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# Soft Photon and Graviton Theorems in Effective Field Theory

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Extensions of the photon and graviton soft theorems are derived in 4d local effective field theories with massless particles of arbitrary spin. We prove that effective operators can result in new terms in the soft theorems at subleading order for photons and subsubleading order for gravitons. The new soft terms are unique and we provide a complete classification of all local operators responsible for such modifications. We show that no local operators can modify the subleading soft graviton theorem. The soft limits are taken in a manifestly on-locus manner using a complex double deformation of the external momenta. In addition to the new soft theorems, the resulting master formula yields consistency conditions such as the conservation of electric charge, the Einstein equivalence principle, supergravity Ward identities, and that particles with spin greater than two cannot couple to those with spin less than or equal to two.

## I. INTRODUCTION

In quantum field theory, scattering amplitudes are key observables for calculation of measurable decay rates and cross-sections. Modern research on the mathematical structure of amplitudes spans a wide range of topics: from new “on-shell methods” implemented in computer codes for efficient calculation of experimentally relevant cross-sections to abstract mathematical interpretations of certain amplitudes as volumes of geometric figures (see e.g. [1–4]).

Interest in soft limits of amplitudes has recently been rekindled due to the derivation of the classic soft photon and graviton theorems [5–11] as Ward identities of asymptotic symmetries [12–17]. An intriguing question is the effect of loop-quantum corrections on these recent results. This is subtle for loops of massless particles because of IR divergences, but loops of massive particles can be integrated out to leave effective local operators. The purpose of this Letter is to completely classify quantum corrections from loops of massive particles to the soft theorems using on-shell methods in the context of effective field theory. Our primary result is a novel generalized soft theorem in effective field theory.

Specifically, we show that in a 4d local effective field theory of only massless particles, the tree-level soft pho-

ton and graviton theorems receive modifications at subleading and subsubleading orders, respectively. These new modified soft theorems for positive-helicity soft photons or gravitons take the form

$$A_{n+1}^{\text{ph}} = \left( \frac{\mathcal{S}^{(0)}}{\epsilon^2} + \frac{\mathcal{S}^{(1)}}{\epsilon} \right) A_n + \frac{\tilde{\mathcal{S}}^{(1)}}{\epsilon} \tilde{A}_n + O(\epsilon), \quad (1)$$

$$A_{n+1}^{\text{grav}} = \left( \frac{\mathcal{S}^{(0)}}{\epsilon^3} + \frac{\mathcal{S}^{(1)}}{\epsilon^2} + \frac{\mathcal{S}^{(2)}}{\epsilon} \right) A_n + \frac{\tilde{\mathcal{S}}^{(2)}}{\epsilon} \tilde{A}_n + O(\epsilon), \quad (2)$$

where  $\mathcal{S}^{(i)}$  and  $\tilde{\mathcal{S}}^{(i)}$  are the standard soft factors, well-known from the work of [5–12], and given explicitly in (4) and (5) below. The new soft terms are

$$\tilde{\mathcal{S}}^{(1)} \tilde{A}_n = \sum_k g_k \frac{[sk]}{\langle sk \rangle} \tilde{A}_n^{(k)}, \quad \tilde{\mathcal{S}}^{(2)} \tilde{A}_n = \sum_k g_k \frac{[sk]^3}{\langle sk \rangle} \tilde{A}_n^{(k)}, \quad (3)$$

where  $g_k$  denotes the couplings of the associated effective operators. The tilde and superscript  $(k)$  on the  $n$ -point amplitude indicate that the particle type of the  $k$ th leg of  $\tilde{A}_n$  may differ from that in  $A_{n+1}$ . Thus, the new soft terms are different from the factorized form of the traditional soft theorems. Only a small set of effective operators can modify the soft theorems and we provide a complete classification. We show that no matter which operator is responsible for the modification, the kinematic soft factor is uniquely fixed to take the form (3). (It would be interesting to know if these new universal modifications are associated with asymptotic symmetries.) In this note we present modifications of the soft photon theorem, though our results naturally generalize to non-abelian gauge theory.

