy collective excitations of energetically stable

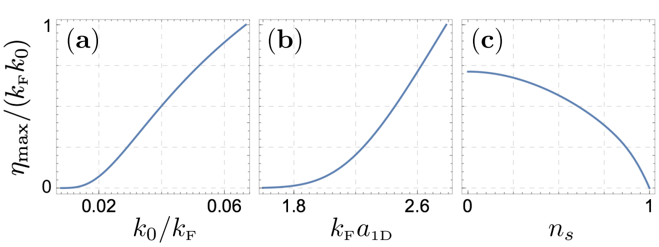

FIG. 3. Maximum instability rate vs (**a**) inverse soliton separation, with $n_s = 0$, $k_{\mathrm{F}}a_{1\mathrm{D}} = 2.6$, (**b**) interaction strength, with $n_s = 0$, $k_0/k_{\mathrm{F}} = 0.05$, and (**c**) spin imbalance, with $k_0/k_{\mathrm{F}} = 0.05$, $k_{\mathrm{F}}a_{1\mathrm{D}} = 2.6$. By making the rate sufficiently small, one can investigate the stable collective modes.

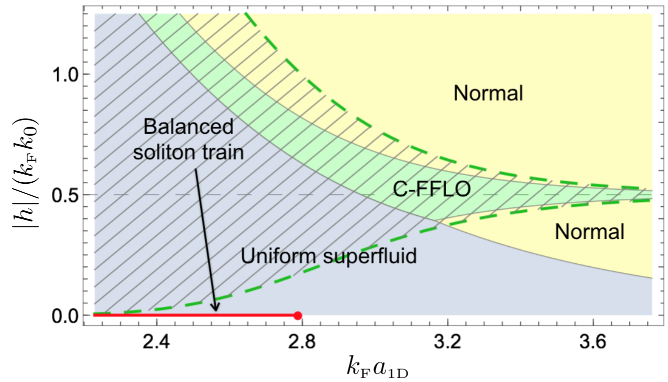

FIG. 4. (Color online.) Phase diagram obtained by comparing mean-field energies of homogeneous phases and soliton train states with $k_0/k_{\mathrm{F}} = 0.05$. Solid regions show lowest-energy states. The C-FFLO phase exists, and is dynamically stable, throughout the hatched region. The balanced soliton train exists above a minimum interaction strength ($w < 4$). To see where the other soliton train solutions exist, see [54].

FFLO states have been explored using different theoretical techniques [45–47, 61], and methods for detecting such states have been proposed [44]. However, we find that a C-FFLO phase is always dynamically stable, even when there are lower-energy states available. To see this, we compare the energies of competing states [54] to arrive at a phase diagram, shown in Fig. 4. Despite its large region of stability, the C-FFLO phase has the lowest energy in a relatively small region. Moreover, the optimal value of k_0/k_{F} varies continuously with h, a feature not apparent in Fig. 4 which is concerned with a fixed value of k_0/k_{F}. The metastability in this system implies that energetic considerations are of less importance than how the cloud is prepared. In particular, one can engineer long-lived FFLO states by phase imprinting. In the Supplemental Material [54] we describe an experimental protocol for creating such states.

Our results carry over to other physical systems where solitons arise in a BdG formalism. This includes quasi-1D superconductors in a magnetic field [8, 9], electron-phonon model of conducting polymers [6, 7], and Gross-Neveu models in quantum field theory [21]. The gapped modes describing width- and grayness oscillations of solitons could be more generic features associated with mesoscale structures, e.g., we find such modes in soliton trains described by a nonlinear Klein-Gordon equation, which also have unstable modes [54]. Although defined by pairing oscillations, these modes should be visible in many different spectroscopic channels. For example, the techniques demonstrated in [26] for observing the oscillation of a single soliton are well-suited for probing the 'elastic' modes. The instabilities can be studied using techniques from [27, 62]. The 'core' modes may be accessible through radio-frequency or modulation spectroscopy [44, 46, 63]. The dynamical stability of the C-FFLO phase should pave the way to its realization via phase imprinting [54]. Other techniques might also be feasible, e.g., in Bose-Einstein condensates, soliton trains spontaneously form in rapid quenches of interaction strength [15] or temperature [16], or when two condensates collide [17]. These processes could have analogs in Fermi superfluids. There exist theoretical methods complementary to BdG such as effective field theories [35, 64] and density-functional theories [33, 36] which could be extended to study soliton trains at strong interactions and finite temperatures. Our analysis provides a useful benchmark for such future investigations.

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