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Gravity Amplitudes as Generalized Double Copies

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Whenever the integrand of a gauge-theory loop amplitude can be arranged into a form where the BCJ duality between color and kinematics is manifest, a corresponding gravity integrand can be obtained simply via the double-copy procedure. However, finding such gauge-theory representations can be challenging, especially at high loop orders. Here we show that we can instead start from generic gauge-theory integrands, where the duality is not manifest, and apply a modified double-copy procedure to obtain gravity integrands that include contact terms generated by violations of dual Jacobi identities. We illustrate this with three-, four- and five-loop examples in $\mathcal{N} = 8$ supergravity.

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Introduction— Gravity and gauge theories are intimately connected by a double-copy relationship that was first brought to light by the Kawai–Lewellen–Tye (KLT) [1] tree-level amplitude relations, and then fleshed out by the Bern–Carrasco–Johansson (BCJ) duality [2, 3] between color and kinematics. Apart from giving remarkably simple means for obtaining loop-level scattering amplitudes in a broad class of (super)gravity theories [3–8], the duality also addresses the construction of black-hole and other classical solutions [9] including those potentially relevant to gravitational-wave detectors [10], corrections to gravitational potentials [11], the relation of supergravity symmetries to gauge-theory ones [7, 8, 12], and the observation of mysterious “enhanced” ultraviolet cancellations in certain supergravity theories [13]. This duality was later found to be applicable to a wider class of quantum field and string theories [14]. For recent reviews, see Ref. [15].

In the tree-level approximation, manifestly BCJ duality-satisfying representations of amplitudes are known for any multiplicity [16]. However, the off-shell duality remains mysterious despite progress related to the infinite-dimensional Lie algebra underlying BCJ duality [17] and interesting connections to gauge symmetries [18]. At loop level the duality continues to be a conjecture, with many known examples [3–8]. There are also various related double-copy constructions [14, 19].

When known, BCJ-dual representations arguably provide the most efficient approach to finding loop amplitudes in gauge, gravity and other double-copy con-

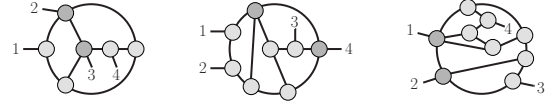


FIG. 1: Sample \mathcal{N}^k -maximal cuts at three, four and five loops. Exposed lines are all on-shell.

structible theories. Gauge-theory BCJ representations are usually found by subjecting an Ansatz to the duality and unitarity constraints. However, as the multiplicity and loop order increases, finding an Ansatz that actually solves the system becomes an increasingly difficult challenge. In particular, no BCJ form of the five-loop four-point amplitude of $\mathcal{N} = 4$ super-Yang–Mills (sYM) theory has yet been found, although a BCJ-dual 1/2-BPS five-loop form factor in this theory has recently been found [20]. In this Letter we explain how to apply the duality to construct gravity integrands from generic (not manifestly BCJ-dual) gauge-theory integrands.

Contact Terms from BCJ Duality— Our derivation of gravity amplitudes uses the method of maximal cuts [21], a refinement of generalized unitarity [22]. In this method, amplitudes are constructed from cuts that reduce integrands to sums of products of tree-level amplitudes, as illustrated in Fig. 1. The cuts are organized according to the number k of propagators that remain off shell. We first find an expression whose maximal ($k = 0$) cuts (MCs) are correct, then correct it such that all next-to-maximal ($k = 1$) cuts (NMCs) are correct and system-

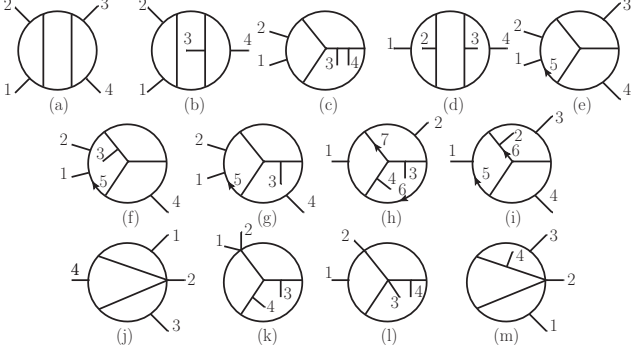


FIG. 2: Diagrams (a)-(i) define the three-loop four-point amplitude of $\mathcal{N} = 4$ sYM theory; diagrams (j)-(m) are the additional contact terms needed for $\mathcal{N} = 8$ supergravity.