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Only effective operators with 3-point interactions can affect the single-particle soft theorems in (1) and (2). If an operator has many derivatives, its interaction is too soft to affect the soft theorems at these orders. For example,  $\text{tr}F^3$  does not modify the soft theorem, but the Pauli dipole operator  $\bar{\chi}\gamma^{\mu\nu}F_{\mu\nu}\chi$  does. All effective operators that can modify the soft theorems (1)-(2) are listed in (22) and (24). Note that our results imply that the *soft graviton theorem is not corrected at subleading order*  $1/\epsilon^2$  in effective field theory. This is important for recent proposals [16, 17] connecting soft graviton theorems to asymptotic symmetries.

To investigate the soft limits, we present a novel approach based on a double complex deformation of the amplitudes. Combining a ‘‘soft shift’’ with two BCFW shifts allows us to identify the parts of the amplitude responsible for the soft theorems as factorization poles. Note that we are not deriving new recursion relations and the results are independent of which lines are shifted along with the soft line. The method allows us to take the soft limit in a manifestly on-locus fashion that emphasizes the path dependence of the soft theorems at subleading order.

The approach yields not only the well-known soft theorems and new soft terms, it also implies non-trivial, though well known, consistency conditions, such as charge conservation, the equivalence principle, and the supersymmetric Ward identities which state that a spin 3/2 particle must be coupled supersymmetrically to a graviton. We also demonstrate the well-known result [18, 19] that no massless spin  $> 2$  particle can couple consistently to massless particles with spin 2 or less [20].

## II. COMPLEX DEFORMATIONS

We work with spinor helicity formalism in 4d following the conventions of [1, 2]. Momenta are assumed to be complex so that angle and square spinors are independent. The momentum  $p_s = -|s\rangle[s]$  is taken soft holomorphically:  $|s\rangle \rightarrow \epsilon|s\rangle$  and  $|s] \rightarrow |s]$ , with  $\epsilon$  a small parameter. The standard soft theorems for soft *positive-helicity* photons and gravitons then take the form (1)-(2) (without the tilde'd modifications), where for a soft photon

$$S^{(0)} = \sum_k g_k \frac{\langle xk \rangle}{\langle xs \rangle \langle sk \rangle}, \quad S^{(1)} = \sum_k \frac{g_k}{\langle sk \rangle} D_{sk}, \quad (4)$$

and for a soft graviton

$$S^{(0)} = \kappa \sum_k \frac{[sk] \langle xk \rangle \langle yk \rangle}{\langle sk \rangle \langle xs \rangle \langle ys \rangle}, \quad S^{(2)} = \frac{\kappa}{2} \sum_k \frac{[sk]}{\langle sk \rangle} D_{sk}^2, \\ S^{(1)} = \frac{\kappa}{2} \sum_k \frac{[sk]}{\langle sk \rangle} \left( \frac{\langle xk \rangle}{\langle xs \rangle} + \frac{\langle yk \rangle}{\langle ys \rangle} \right) D_{sk}. \quad (5)$$

Here  $|x\rangle$  and  $|y\rangle$  are arbitrary reference spinors and  $D_{sk} \equiv |s]_a \partial_{|k]_a}$ . When the amplitudes have their

momentum-conserving delta functions stripped off, the derivatives are taken with a prescription where one uses momentum conservation to eliminate a choice of two square spinors [12].

In this note we use a prescription in which the soft limit is taken along a path on the algebraic locus in momentum space defined by requiring that the external momenta are on-shell and satisfy  $(n+1)$ -particle momentum conservation. Start with  $n$  *unshifted* momenta  $p_k = -|k\rangle[k]$  satisfying  $n$ -particle momentum conservation,  $\sum_{k=1}^n p_k = 0$ . Introduce the soft momentum  $p_s = -|s\rangle[s]$  such that the *shifted* momenta  $\hat{p}_k = -|\hat{k}\rangle[\hat{k}]$ , defined as

