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New easy-plane \mathbb{CP}^{N-1} fixed points

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We study fixed points of the easy-plane \mathbb{CP}^{N-1} field theory by combining quantum Monte Carlo simulations of lattice models of easy-plane $SU(N)$ superfluids with field theoretic renormalization group calculations, by using ideas of deconfined criticality. From our simulations, we present evidence that at small N our lattice model has a first order phase transition which progressively weakens as N increases, eventually becoming continuous for large values of N . Renormalization group calculations in $4 - \epsilon$ dimensions provide an explanation of these results as arising due to the existence of an N_{ep} that separates the fate of the flows with easy-plane anisotropy. When $N < N_{ep}$ the renormalization group flows to a discontinuity fixed point and hence a first order transition arises. On the other hand, for $N > N_{ep}$ the flows are to a new easy-plane \mathbb{CP}^{N-1} fixed point that describes the quantum criticality in the lattice model at large N . Our lattice model at its critical point, thus gives efficient numerical access to a new strongly coupled gauge-matter field theory.

Introduction: The emergence of gauge theories in quantum spin Hamiltonians has played an important role in theoretical descriptions of novel magnetic phenomena over the past few decades. Recently, by exploiting advances in simulation algorithms for quantum antiferromagnets [1], the connections between magnetism and gauge theories have facilitated controlled numerical access to otherwise poorly understood strongly coupled gauge theories; the most prominent example is the study of N -component scalar electrodynamics (also called \mathbb{CP}^{N-1}) that emerges right at the direct continuous transitions between magnetic and translational-symmetry breaking “valence bond solid” (VBS) states in $SU(N)$ magnets, a phenomena popularly called “deconfined critical points” (DCP) [2].

Two prominent physical systems where the ideas of DCP apply are lattice superfluids and antiferromagnets, each giving rise to its own variant of the \mathbb{CP}^{N-1} theory. In the context of superfluids (SF), the appropriate description is a “easy-plane” \mathbb{CP}^1 field theory [2, 3] which applies to the superfluid SF-VBS critical point in $S = 1/2$ XY models [4]. The easy-plane case was studied intensely initially since a self-duality suggested that this could be the best candidate for a DCP [5]. Subsequent numerical work has concluded however that this transition is first order, both in direct discretizations of the field theory [6, 7] as well as in simulations of the quantum antiferromagnet [8]. The easy-plane case is in sharp contrast to the case of antiferromagnets with $SU(N)$ symmetry, where striking quantitative agreement between detailed field theoretic calculations [9–12] and numerical simulations has been demonstrated [13–15].

The persistent first order behavior in the XY-like models and its striking difference from the continuous transitions found in the $SU(N)$ symmetric case has been unexplained so far, despite the central role of both systems in our understanding of the DCP phenomena. In this work we address this issue by formulating an extension of the “easy-plane” XY symmetry of $SU(2)$ to general

$SU(N)$. Our approach allows for a study of the first order transition for arbitrary N using both lattice simulations of an easy plane- $SU(N)$ [ep- $SU(N)$] model as well as renormalization group (RG) calculations on a proposed easy-plane- \mathbb{CP}^{N-1} [ep- \mathbb{CP}^{N-1}]: We find the first order transition in the ep- $SU(N)$ models found for $N = 2$ in previous work persists for larger N . A careful analysis however shows that the first order jump quantitatively weakens as N increases. RG ϵ -expansion calculations find that the field theory hosts a new ep- \mathbb{CP}^{N-1} fixed point only for $N > N_{ep}$, suggesting that the transition can eventually become continuous. Consistent with this result, we find that the transition in our lattice model turns continuous around $N \approx 20$. For $N = 21$ we provide a detailed scaling analysis of our numerical data that confirms a continuous transition in a new universality class. Our work clarifies and significantly extends the discussion of the DCP phenomena in easy-plane magnets and its relation to the symmetric case.