atically proceed through the N^k MCs, until no further corrections exist. The maximal k depends on the power counting of the theory and on choices made at earlier levels. The corrections coming from N^k MC cuts are assigned to contact terms corresponding to each cut. For example, the first cut in Fig. 1 determines the double-contact diagram (l) in Fig. 2. The contact terms are taken off shell in a manner that preserves the diagram symmetry. This process introduces an ambiguity that can then be absorbed into changes in subsequent $(k+1)$ -level contact terms.

Generically, an L -loop m -point gravity amplitude can be organized in terms of diagrams that have cubic and higher-point (contact term) vertices,

$$\mathcal{M}_m^{(L)} = i^L \left(\frac{\kappa}{2}\right)^{m-2+2L} \sum_{j \in \Gamma_L} \int \frac{d^L p}{(2\pi)^{LD}} \frac{1}{S_j} \frac{N_j}{\prod_{\lambda_j} d_j^{(\lambda_j)}}, \quad (1)$$

where Γ_L is the set of all L -loop m -point graphs with labeled external lines, λ_j labels the edges of graph j , and $1/d_j^{(\lambda_j)}$ are the corresponding propagators. For example, for the three-loop four-point $\mathcal{N} = 8$ amplitude, the diagrams needed are given in Fig. 2. The symmetry factors S_j remove over-counts arising from automorphisms of each graph. The integration is over L independent D -dimensional loop momenta. The gravity numerators N_j will be obtained through our double-copy procedure, in a manner that respects the graph's symmetries.

BCJ duality [2, 3] is manifest if the kinematic numerators n of a gauge-theory amplitude's graphs have the same algebraic properties as its color factors c . The basic Jacobi identity of the gauge-group structure constants can be embedded in arbitrary multiloop diagrams and leads to relations between the color factors c of triplets $\{A, B, C\}$ of graphs. For generic theories with only fields in the adjoint representation, the duality implies the functional relations

$$c_A + c_B + c_C = 0 \iff n_A^{\text{BCJ}} + n_B^{\text{BCJ}} + n_C^{\text{BCJ}} = 0, \quad (2)$$

for all such triplets. Generalized gauge transformations—

shifts of n that cancel in the amplitude because of color Jacobi relations—bring the kinematic numerators to this form. Numerators that satisfy Eq. (2) lead directly to gravity integrands by replacing color factors of the gauge-theory amplitude with the kinematic numerators of a second gauge-theory amplitude [2, 3, 23].

At loop level, the functional relations (2) are typically solved through Ansätze obeying additional simplifying assumptions [4]. While the resulting expressions are compact, sufficiently unconstrained Ansätze can be prohibitively large. This motivates us to find an alternative efficient double-copy construction of gravity integrands that evades the need for explicit duality-satisfying representations of gauge-theory amplitudes.

The starting point of our construction is a “naive double copy” of amplitudes of two possibly distinct gauge theories, written in terms of cubic diagrams with numerator factors n_i and \tilde{n}_i for which duality (2) is not manifest:

$$N_i = n_i \tilde{n}_i \quad (\text{cubic only}). \quad (3)$$

This naive double copy automatically satisfies all the gravity MCs and NMCs, because BCJ duality always holds for on-shell four-point tree amplitudes [2]. However, because BCJ duality is not manifest, Eq. (3) is not the complete answer, as can be checked by evaluating N^2 MCs. We need a systematic determination of the contact-term corrections that lead to the correct N^k MCs, matching the result that could be independently obtained by less efficient methods, such as applying KLT relations directly on the cuts.