$$|\hat{s}\rangle = \epsilon|s\rangle - z|X\rangle, \\ |\hat{i}\rangle = |i\rangle - \epsilon \frac{\langle js \rangle}{\langle ji \rangle} |s\rangle + z \frac{\langle jX \rangle}{\langle ji \rangle} |s\rangle, \\ |\hat{j}\rangle = |j\rangle - \epsilon \frac{\langle is \rangle}{\langle ij \rangle} |s\rangle + z \frac{\langle iX \rangle}{\langle ij \rangle} |s\rangle, \quad (6)$$

with no other spinors shifted, satisfy  $(n+1)$ -particle momentum conservation,  $\hat{p}_s + \sum_{k=1}^n \hat{p}_k = 0$ . The spinor  $|X\rangle$  is completely arbitrary. The complex deformation (6) can be viewed as the combination of a *soft  $\epsilon$ -shift* [21] and two BCFW shifts with parameters  $z_1 = \frac{\langle jX \rangle}{\langle ji \rangle} z$  and  $z_2 = \frac{\langle iX \rangle}{\langle ij \rangle} z$ , and  $z|X\rangle = z_1|i\rangle + z_2|j\rangle$ . The choice of the two lines  $i$  and  $j$  is arbitrary and does not affect the physics conclusions.

For any momentum  $k = 1, \dots, n$  we have

$$\hat{P}_{sk}^2 = (\hat{p}_k + \hat{p}_s)^2 = (\epsilon - \epsilon_k) P_{sk}^2, \quad \epsilon_k = z \frac{\langle Xk \rangle}{\langle sk \rangle}. \quad (7)$$

Evaluating  $\hat{P}_{sk}$  at  $\epsilon = \epsilon_k$ , we obtain  $\hat{P}_{sk} = -|k\rangle[\hat{P}_{sk}]$  with

$$|\hat{P}_{is}\rangle = |i\rangle, \quad |\hat{P}_{js}\rangle = |j\rangle, \\ |\hat{P}_{sk}\rangle = |k\rangle + z \frac{\langle Xs \rangle}{\langle sk \rangle} |s\rangle, \quad \text{for } k \neq i, j. \quad (8)$$

We are interested in poles at  $\epsilon = 0$  in the  $(n+1)$ -particle amplitude. With  $z = 0$ , there are multiple contributions to such poles, since (as is obvious from (7)) all 2-particle channels with a soft line  $s$  contribute. The role of  $z \neq 0$  is to separate these poles to different locations in the complex  $\epsilon$ -plane and exploit that the amplitude factorizes on simple poles. Since the only possible poles in  $\epsilon$  come from the 2-particle channels, we can write

$$\hat{A}_{n+1}(z, \epsilon) = \sum_{k, h_P, c} \hat{A}_3(\hat{s}, \hat{k}, -\hat{P}_{sk, c}^{h_P}) \frac{1}{\hat{P}_{sk}^2} \hat{A}_n^{(k)}(z) + O(\epsilon^0), \quad (9)$$

because when a propagator goes on shell, the amplitude factorizes into a product of on-shell amplitudes [22]. The sum is over all relevant momentum channels  $k$  as well as over the spectrum of particles on the internal line, as indicated with the helicity label  $h_P$  and a collective index  $c$  of other quantum numbers. The superscript on the  $n$ -point amplitude indicates that it in general depends on the channel momentum  $k$ :  $\hat{A}_n^{(k)}(z) = \hat{A}_n(\hat{P}_{sk, c}^{-h_P}, \dots)$ .

Little-group scaling fixes the 3-particle amplitude up to a constant which we absorb in the associated coupling  $g_{H_k}$ , where  $H_k = \{h_s, h_k, h_P; a, b, c\}$  labels helicities and possible quantum numbers:

$$\hat{A}_3(\hat{s}, \hat{k}, -\hat{P}_{sk}^{h_P}) = g_{H_k} [\hat{s}\hat{k}]^{x_1} [\hat{k}\hat{P}_{sk}]^{x_2} [\hat{P}_{sk}\hat{s}]^{x_3}, \quad (10)$$

where  $x_1 = h_s + h_k - h_P$ ,  $x_2 = h_k + h_P - h_s$ , and  $x_3 = h_P + h_s - h_k$ . In special 3-particle kinematics, another option is that  $A_3$  could depend on angle brackets only; however, the shifted angle brackets vanish. The mass dimension of the coupling is

$$[g_{H_k}] = \mathbf{a} - 2h_s, \quad \text{with } \mathbf{a} \equiv h_s - h_k - h_P + 1. \quad (11)$$