Easy-plane model & field theory: We introduce a family of bipartite ep- $SU(N)$ spin models that are extensions of the quantum XY model to larger N akin to those studied by us recently [8]. They are written in terms of the T_i^a , the fundamental generators of $SU(N)$ on site i :

$$H_{ep} = -\frac{J_{1\perp}}{N} \sum_{a, \langle ij \rangle} ' T_i^a T_j^{a*} - \frac{J_{2\perp}}{N} \sum_{a, \langle\langle ij \rangle\rangle} ' T_i^a T_j^a. \quad (1)$$

the \sum' denotes the sum on a is restricted to the $N^2 - N$ off-diagonal generators (a sum on all a would give the $SU(N)$ model). The $\langle ij \rangle$ ($\langle\langle ij \rangle\rangle$) indicates nearest (next nearest) neighbors on the square lattice which are on opposite (same) sublattices and in conjugate (same) representations. H_{ep} is an easy plane deformation of the $SU(N)$ J_1 - J_2 model [13], it has a global $U(1)^{N-1} \times S_N$ (we call this ep- $SU(N)$) in addition to time reversal and lattice symmetries. As we shall show the model harbors in its phase diagram the SF-VBS transition for all $N > 5$. H_{ep} is Marshall positive for $J_{1\perp}, J_{2\perp} > 0$ [16]; we

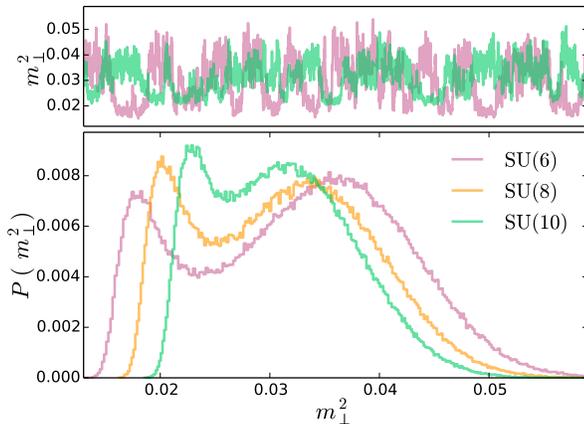


FIG. 1. First order transitions for moderate values of N . The upper panels shows MC histories (arbitrary units) of the estimator for m_{\perp}^2 for $N = 6$ and 10 . The bottom panel shows histograms of m_{\perp}^2 taken at $L = 50$ for $J_2/J_1 \equiv g = 0.250, 0.876, 1.58$ for $N = 6, 8, 10$ respectively clearly show double peaked behavior. The double peaked behavior persists in the $T = 0$ and thermodynamics limit [16].

simulate it with stochastic series Monte Carlo on $L \times L$ lattice at an inverse temperature β [17].

To obtain the effective field theory (\mathcal{L}_{ep}) for Eq. 1, we start with the $\mathbb{C}\mathbb{P}^{N-1}$ model (\mathcal{L}_s) proposed to describe DCP in anti-ferromagnets with $SU(N)$ symmetry [2, 3] and deform it using an anisotropy operator (\mathcal{L}_v), so that $\mathcal{L}_{ep} = \mathcal{L}_s + \mathcal{L}_v$ with,

$$\begin{aligned} \mathcal{L}_s &= \sum_{\alpha} |(\partial_{\mu} - ieA_{\mu})z_{\alpha}|^2 + \frac{1}{2}(\vec{\nabla} \times \vec{A})^2 \\ &+ r \sum_{\alpha} |z_{\alpha}|^2 + \frac{u}{2} \left(\sum_{\alpha} |z_{\alpha}|^2 \right)^2 \\ \mathcal{L}_v &= \frac{v}{2} \sum_{\alpha} |z_{\alpha}|^4, \end{aligned} \quad (2)$$

where the z_{α} are N complex fields coupled to a $U(1)$ gauge field, A_{μ} . The term v breaks the $SU(N)$ symmetry of \mathcal{L}_s to the same ep- $SU(N)$ symmetry found above for Eq. 1. It is known from the large- N expansion that for N larger than some finite N_s , in $d = 3$ the $\mathbb{C}\mathbb{P}^{N-1}$ field theory \mathcal{L}_s has a finite coupling fixed point (FP) [18, 19]. Below we will address the fate of these FPs for \mathcal{L}_{ep} .