The key observation is that the additional contact contributions should be related to the violation of the kinematic Jacobi relations (2) by the gauge-theory amplitude numerators. Together with generalized gauge invariance and properties of BCJ numerators in the generalized cuts, this observation provides the building blocks for the construction of the missing terms. To describe their construction we need a labeling of a general cut \mathcal{C} , made of q factors of $4 \leq m$ -point tree amplitudes. We choose an ordering, $1, \dots, q$, of these amplitude factors, an ordering of the graphs contributing to each such factor and label numerators by the labels of the graph in each amplitude factor, n_{i_1, i_2, \dots, i_q} with i_1 running over the graphs in the first amplitude factor, *etc.* For example, the numerators of the nine diagrams (some of which can vanish) appearing in the first cut in Fig. 1 are labeled by n_{i_1, i_2} , where i_1 and i_2 each runs over the three graphs in a four-point tree amplitude.

For every propagator of every graph contributing to a generalized cut there is a kinematic Jacobi relation. Choosing an ordering for the propagators in every graph, we define the violation of the kinematic Jacobi relation on the λ_A -th propagator of graph A of v -th amplitude

factor:

$$J_{i_1, \dots, i_{v-1}, \{A, \lambda_A\}, i_{v+1}, \dots, i_q} = n_{i_1, \dots, i_{v-1}, A, i_{v+1}, \dots, i_q} + n_{i_1, \dots, i_{v-1}, B, i_{v+1}, \dots, i_q} + n_{i_1, \dots, i_{v-1}, C, i_{v+1}, \dots, i_q}, \quad (4)$$

where the graphs B and C are connected to graph A by the color Jacobi relation on the λ_A -th propagator of graph A . (The relative signs between terms should match those of the corresponding color Jacobi relation.) We can define, in the obvious way, violations of multiple Jacobi relations; they are linear combinations of these.

Not all such J s are independent. First, there is a triple over-count, since the same J can be defined for each of the three diagrams connected by a Jacobi relations. Furthermore, there are linear relations, some from the definition (4) of J in terms of n and some due to BCJ amplitude relations [2]. Tree-level examples of these latter J relationships are derived in Refs. [24, 25].

With this notation, the generalized gauge shift, Δ , that relates arbitrary kinematic numerators n to BCJ ones is

$$n_{i_1, i_2, \dots, i_q} = n_{i_1, i_2, \dots, i_q}^{\text{BCJ}} + \Delta_{i_1, i_2, \dots, i_q}, \quad \Delta_{i_1, i_2, \dots, i_q} = \sum_v \sum_j d_{i_v}^{(v,j)} \alpha_{i_1, \dots, \hat{i}_v, \dots, i_q}^{(v,j)}, \quad (5)$$

where the hat notation indicates that the index is omitted, and j runs over the labels of the ordered set of inverse propagators of the graph i_v of the v -th amplitude factor and $d_{i_v}^{(v,j)}$ is the j -th element of this set. The Δ are constrained so they do not alter the gauge-theory cut integrand [2, 3, 23]

$$\sum_{i_1, \dots, i_q} \frac{\Delta_{i_1, i_2, \dots, i_q} c_{i_1, i_2, \dots, i_q}}{D_{i_1} \dots D_{i_q}} = 0, \quad (6)$$

where D_{i_v} is the product of all inverse propagators of the graph i_v in the v -th amplitude factor.

Using Eq. (5), the gravity cut is given by

$$\mathcal{C}_G = \sum_{i_1, \dots, i_q} \frac{(n_{i_1, i_2, \dots, i_q}^{\text{BCJ}})^2}{D_{i_1} \dots D_{i_q}} = \sum_{i_1, \dots, i_q} \frac{n_{i_1, i_2, \dots, i_q}^2}{D_{i_1} \dots D_{i_q}} + \mathcal{E}_G, \quad \mathcal{E}_G = - \sum_{i_1, \dots, i_q} \frac{\Delta_{i_1, i_2, \dots, i_q}^2}{D_{i_1} \dots D_{i_q}}. \quad (7)$$

For simplicity we have taken the two gauge-theory numerators to be identical. The key to the cancellation of the $n^{\text{BCJ}} \cdot \Delta$ cross terms is Eq. (6), given that n^{BCJ} satisfy Eq. (2). We stress that this argument relies only on the existence, but not explicit construction, of tree-level BCJ representations used in the generalized cuts.

To express \mathcal{E}_G in terms of J s requires inverting, on a case by case basis, the relations $J(\Delta)$ obtained by plugging Eqs. (5) in Eq. (4). Since not all J s are independent, only some gauge shifts can be determined. The remaining ones preserve the BCJ form of the gauge-theory cut.