Using the kinematics above, (9) becomes

$$\hat{A}_{n+1}(z, \epsilon) = \sum_{k, h_P, c} \frac{g_{H_k} [sk]^{2h_s - \mathbf{a}} \langle Xs \rangle^{1-\mathbf{a}} \hat{A}_n^{(k)}(z)}{\epsilon z^{\mathbf{a}-1} \langle sk \rangle^{2-\mathbf{a}} \left(1 - \frac{z}{\epsilon} \frac{\langle Xk \rangle}{\langle sk \rangle}\right)} + O(\epsilon^0). \quad (12)$$

This is the ‘‘master formula’’ for the following analysis. (For comments about signs, see footnote [19].) We work with the Laurent expansion (12) for sufficiently small  $z \ll \epsilon$  and, as we shall see, the soft theorems then follow from the  $O(z^0)$  terms.

### III. PHOTON AND GRAVITON CONSISTENCY CONDITIONS

At tree level, locality requires that an amplitude can be singular only on a factorization channel. For  $z = 0$  and generic  $\epsilon \neq 0$  there is no associated channel, so the appearance of such a pole violates locality. Therefore, if the value of  $\mathbf{a}$  is greater than 1 in (12), the sum of residues of the apparent poles at  $z = 0$  must vanish. This imposes non-trivial constraints on the amplitudes.

Two non-trivial constraints arising from this requirement are

$$\begin{aligned} h_s = 1, \mathbf{a} = 2 &\implies \sum_k g_{H_k} = 0, \\ h_s = 2, \mathbf{a} = 3 &\implies \sum_{k=1}^n g_{H_k} [sk] \langle sk \rangle = 0. \end{aligned} \quad (13)$$

The first condition is simply *charge conservation*. The second condition can be satisfied only when the graviton couples identically to all particles; we recognize this as the *equivalence principle*. These results were first obtained by a different argument by Weinberg [18].

We now prove that a unitary local theory can have no interactions with  $\mathbf{a} \geq 4$ . Let the highest value of  $\mathbf{a}$  in a theory be  $\mathbf{a}_{\max} \geq 4$ . The kinematic structure of the corresponding 3-particle amplitude  $A_3(s_a^{h_s}, 1_b^{h_1}, P_c^{h_P})$  is uniquely determined by little group scaling as in (10). Denote the coupling by  $f_{abc}$ , where  $a, b, c$  are collective indices for all internal quantum numbers. CPT invariance requires that the theory also includes the amplitude of the CP conjugate states; its coupling is  $f_{\overline{abc}} =$

$f_{abc}^*$ . Consider the soft limit of the 4-particle amplitude  $A_4(s_a^{h_s}, 1_b^{h_1}, 2_{\overline{a}}^{-h_s}, 3_{\overline{b}}^{-h_1})$ , whose  $s_1$ -channel diagram includes the 3-particle interaction with  $\mathbf{a}_{\max} \geq 4$  and its conjugate, as well as the  $s_2$ - and  $s_3$ -channel diagrams, if relevant. The consistency condition arising from the absence of the pole  $1/z^{\mathbf{a}_{\max}-1}$  in (12) implies

$$\sum_{k=1}^3 \langle sk \rangle^{\mathbf{a}_{\max}-2} B_k = 0, \quad (14)$$

where  $B_1 = \sum_c f_{abc} [sk]^{2h_s - \mathbf{a}_{\max}} \hat{A}_3^{(k)}(0)$  and similarly for  $B_2$  and  $B_3$  (if present). Importantly, the  $B_i$  are independent of  $|s\rangle$ . Applying the operator  $|p\rangle^{\hat{a}} \partial_{|s\rangle^{\hat{a}}}$  to (14) gives

$$\sum_{k=1}^3 \langle kp \rangle \langle sk \rangle^{\mathbf{a}_{\max}-3} B_k = 0. \quad (15)$$