Weakening first-order transition: We begin with a numerical study of Eq. (1). We have shown [8] that the ep- $SU(N)$ models map to a certain loop model. We can hence calculate two useful quantities to probe magnetic ordering: the average of the square of the spatial winding number of the loops $\langle W^2 \rangle$ and a normalized magnetic order parameter $m_{\perp}^2 = \frac{1}{(1-\frac{1}{N})L^2} \sum_a \sum_{i,j} \tilde{T}_i^a \tilde{T}_j^a$ [where the sum on a is on the off-diagonal generators, i and j are summed on the entire lattice and $\tilde{T} = T(T^*)$ on the A(B)

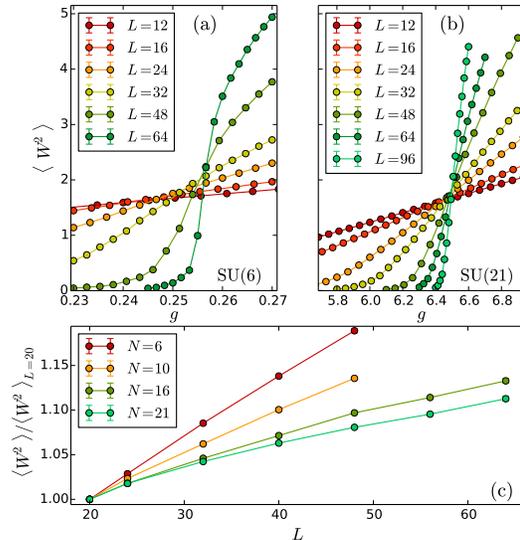


FIG. 2. Scaling of the spatial winding number square $\langle W^2 \rangle$. (a) Crossing for $N = 6$. (b) Crossing for $N = 21$. (c) Value at the L and $L/2$ crossing of $\langle W^2 \rangle$ for a range of N normalized to the crossing value at $L = 20$ for each N . For the smaller N a clear linear divergence is seen as expected for a first-order transition (ergodicity issues limit the system sizes here). For larger N a slow growth is observed very similar to what has been studied in detail for the $SU(2)$ case and interpreted as evidence for a continuous transition with two length scales [20], like we have here. The data was taken at $\beta = 6L$ which is in the $T = 0$ regime [16].

sublattice], which although off-diagonal in the $|\alpha\rangle$ basis can be estimated by measuring a particular statistical property of the loops [16]. We have normalized m_{\perp}^2 so that the maximum value it can take is 1 for all N , allowing for a meaningful comparison across different N .

Previously we found that the SF-VBS transition is first order for $N \leq 5$ [8]. In Fig. 1 we present data that shows the first order behavior persists as N is increased up to $N = 10$. A hitherto unanswered but important question is whether the first order jump weakens as N increases. We find evidence in favor of this assertion, since the histogram peaks get closer as N is increased. Beyond $N \approx 16$ we have found no evidence for double peaked histograms. To carry out a more quantitative analysis, which has been popular in the study of the DCPs [6], we turn to $\langle W^2 \rangle$ (which is related to the spin stiffness as $\beta\rho_s$). At a first order transition one expects a linear divergence of $\langle W^2 \rangle$ as one approaches the phase transition since ρ_s stays finite. Any sub-linear behavior indicates that the transition is continuous since ρ_s vanishes in the thermodynamic limit [21]. In Fig. 2 we present a study of the crossing of $\langle W^2 \rangle$. We find clear evidence for the expected linear behavior at moderate values of N . As N is increased beyond about $N \approx 16$ we find a very slow