The resulting expression in a complete J -basis superficially has spurious singularities; they may be eliminated explicitly by using the remaining gauge freedom.

We illustrate the general construction described here by discussing in some detail the N^2 MCs made of two four-point amplitudes. The numerators are labeled as n_{i_1, i_2} where i_1 and i_2 run over the three graphs in the first and second four-point amplitude, respectively. Each graph has a single propagator; the second upper index on inverse propagators is therefore redundant so we do not include it. The generalized gauge transformation (5) is

$$\Delta_{i_1, i_2} = d_{i_1}^{(1)} \alpha_{i_2}^{(1)} + d_{i_2}^{(2)} \alpha_{i_1}^{(2)}. \quad (8)$$

After use of momentum conservation $\sum_{i_1} d_{i_1}^{(v)} = 0$, this gives the violations of kinematic Jacobi relations as

$$J_{\{u_1, 1\}, i_2} \equiv \sum_{i_1} n_{i_1, i_2} = d_{i_2}^{(2)} \sum_{i_1} \alpha_{i_1}^{(2)}, \quad J_{i_1, \{u_2, 1\}} \equiv \sum_{i_2} n_{i_1, i_2} = d_{i_1}^{(1)} \sum_{i_2} \alpha_{i_2}^{(1)}. \quad (9)$$

The three-fold degeneracy of J implies independence on the labels u_1 or u_2 . We see that only particular combinations of gauge shifts $\alpha_i^{(v)}$ are determined. We also note that $J_{i_1, \{u_2, 1\}}/d_{i_1}^{(1)}$ and $J_{\{u_1, 1\}, i_2}/d_{i_2}^{(2)}$ are independent of the graph in the second and first amplitude, respectively.

Combining Eqs. (7), (8) and (9), the additional contact term in N^2 MC with two four-point amplitudes is

$$\mathcal{E}_G^{4 \times 4} = -2 \sum_{i_1, i_2} \alpha_{i_1}^{(1)} \alpha_{i_2}^{(2)} = -2 \frac{J_{\{1, 1\}, 1} J_{1, \{1, 1\}}}{d_1^{(1)} d_1^{(2)}}. \quad (10)$$

The two denominators cancel against the numerator, yielding a local expression.

Similar but more involved analysis gives formulae correcting any cut of a naive double copy. Unlike the example above, the nontrivial constraints between J s as well as the requirement that generalized gauge transformations should not modify gravity amplitudes require nontrivial disentangling. At N^2 MC level the extra term corresponding to a five-point contact term is

$$\mathcal{E}_G^5 = -\frac{1}{3} \sum_{i=1}^{15} \frac{J_{\{i, 1\}} J_{\{i, 2\}}}{d_i^{(1,1)} d_i^{(1,2)}}, \quad (11)$$

where the sum runs over all 15 cubic graphs of the five-point amplitude and $d_i^{(1,1)}$ and $d_i^{(1,2)}$ are the two propagators of graph i . This formula is a symmetric loop-generalization of the one given in Ref. [25] for tree amplitudes.

At N^3 MC level, the extra terms are more involved. For example, the correction terms relevant to three four-point

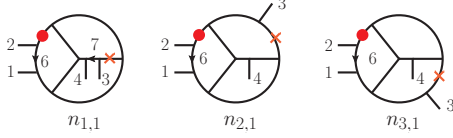


FIG. 3: The three diagrams whose kinematic numerators contribute to $J_{\{1,1\},1}$. The thick shaded (red) cross marks the off-shell legs participating in the dual Jacobi relation. The shaded (red) dot indicates the off-shell leg of the second amplitude factor.