Since  $|s\rangle$  and  $|p\rangle$  are arbitrary, we can choose them to be  $|2\rangle$  and  $|3\rangle$  in which case (15) requires  $B_1 = 0$ . (Similarly, one can show  $B_2 = B_3 = 0$ .) Since

$$B_1 \propto \sum_c f_{abc} f_{\overline{abc}} = \sum_c |f_{abc}|^2, \quad (16)$$

it can vanish only if  $f_{abc} = 0$ . This shows that any couplings of interactions with  $\mathbf{a} \geq 4$  must vanish.

For  $\mathbf{a} = 3$ , the above argument fails because the power of  $\langle sk \rangle$  in (15) is no longer strictly positive. Indeed,  $\mathbf{a} = 3$  is perfectly fine for gravitons. For soft photons, however, we have proven that there are no interactions with  $\mathbf{a}_{\max} > 2$ .

Consider two examples of excluded interactions:

- A 3-particle interaction with  $h_s = 1$  and  $h_k = h_P = -1$  gives  $\mathbf{a} = 4$ . It may appear strange that such an interaction is excluded here, since the gluon amplitude  $A_3(1^+, 2^-, 3^-)$  certainly exists and is non-vanishing in Yang-Mills theory. However, this 3-gluon amplitude is non-vanishing in terms of angle brackets only. To produce such an amplitude in terms of square brackets only would require a non-local interaction  $A^2 \square A$  [1, 2] and is therefore not allowed in a local theory.

- Consider a soft photon case of  $\mathbf{a} = 3$ : take  $h_s = 1$ ,  $h_k = -1$ ,  $h_P = 0$ . This matrix element can be obtained from the operator  $\square^{-2} F_{\mu\nu} F^\nu{}_\rho \partial^\mu \partial^\rho \phi$  which clearly is not local.

Since  $h_s + h_k + h_P = 0$  implies  $\mathbf{a} = 2h_s + 1$ , we conclude from the above bounds on  $\mathbf{a}$  that no 3-point interactions involving photons, gravitinos, or gravitons are allowed if the sum of the three helicities vanishes.

### IV. SOFT PHOTON THEOREMS

**Standard soft photon theorem.** Set  $\mathbf{a} = 2$  in the master formula (12) for a soft positive-helicity photon

( $h_s = 1$ ). Expanding the  $n$ -point amplitude and the denominator factor in small  $z$ , there are two contributions at order  $z^0$ . One goes as  $1/\epsilon^2$  and takes the form

$$\hat{A}_{n+1}(z, \epsilon)|_{z^0, 1/\epsilon^2} = \frac{1}{\epsilon^2} \sum_{k,c} g_{H_k} \frac{\langle Xk \rangle}{\langle Xs \rangle \langle sk \rangle} A_n, \quad (17)$$

where  $A_n$  is the unshifted amplitude, which is a function of the momenta  $p_k$  that satisfy  $n$ -particle momentum conservation. The result (17) is the standard leading soft factor  $S^{(0)}$ .

The other  $O(z^0)$  contribution is order  $1/\epsilon$ :

$$\hat{A}_{n+1}(z, \epsilon)|_{z^0, 1/\epsilon} = \frac{1}{\epsilon} \sum_{k,c} \frac{g_{H_k}}{\langle Xs \rangle} \partial_z \hat{A}_n(z)|_{z=0}. \quad (18)$$

The shifted amplitude  $\hat{A}_n(z)$  depends on  $z$  through the momentum line  $\hat{P}_{sk}$  as well as potentially through the shifted momenta  $\hat{p}_i$  and  $\hat{p}_j$ . In the momentum channel with  $\hat{P}_{sk}^2 = 0$ , one uses the chain rule to find  $\partial_z \hat{A}_n(z) = \frac{\langle Xs \rangle}{\langle sk \rangle} \nabla_{sk} \hat{A}_n(z)$  with

$$\nabla_{sk} \equiv |s]_a \left( \partial_{[k]_a} + \frac{\langle ki \rangle}{\langle ij \rangle} \partial_{[j]_a} - \frac{\langle kj \rangle}{\langle ij \rangle} \partial_{[i]_a} \right). \quad (19)$$