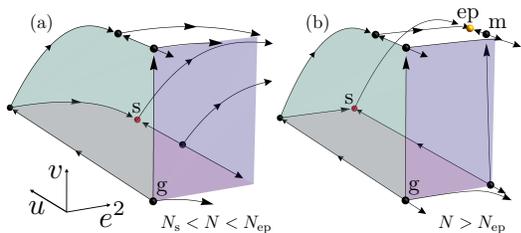


FIG. 3. RG flows of the ep- \mathbb{CP}^{N-1} model for (a) $N_s < N < N_{ep}$ and (b) $N > N_{ep}$ at leading order in $4 - \epsilon$ dimensions obtained by numerical integration of Eq. (3). Fixed points are shown as bold dots, we have only labeled a few significant to our discussion. The flows in the $v = 0$ plane have been obtained previously [18] and include the “s” fixed point that describes DCP in $SU(N)$ models (red dot). While the flows have many FPs [16], a DCP of the ep- $SU(N)$ spin model must have all three eigen-directions in the e^2 - u - v irrelevant. For $N < N_{ep}$ there are no such FPs; there is hence a runaway flow to a first order transition. For $N > N_{ep}$ two FPs emerge: “m” is multicritical and “ep” is the new ep-DCP that describes the SF-VBS transition (yellow dot). The gaussian fixed point at the origin has been labeled “g” for clarity. See [16] for further details.

growth of $\langle W^2 \rangle$ inconsistent with linear behavior but consistent with what has been found in $SU(N)$ models, where the transition is believed to be continuous [20, 21]. This study provides clear evidence that the first-order jump decreases as N increases, possibly becoming continuous.

RG analysis: The weakening of the first-order SF-VBS transitions at larger N raises important questions: Is the transition first order for all N or does it become continuous beyond some finite N_{ep} ? If the transition becomes continuous: Is it a new universality class of an ep- \mathbb{CP}^{N-1} or is the anisotropy irrelevant resulting in \mathbb{CP}^{N-1} criticality for the “easy-plane” models?

To answer these questions, we compute the RG flows of Eq. (2) in $4 - \epsilon$ dimensions. We will work in the critical plane where $r = 0$, the r operator being strongly relevant at tree level will continue to be relevant in the ϵ -expansion. To leading order (assuming u, v and e^2 are $\mathcal{O}(\epsilon)$), we find the following RG equations,

$$\begin{aligned} \frac{de^2}{d\ln s} &= \epsilon e^2 - \frac{N}{3} e^4, \\ \frac{du}{d\ln s} &= \epsilon u - (N + 4)u^2 - 4uv - 6e^4 + 6e^2 u, \\ \frac{dv}{d\ln s} &= \epsilon v - 5v^2 - 6uv + 6e^2 v, \end{aligned} \quad (3)$$

which for $v = 0$ reduce to the well known RG equations for the \mathbb{CP}^{N-1} model [18, 22]. Given the relevance of r , a generic critical point of ep- \mathbb{CP}^{N-1} would be a fixed point of Eq. (3) with all three eigen-directions in e^2 - u - v -space irrelevant. The FP structure and flows of Eq. 3 (shown in Fig. 3) change at two values of N : N_s and N_{ep} with $N_s < N_{ep}$. For $N < N_s$ [not shown] there are no FPs with $e^2 \neq 0$ and a generic flow runs away to a first order

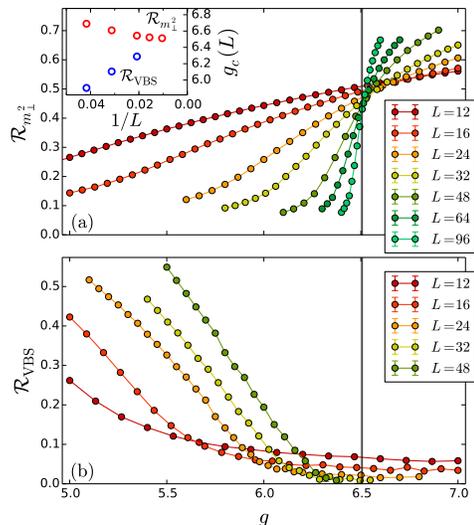


FIG. 4. Correlation ratios close to the phase transition for $N = 21$. (a) The SF order parameter ratio, R_{m_2} shows good evidence for a continuous transition with a nicely convergent crossing point of $g = 6.505(5)$. (b) R_{VBS} shows a crossing point that converges to the same value of the critical coupling. We note however that the crossing converges much more slowly (see text). The inset shows the convergence of the crossings points of L and $L/2$ of SF and VBS ratios. Note their convergence to a common critical coupling indicating a direct transition. The data shown in Fig. 4 and 5 was taken at $\beta = 6L$ which is in the $T = 0$ regime [16].