amplitude cuts are:

$$\mathcal{E}_G^{4 \times 4 \times 4} = 2 \frac{J_{1,1}^{(1)} J_1^{(2,3)} + J_{1,1}^{(2)} J_1^{(1,3)} + J_{1,1}^{(3)} J_1^{(1,2)}}{d_1^{(1)} d_1^{(2)} d_1^{(3)}} \quad (12)$$

$$- \sum_{i_3} \frac{2 J_{1,i_3}^{(1)} J_{i_3}^{(2)}}{d_1^{(1)} d_1^{(2)} d_{i_3}^{(3)}} - \sum_{i_2} \frac{2 J_{i_2,1}^{(1)} J_{i_2}^{(3)}}{d_1^{(1)} d_{i_2}^{(2)} d_1^{(3)}} - \sum_{i_1} \frac{2 J_{i_1,1}^{(2)} J_{i_1}^{(3)}}{d_{i_1}^{(1)} d_1^{(2)} d_1^{(3)}},$$

where we used the shorthand notations $J_{1,i_3}^{(1)} \equiv J_{\{1,1\},1,i_3}$, *etc.* and $J_{i_3}^{(1,2)} \equiv \sum_{i_1} J_{i_1,\{1,1\},i_3}$, *etc.* As in Eq. (8), we have suppressed the second upper index on $d_i^{(v,j)}$ because it takes a single value. We have also derived general formulae for cuts with 4×5 and 6-point amplitude factors, which we will present together with the $\mathcal{N} = 8$ supergravity five-loop four-point integrand [26].

One subtlety is that, in special cuts, momentum conservation can force on shell an internal propagator of a tree amplitude, leading to a $1/0$ divergence. This is associated with bubble on external leg or tadpole diagrams, which in dimensionally regulated massless theories integrate to zero. In $\mathcal{N} = 8$ supergravity the simplest prescription is to take such contributions to vanish whenever a corresponding numerator vanishes and, if the cancellation occurs between terms (which can leave finite pieces behind) to take advantage of the asymmetry of the formulae to choose a labeling that avoids this situation.

Locality and dimension counting imply that contact terms become simpler as the level increases. For example, the contact-term numerators at the \mathcal{N}^6 MCs level in the five-loop four-point amplitude of $\mathcal{N} = 8$ supergravity are just a linear combination of s^2 , st and t^2 . Thus, in practical calculations, it is more efficient to determine the high-level contact terms by numerically evaluating the generalized cuts.

Examples— The three-loop four-point amplitude of $\mathcal{N} = 8$ supergravity is well studied [3, 4, 27, 28] and serves as a useful illustration. We will reconstruct it here from the corresponding $\mathcal{N} = 4$ sYM amplitude of Ref. [27] whose numerators are displayed in Table I with the momentum labeling in Fig. 2(a)-(i) (corresponding to the one of Ref. [3]). As in the normalization of Ref. [3], an overall factor of stA_4^{tree} is removed. Following our procedure, the $\mathcal{N} = 8$ supergravity numerators of diagrams

TABLE I: A non-BCJ form of the three-loop four-point $\mathcal{N} = 4$ sYM from Ref. [27]. We define $\tau_{ij} = 2p_i \cdot p_j$, $s = (p_1 + p_2)^2$, $t = (p_2 + p_3)^2$ and $u = (p_1 + p_3)^2$.

Graph	$\mathcal{N} = 4$ sYM numerators.
(a)-(d)	s^2
(e)-(g)	$s(p_5^2 + \tau_{45})$
(h)	$s(\tau_{26} + \tau_{36}) - t(\tau_{17} + \tau_{27}) + st$
(i)	$s(p_5^2 + \tau_{45}) - t(p_5^2 + \tau_{56} + p_6^2) - (s - t)p_6^2/3$

(a)–(l) are squares of the corresponding $\mathcal{N} = 4$ sYM ones:

$$N_{(x)}^{\mathcal{N}=8} = n_{(x)}^2, \quad x \in \{\text{a-i}\}. \quad (13)$$

Contact diagrams can appear only at the \mathcal{N}^2 MC level. There are a total of 62 possible independent such contact terms. Of these, all but the four diagrams (j)–(m) in Fig. 2 vanish. As an example, consider the contact diagram in Fig. 2(l), composed of two four-point vertices. We obtain it from Eq. (10). First, we identify the nine cubic diagrams that contribute to it (some are vanishing) and pick one whose numerator we label as $n_{1,1}$; we choose diagram (c) in Fig. 2. The two J -functions are calculated by relabeling the appropriate numerators to the labels of Fig. 3. For example, $J_{\{1,1\},1}$ is obtained from the $\mathcal{N} = 4$ sYM numerators of the three diagrams shown in Fig. 3,