The first term gives the familiar subleading soft factor  $S^{(1)}$ . The two other terms are consequences of our prescription for taking the soft limit. In contrast to [12] where the soft limit is taken by defining an *extrinsic* continuation of the amplitude off-locus (away from the support of the momentum conserving delta function), our soft limit is calculated along an on-locus path defined by the  $z = 0$  deformation (6). The corresponding soft theorems can therefore depend only on *intrinsic* on-locus data. The modified differential operators can be understood as an element of tangent space of the momentum conserving locus.

The strategy employed by Cachazo and Strominger [12] for acting with the subleading soft-theorem differential operator on the delta-function-stripped amplitude is to use momentum conservation to eliminate two square brackets  $[i]$  or  $[j]$  from the expression via  $[i] = -\sum_{k \neq i} \frac{\langle jk \rangle}{\langle ji \rangle} |k]$  and similarly for  $[j]$ . Now acting with  $|s]_a \partial_{[k]_a}$  on

$$A_{n+1}^{[i,j]} = A_{n+1} \left( \dots, [i-1], -\sum_{k \neq i} \frac{\langle jk \rangle}{\langle ji \rangle} |k], \dots; |1], \dots \right) \quad (20)$$

gives the same result as acting with our  $\nabla_{sk}$  of (19) on  $A_{n+1}$  without  $[i]$  and  $[j]$  eliminated by momentum conservation. One sees this by considering the cases  $k = i, j$  or  $k \neq i, j$  separately. Therefore the prescription used in [12] is equivalent to our on-locus differential operator (19).

**Modification of the subleading soft photon theorem.** The only other  $1/\epsilon$  contributions from (12) for

$h_s = 1$  arise from interactions with  $\mathbf{a} = 1$ . These give

$$\hat{A}_{n+1}(z, \epsilon)|_{z^0, 1/\epsilon} = \frac{1}{\epsilon} \sum_{k,c} g_{H_k} \frac{[sk]}{\langle sk \rangle} \tilde{A}_n^{(k)} + O(\epsilon^0), \quad (21)$$

which yields the new subleading soft factor  $\tilde{S}^{(1)} \tilde{A}_n$  in (3). By (11), the coupling must have mass dimension  $-1$  and  $h_k + h_P = 1$ . The new  $\mathbf{a} = 1$  contribution to the subleading soft theorem involves an  $n$ -point amplitude  $\tilde{A}_n$  whose external states may differ from the  $n$  hard states of  $A_{n+1}$ . To determine which theories can have these corrections, one simply goes through the options to find that the only possible operators are

$$\begin{aligned} & \bar{\chi} \gamma^{\mu\nu} F_{\mu\nu} \chi, \quad \phi F_{\mu\nu} F^{\mu\nu}, \quad \phi F_{\mu\nu} \tilde{F}^{\mu\nu}, \\ & \bar{\psi}_\mu F_{\nu\rho} \gamma^{\mu\nu\rho} \chi, \quad h F^2, \end{aligned} \quad (22)$$

where  $\chi$  is a spin 1/2 field and  $\psi_\mu$  is the gravitino field. The operator  $h F^2$  is shorthand for the 3-particle interaction that arises from the metric expansion of  $F_{\mu\nu} F^{\mu\nu}$ .

To summarize, we have shown that in effective field theory the soft theorem for a positive-helicity soft photon takes the form (1), where  $S^{(0)}$  and  $S^{(1)}$  are as given in (4) with  $D_{sk} \rightarrow \nabla_{sk}$  on the momentum conserving locus. The new soft factor  $\tilde{S}^{(1)}$  in (3) is unique no matter which of the possible effective operators in (22) are responsible for the modification of the soft theorems.