transition. For $N > N_s$ a $v = 0$ FP “s” appears, which describes the $SU(N)$ DCP phenomena, but at which v is always relevant. There are two distinct fates of the flow with $v \neq 0$: For $N_s < N < N_{ep}$ [see Fig. 3(a)] v causes a runaway flow to a discontinuity FP, i.e. the phase transition turns first order. On the other hand, for $N > N_{ep}$ [see Fig. 3(b)] a new fixed point “ep” appears. At this FP all eigen-directions in the $e^2 - u - v$ space are irrelevant and hence r is the only relevant perturbation. “ep” hence describes a generic continuous deconfined SF-VBS transition in models of the form Eq. (1). In the leading order of the ϵ -expansion we have $N_s \approx 183$ [18] and $N_{ep} \approx 5363$ (independent of ϵ). From previous work on the symmetric case, it is well known that these leading order estimates are unreliable in $d = 3$: Indeed, in the next to leading order, N_s becomes negative for $\epsilon = 1$ [23, 24]. Ultimately the values of $N_{s,ep}$ must be obtained from numerical simulations. Nonetheless, it is expected that the basic structure of fixed points and flows obtained here using the ϵ -expansion are reliable. Based on our study, we make the following conclusions: Even in a regime where there is a symmetric fixed point ($N > N_s$), for $N_s < N < N_{ep}$, easy-plane anisotropy will drive the DCP first-order. For $N > N_{ep}$ a new FP emerges. Easy-plane anisotropy

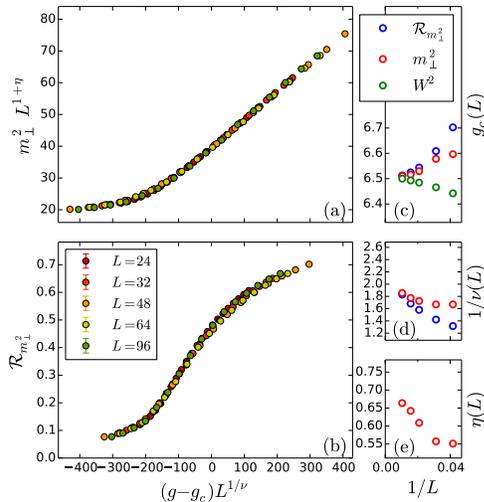


FIG. 5. Data collapse for the SF order parameter at $N = 21$. (a) Finite size data collapsed to $m_{\perp}^2 = L^{-(1+\eta_{\text{SF}})} \mathbb{M}[(g - g_c)L^{1/\nu}]$ with parameters $g_c=6.511$, $\nu=0.556$ ($1/\nu = 1.795$), $\eta=0.652$. (b) Collapse of ratio $R_{m_{\perp}^2} = \mathbb{R}[(g - g_c)L^{1/\nu}]$ with parameters $g_c=6.518$, $\nu=0.582$ ($1/\nu = 1.719$). The side panels show convergence of estimates for various quantities from the collapse of L and $L/2$ data: (c) the critical coupling $g_c = 6.505(1)$ from pair-wise collapses of m_{\perp}^2 and $R_{m_{\perp}^2}$, as well as crossings of $\langle W^2 \rangle$ (see Fig. 2) for data. Panel (d) shows $1/\nu(L)$, which we estimate to converge to $1/\nu = 2.3(2)$. Likewise we estimate $\eta(L)$ to converge to $\eta = 0.72(3)$, which is shown in panel (e).

then results in a continuous SF-VBS transition in a new ep- \mathbb{CP}^{N-1} universality class.

Study of fixed point: Having presented evidence from the ϵ -expansion that with increasing N the transition should turn continuous and in a new universality class, it is of interest to study the scaling behavior at large N . We will focus on $N = 21$ where we have found no evidence for first order behavior on the largest system sizes that we have access to. We construct dimensionless ratios $\mathcal{R}_{m_{\perp}^2}$ and \mathcal{R}_{VBS} which go to 1(0) in their respective ordered (disordered) phases. Fig. 4 shows our data for $N = 21$. The large correction to scaling observed in the VBS data are expected: according to the DCP theory the VBS anomalous dimension $\eta_{\text{VBS}} \propto N$ which causes the leading VBS correlation functions to decay very rapidly at this large value of N . This makes it hard to separate the leading and sub-leading behavior on the available system sizes. Since the SF data shows a good crossing, we carry out a full scaling analysis in Fig. 5. The data for both m_{\perp}^2 and $R_{m_{\perp}^2}$ collapse nicely without the inclusion of corrections to scaling [25, 26]. They lead to consistent values of critical couplings and scaling dimensions lending support for a continuous transition ep- \mathbb{CP}^{N-1} fixed

point emerging at large N .