$$n_{1,1} = s^2, \quad n_{2,1} = s(t + \tau_{26} + \tau_{36}), \quad n_{3,1} = s(u - \tau_{36}), \quad (14)$$

corresponding to relabeling of diagrams (c) and (g) in Fig. 2. Summing and applying momentum conservation gives $J_{\{1,1\},1} = s\tau_{26}$. Similarly, $J_{1,\{1,1\}} = s\tau_{37}$. With these labels the two off-shell inverse propagators are τ_{26} and τ_{37} , so that from Eq. (10) the $\mathcal{N} = 8$ supergravity contact term numerator for diagram (l) is

$$N_{(l)}^{\mathcal{N}=8} = -2 \frac{J_{\{1,1\},1} J_{1,\{1,1\}}}{\tau_{26} \tau_{37}} = -2s^2. \quad (15)$$

The other three independent contact terms corresponding to diagrams (j), (k) and (m), can similarly be obtained from Eq. (11), with the result

$$N_{(j)}^{\mathcal{N}=8} = -\frac{1}{9}(s - u)^2, \quad N_{(k)}^{\mathcal{N}=8} = N_{(m)}^{\mathcal{N}=8} = -2s^2. \quad (16)$$

All nonvanishing contact terms are relabelings of these.

We have also computed the four-loop four-point amplitude of $\mathcal{N} = 8$ supergravity using the contact-term method described above. The results are included as a mathematica attachment [29]. Power counting dictates that this $\mathcal{N} = 8$ amplitude can have no contact terms beyond level $k = 4$, which we checked explicitly.

Generating contact term diagrams by collapsing the propagators of the cubic-contributions in all possible ways, we find the result is surprisingly simple. The vast majority of contacts, 2353 of 2621, vanish outright due

to vanishing J 's. (In this count we drop cuts where a leg of a tree-amplitude is directly sewn to another one of the same tree, since these do not appear in $\mathcal{N} = 8$ supergravity.) Even the nonvanishing 268 contact terms are remarkably simple. For example, as for the three-loop cut (1), we evaluated the four-loop contact term given by the second cut in Fig. 1 (corresponding to the 48th N^2 MC in the attachment [29]) using Eq. (10) and found that its numerator is $(-2s^4)$. All remaining contact terms are included in the attached file [29].

By five loops, even promising methods may prove ineffective due to a combinatorial proliferation of terms. We therefore perform extensive checks at five loops to ensure that the methods presented here remain practical. For example, starting from the $\mathcal{N} = 4$ sYM expression in Ref. [30], we find that the overwhelming fraction of contact terms through N^6 MCs are zero, such as the rather nontrivial third cut in Fig. 1. This is consistent with lower loops and enormously simplifies the construction and structure of the $\mathcal{N} = 8$ supergravity five-loop four-point amplitude, to be described elsewhere [26].

Conclusions and Outlook—Some open problems remain. The power counting of the gravity integrands given by our modified double-copy construction depends on the choices of numerators in the sYM amplitude; generic representations of the latter typically lead to higher-than-optimal power counting of the former. For example, the known five-loop $\mathcal{N} = 4$ sYM integrand is of this type. Gauge-theory integrands designed to minimize the power count in the double copy, in particular of its naive part, are therefore desirable; finding them is an important problem.

Although we focused here on $\mathcal{N} = 8$ supergravity and $\mathcal{N} = 4$ sYM, the construction generalizes in the obvious way to different gauge and other theories with adjoint matter which obey BCJ duality, and thus to all gravitational and non-gravitational [14] double-copy theories obtained from them. If the two theories in the double copy are distinct, the contact terms are obtained by simply replacing in our expressions $J_i J_j \rightarrow (J_i \tilde{J}_j + \tilde{J}_i J_j)/2$, where J and \tilde{J} are the violations of the kinematic Jacobi relations in each theory.

Similar ideas to the ones presented in this Letter should hold in all double-copy theories whose single-copies include fields in the fundamental representation of the gauge group [6, 8]. Our results suggest that it may be possible to generically convert any gauge-theory classical solution to a gravitational one without choosing special generalized gauges. We expect that the ideas presented in this paper will be useful not only for investigating the ultraviolet behavior of perturbative quantum gravity but also for understanding general physical properties of gravity theories.

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