## V. SOFT GRAVITON THEOREMS

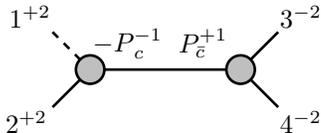
**Standard soft graviton theorem.** The familiar terms (5) of the graviton soft theorem (2) follow from the master equation (12) by setting  $h_s = 2$  and  $\mathbf{a} = 3$ . As we have already seen in (13), the absence of the  $1/z^2$  pole in this expression implies the equivalence principle: the graviton couples uniformly to all particles with a universal coupling  $\kappa = g_{H_k}$ . Using this, the  $1/z$  terms can be rewritten in terms of the Lorentz generators  $J_{ab} = \frac{i}{2} \sum_k (|k]_a \partial_{[k]_b} + |k]_b \partial_{[k]_a})$  and terms that vanish by momentum conservation. Since  $J_{ab}$  annihilates the on-shell amplitudes, the residue of the  $1/z$  pole vanishes without imposing further constraints. The  $O(z^0)$  terms give the soft theorem (2) in a form with

$$\begin{aligned} \mathcal{S}^{(0)} &= \kappa \sum_k [sk] \langle sk \rangle \frac{\langle Xk \rangle^2}{\langle Xs \rangle^2 \langle sk \rangle^2}, \quad \mathcal{S}^{(2)} = \frac{\kappa}{2} \sum_k \frac{[sk]}{\langle sk \rangle} \nabla_{sk}^2, \\ \mathcal{S}^{(1)} &= \kappa \sum_k \frac{\langle Xk \rangle [sk]}{\langle Xs \rangle \langle sk \rangle} \nabla_{sk}. \end{aligned} \quad (23)$$

Using the Schouten identity to write e.g.  $\frac{\langle Xk \rangle}{\langle Xs \rangle \langle sk \rangle} = \frac{\langle Xy \rangle}{\langle Xs \rangle \langle sy \rangle} - \frac{\langle ky \rangle}{\langle ks \rangle \langle sy \rangle}$  as well as using momentum conservation and annihilation of the amplitude by  $J_{ab}$ , one can show that the soft factors (23) are equivalent to those in (5) with the replacement  $D_{sk} \rightarrow \nabla_{sk}$  as discussed for the photon soft theorem above.

### Subleading soft graviton theorem unchanged.

The only way to get a modification to the soft graviton theorem at order  $1/\epsilon^2$  is via interactions with  $\mathbf{a} = 2$ . The responsible local operators would have couplings of mass dimension  $-2$  and give rise to 3-particle amplitudes  $A_3(1^{+2}, 2^{h_2}, 3^{h_P})$  with  $h_2 + h_P = 1$ . Restricting to  $\text{spin} \leq 2$ , the options are  $(h_2, h_P) = (2, -1), (\frac{3}{2}, -\frac{1}{2}), (1, 0), (\frac{1}{2}, \frac{1}{2})$ . The requirement that the  $1/z$  pole in (12) vanishes implies that no such *local* operators exist. For the case  $(h_2, h_P) = (2, -1)$ , consider the 4-graviton amplitude  $A_4(1^{+2}, 2^{+2}, 3^{-2}, 4^{-2})$  at quadratic order in the non-standard effective coupling  $g_c$ . Only one factorization channel contributes to  $1/z$  in (12), namely



with implicit sum over possible internal quantum numbers  $c$  of the exchanged spin-1 state. CPT invariance requires the couplings of the two interactions to be conjugate, so the  $1/z$  pole in (12) will be proportional to  $\sum_c |g_c|^2$ . Absence of this pole requires  $g_c = 0$ . The three other cases of  $\mathbf{a} = 2$  interactions can be similarly excluded. In conclusion, in a unitary CPT-invariant theory there can exist no local operators that modify the subleading soft graviton theorem.

This result may have relevance to recent discussions of asymptotic symmetries. In [13] it was shown that the universality of the subleading soft graviton theorem (2) is equivalent to the Ward identity of a Virasoro symmetry of the quantum gravity S-matrix. Our result implies that the subleading soft graviton theorem, and consequently the Virasoro symmetry, is unmodified at tree-level in the presence of local effective operators. In particular this includes curvature corrections.