In conclusion, we have studied new lattice models for deconfined criticality with easy-plane $\text{SU}(N)$ symmetry. We find persistent first order behavior in these lattice models at small to intermediate N , in sharp contrast to the continuous transitions found in the symmetric models for the same range of N . As N increases the first order easy-plane transition weakens and eventually becomes continuous. Our RG flows provide a way to understand both the first-order and shift to continuous transitions: The easy-plane anisotropy is always relevant at the symmetric \mathbb{CP}^{N-1} fixed point, for $N < N_{ep}$ there is no easy-plane fixed point and hence the anisotropy drives the transition first order. For $N > N_{ep}$ a new fixed point emerges resulting in a continuous transition in a new “easy-plane”- \mathbb{CP}^{N-1} universality class which is an example of a strongly coupled gauge-matter field theory. Our lattice model provides a sign-free discretization of this field theory that is amenable to efficient numerical simulations. We leave for future work the determination of a precise value of N_{ep} , comparisons of the universal quantities with easy-plane large- N expansions, and a comparative study of the scaling corrections between the easy-plane and symmetric cases. It would be of interest to complement our work with studies of field theories such as Eq. (2) using the conformal bootstrap [27].

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- [1] R. K. Kaul, R. G. Melko, and A. W. Sandvik, Annu. Rev. Cond. Matt. Phys **4**, 179 (2013), URL <http://www.annualreviews.org/doi/abs/10.1146/annurev-conmatphys-030212-184215>.
 - [2] T. Senthil, A. Vishwanath, L. Balents, S. Sachdev, and M. P. A. Fisher, Science **303**, 1490 (2004), URL <http://www.sciencemag.org/content/303/5663/1490.abstract>.
 - [3] T. Senthil, L. Balents, S. Sachdev, A. Vishwanath, and M. P. A. Fisher, Phys. Rev. B **70**, 144407 (2004), URL <http://link.aps.org/doi/10.1103/PhysRevB.70.144407>.
 - [4] A. W. Sandvik, S. Daul, R. R. P. Singh, and D. J. Scalapino, Phys. Rev. Lett. **89**, 247201 (2002), URL <http://link.aps.org/doi/10.1103/PhysRevLett.89.247201>.
 - [5] O. I. Motrunich and A. Vishwanath, Phys. Rev. B **70**, 075104 (2004), URL <http://link.aps.org/doi/10.1103/PhysRevB.70.075104>.
 - [6] A. Kuklov, N. Prokofev, B. Svistunov, and M. Troyer, Annals of Physics **321**, 1602 (2006), ISSN 0003-4916, july 2006 Special Issue, URL <http://www.sciencedirect.com/science/article/>

- [pii/S0003491606000789](http://link.aps.org/doi/10.1103/PhysRevLett.97.247201).
- [7] S. Kragset, E. Smørgrav, J. Hove, F. S. Nogueira, and A. Sudbø, Phys. Rev. Lett. **97**, 247201 (2006), URL <http://link.aps.org/doi/10.1103/PhysRevLett.97.247201>.
- [8] J. D'Emidio and R. K. Kaul, Phys. Rev. B **93**, 054406 (2016), URL <http://link.aps.org/doi/10.1103/PhysRevB.93.054406>.
- [9] G. Murthy and S. Sachdev, Nucl. Phys. B **344**, 557 (1990).
- [10] E. Dyer, M. Mezei, S. S. Pufu, and S. Sachdev, Journal of High Energy Physics **2015**, 1 (2015), ISSN 1029-8479, URL [http://dx.doi.org/10.1007/JHEP06\(2015\)037](http://dx.doi.org/10.1007/JHEP06(2015)037).
- [11] R. K. Kaul and S. Sachdev, Phys. Rev. B **77**, 155105 (2008), URL <http://link.aps.org/doi/10.1103/PhysRevB.77.155105>.
- [12] T. Senthil and M. P. A. Fisher, Phys. Rev. B **74**, 064405 (2006), URL <http://link.aps.org/doi/10.1103/PhysRevB.74.064405>.
- [13] R. K. Kaul and A. W. Sandvik, Phys. Rev. Lett. **108**, 137201 (2012), URL <http://link.aps.org/doi/10.1103/PhysRevLett.108.137201>.
- [14] M. S. Block, R. G. Melko, and R. K. Kaul, Phys. Rev. Lett. **111**, 137202 (2013), URL <http://link.aps.org/doi/10.1103/PhysRevLett.111.137202>.
- [15] A. Nahum, P. Serna, J. T. Chalker, M. Ortuño, and A. M. Somoza, Phys. Rev. Lett. **115**, 267203 (2015), URL <http://link.aps.org/doi/10.1103/PhysRevLett.115.267203>.
- [16] Please refer to Supplemental Materials for further details of the numerical simulations.
- [17] A. W. Sandvik, AIP Conf. Proc. **1297**, 135 (2010), URL <http://scitation.aip.org/content/aip/proceeding/aipcp/10.1063/1.3518900>.
- [18] B. I. Halperin, T. C. Lubensky, and S.-k. Ma, Phys. Rev. Lett. **32**, 292 (1974), URL <http://link.aps.org/doi/10.1103/PhysRevLett.32.292>.
- [19] A. Nahum, J. T. Chalker, P. Serna, M. Ortuño, and A. M. Somoza, Phys. Rev. X **5**, 041048 (2015), URL <http://link.aps.org/doi/10.1103/PhysRevX.5.041048>.
- [20] H. Shao, W. Guo, and A. W. Sandvik, **352**, 213 (2016), ISSN 0036-8075, URL <http://science.sciencemag.org/content/352/6282/213>.
- [21] R. K. Kaul, Phys. Rev. B **84**, 054407 (2011), URL <http://journals.aps.org/prb/abstract/10.1103/PhysRevB.84.054407>.
- [22] I. Herbut, *A Modern Approach to Critical Phenomena* (Cambridge University Press, 2010).
- [23] S. Kolnberger and R. Folk, Phys. Rev. B **41**, 4083 (1990), URL <http://link.aps.org/doi/10.1103/PhysRevB.41.4083>.
- [24] I. F. Herbut and Z. Tesčanović, Phys. Rev. Lett. **78**, 980 (1997), URL <http://link.aps.org/doi/10.1103/PhysRevLett.78.980>.
- [25] K. Harada, Phys. Rev. E **84**, 056704 (2011), URL <http://link.aps.org/doi/10.1103/PhysRevE.84.056704>.
- [26] K. Harada, Phys. Rev. E **92**, 012106 (2015), URL <http://link.aps.org/doi/10.1103/PhysRevE.92.012106>.
- [27] S. El-Showk, M. F. Paulos, D. Poland, S. Rychkov, D. Simmons-Duffin, and A. Vichi, Phys. Rev. D **86**, 025022 (2012), URL <http://link.aps.org/doi/10.1103/PhysRevD.86.025022>.
- [28] See Supplemental Material [[put_url_in_"career.bib"_under_"suppmat_cite"](#)] which includes Refs. [29-32].
- [29] O. F. Syljuåsen and A. W. Sandvik, Phys. Rev. E **66**, 046701 (2002).
- [30] A. Dorneich and M. Troyer, Phys. Rev. E **64**, 066701 (2001), URL <http://link.aps.org/doi/10.1103/PhysRevE.64.066701>.
- [31] M. Peskin and D. Schroeder, *An Introduction to Quantum Field Theory*, Advanced book classics (Addison-Wesley Publishing Company, 1995), ISBN 9780201503975, URL <https://books.google.com/books?id=i35LALNOGosC>.
- [32] P. M. Chaikin and T. C. Lubensky, *Principles of Condensed Matter Physics* (Cambridge University Press, 2000).