**Modification of the subleading soft graviton theorem.** The only other  $1/\epsilon$  contributions from (12) for  $h_s = 2$  arise from interactions with  $\mathbf{a} = 1$ . By (11), the coupling must have mass dimension  $-3$  and  $h_k + h_P = 2$ . The corresponding operators in effective field theory are

$$\begin{aligned} \phi R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma}, \quad \phi R_{\mu\nu\rho\sigma} \tilde{R}^{\mu\nu\rho\sigma}, \\ R^{\mu\nu\rho\sigma} \bar{\psi}_\rho \gamma_{\mu\nu} \partial_\sigma \chi, \quad R_{\mu\nu\rho\sigma} F^{\mu\nu} F^{\rho\sigma}. \end{aligned} \quad (24)$$

All of these operators, up to constants, give the same correction  $\tilde{S}^{(2)} \tilde{A}_n$  in (3) to the soft theorem. The modification due to the operator  $\phi R^2$  was previously noted by [23, 24].

## VI. HIGHER SPIN AND SUPERGRAVITY

**Higher spin.** The equivalence principle mandates that any theory containing a massless spin 2 boson and a particle  $X$  of spin  $j$  must include a coupling

$A_3(1^{+2}, 2_X^{+j}, 3_X^{-j})$  with the universal coupling constant. Taking the soft limit of the helicity  $+j$  particle, this coupling has  $\mathbf{a} = 2j - 1$ . As discussed in Section III, the condition for the vanishing of poles in  $z$  has no non-trivial solutions for  $\mathbf{a} > 3$ , which implies  $j \leq 2$ . Thus, by demanding locality and unitarity in the soft limit, we find that massless higher spin particles cannot interact in any way with particles of  $\text{spin} \leq 2$  in a theory of gravity. This is an on-shell version of the results presented previously in [18, 19, 25].

**Supergravity.** We learned above that the usual soft graviton theorems arise from interactions with  $\mathbf{a} = 3$ , for which  $h_P = -h_k$ . Such interactions include the standard graviton self interactions, and if we have spin  $3/2$  massless fields, the equivalence principle implies that the coupling of  $A_3(1^{+2}, 2^{+\frac{3}{2}}, 3^{-\frac{3}{2}})$  must be the same as that of  $A_3(1^{+2}, 2^{+2}, 3^{-2})$ . Let us explore the soft limit of a positive-helicity spin  $3/2$  particle. With  $h_s = \frac{3}{2}$ , the interaction  $A_3(1^{+2}, 2^{+\frac{3}{2}}, 3^{-\frac{3}{2}})$  has  $\mathbf{a} = 2$ , for which (12) yields a non-trivial constraint from the absence of a  $1/z$  pole. Consider for example  $A_{n+1}(s^{+\frac{3}{2}}, 1^{-\frac{3}{2}}, 2^{+2}, 3^{+2}, 4^{-2}, \dots, n^{-2})$ . The  $1/z$ -pole in (12) has three contributions with  $\mathbf{a} = 2$ , namely from  $k = 1, 2, 3$ . (Lines  $k = 4, \dots, n$  give  $\mathbf{a} = 6$ .) The sum over the three channels  $k = 1, 2, 3$  gives the consistency condition

$$\begin{aligned} 0 = [s1] A_n(1^{-2}, 2^{+2}, 3^{+2}, 4^{-2}, \dots, n^{-2}) \\ - [s2] A_n(1^{-\frac{3}{2}}, 2^{+\frac{3}{2}}, 3^{+2}, 4^{-2}, \dots, n^{-2}) \\ - [s3] A_n(1^{-\frac{3}{2}}, 2^{+2}, 3^{+\frac{3}{2}}, 4^{-2}, \dots, n^{-2}). \end{aligned} \quad (25)$$

This is precisely the MHV version of the  $N = 1$  supersymmetric Ward identities [26, 27] (see also [28]). Thus we reached the well known conclusion that particles of spin  $3/2$  couple to gravity supersymmetrically. The role of the usual reference spinor in the SUSY Ward identities is here played by the soft momentum.

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