# Parafermionic Wires at the Interface of Chiral Topological States

Luiz H. Santos and Taylor L. Hughes

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# Parafermionic wires at the interface of chiral topological states

Luiz H. Santos[1] and Taylor L. Hughes[1]

[1] *Department of Physics and Institute for Condensed Matter Theory, University of Illinois at Urbana-Champaign, 1110 West Green Street, Urbana, Illinois, 61801-3080, USA*

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We explore a scenario where local interactions form one-dimensional gapped interfaces between a pair of distinct chiral two-dimensional topological states – referred to as phases 1 and 2 – such that each gapped region terminates at a domain wall separating the chiral gapless edge states of these phases. We show that this type of T-junction supports point-like fractionalized excitations obeying parafermion statistics, thus implying that the one-dimensional gapped interface forms an effective topological parafermionic wire possessing a non-trivial ground state degeneracy. The physical properties of the anyon condensate that gives rise to the gapped interface are investigated. Remarkably, this condensate causes the gapped interface to behave as a type of anyon "Andreev reflector" in the bulk, whereby anyons from one phase, upon hitting the interface, can be transformed into a combination of reflected anyons and outgoing anyons from the other phase. Thus, we conclude that while different topological orders can be connected via gapped interfaces, the interfaces are themselves topological.

## INTRODUCTION

Topological phases (TPs) of matter in two dimensions (2D) are often characterized by a "bulk-boundary" correspondence. Bulk properties such as a topological band structure, quasiparticles exhibiting fractional statistics, or topological ground state degeneracy on manifolds with non-zero genus, go hand in hand with an associated set of boundary/interface states where a TP meets a different one such as the vacuum. [1]

TPs appear in two general classes: symmetry protected [2–15], or those that have "intrinsic" topological order [16]. There are several important distinctions between these classes, e.g., differing constraints on the ability to open a gap in the edge state spectrum. For the first class, gapped boundaries can exist when the symmetry is broken explicitly or spontaneously. In the latter, interface states with non-vanishing chirality cannot be completely gapped, and, surprisingly, even in the absence of any symmetries, some interfaces with *vanishing* chirality cannot be completely gapped either.[18] This observation may directly impact experiment since such an ungappable edge may exist in the $\nu = 2/3$ fractional quantum Hall effect, or at the interface between two fractional quantum Hall states with, e.g., filling factors $\nu = 1/3$ and $\nu = 1/5$. The latter interface cannot be gapped by any local interaction, essentially due to the completely incompatible bulk properties of the two TPs.

In this article we focus on the complementary effect that allows disparate TPs to support *gapped* interfaces (GIs), as they provide a domain for a wide-range of interesting physics. The existence of such an interface requires that a local gapping condition be satisfied [see discussion around Eq. (2)], which, physically amounts to the allowed formation of an "anyon condensate" (AC) at the interface. It has been established, for two dimensional Abelian TPs, that each AC is in one-to-one

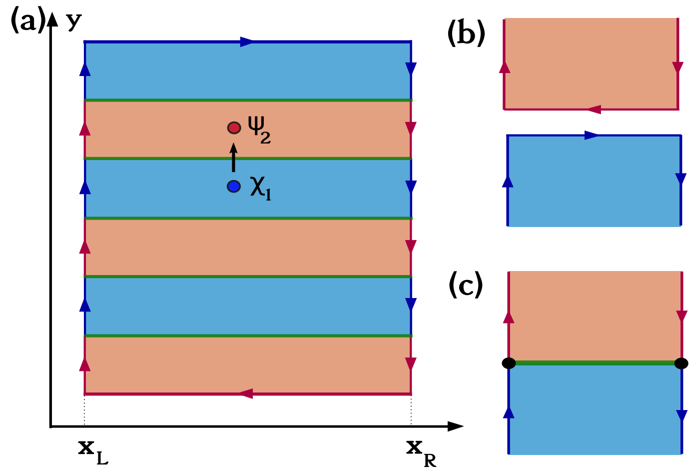


FIG. 1: (color online) (a) An array of phases 1 (blue) and 2 (brown) showing the gapped interfaces along the x direction (green). Each gapped interface constitutes an AC that acts as an anyon Andreev reflector whereby certain quasiparticles of phase 1 are transformed into quasiparticles of phase 2 (and vice-versa) as they cross the interface; and quasi-particles can be reflected into different quasiparticle types. (b) Original chiral gapless edge states of the two phases. (c) Parafermion zero modes (black dots) are located at the T-junctions where the end points of the gapped interface define a domain wall between the chiral gapless edge modes of phases 1 and 2.

correspondence with a mathematical structure called a "Lagrangian subgroup", [17–19] which is a subset $\mathcal{M}$ of the set of anyons wherein (i) all quasiparticles have mutual bosonic statistics, and (ii) every quasiparticle not in $\mathcal{M}$ has non-trivial statistics with at least one quasiparticle of $\mathcal{M}$. Hence, the simultaneous condensation of the quasiparticles in $\mathcal{M}$ is allowed by (i), and will confine *all* the anyons of the theory by (ii). Of great interest are configurations where inequivalent ACs, corresponding to inequivalent choices of $\mathcal{M}$, are formed in adjoining regions of a topological interface. Indeed, domain walls between these gapped regions have been shown to host non-Abelian defect bound states with parafermionic statistics.[20–27] Such bound states could be used as a platform for realizing topological quantum computation. [28]

In this letter we characterize a family of 1D gapped topological systems that can be formed at the interface between different 2D Abelian TPs. For our examples, we choose single-component chiral phases characterized by the topological invariants (K-matrices) $k_1$ and $k_2$, respectively. Hereafter we refer to these as phase 1 and phase 2. If these phases arise from charge conserving quantum Hall states then we have $k_{1,2} = \nu_{1,2}^{-1}$, where $\nu$ is the filling fraction that measures the Hall conductance in fundamental units. More generally, for systems without U(1) (electromagnetic) charge conservation symmetry, e.g., chiral spin liquids [1, 29], $k_{1,2}$ count the number of distinct bulk quasiparticle types in each phase, and give the topological ground state degeneracy $g^{k_i}$ of each system defined on a spatial manifold of genus $g$. For our discussion we will adopt the interface geometry in Fig. 1. The bulk TPs share a GI with each other, and they have a boundary with the vacuum that contains propagating chiral edge modes, such that the GI terminates at points separating gapless edge states of distinct phases.

Our main finding is that such an interface forms a topologically non-trivial, 1D gapped system with a degenerate ground state manifold associated with parafermionic end states. We stress that, instead of being located at the domain walls between different GIs, the parafermions discussed here are situated at domain walls between gapless edge states of phases 1 and 2, as shown in Fig. 1. Therefore this physical scenario departs significantly from those of Refs. [20–24, 26, 27], and more closely matches the setup of Ref. 30, though here we are focused more on what is happening in the bulk, rather than the edge as in their discussion. Ultimately, our results identify that, while one can find gapped interpolations between 2D phases with different topological order, these are not trivial gapped regions; they are instead *topological* themselves.

We shall support our result with a bosonization description of the edge containing a pair of counter propagating modes from the two phases. We will (1) construct the explicit form of the local, gap-opening interaction, (2) provide a description of the interface AC, (3) discuss the onset of the topologically degenerate ground state manifold associated with the expectation value of a non-local operator, and (4) discuss the connection between bulk confinement-deconfinement transitions, edge-state transitions, and the bound parafermion modes.

***1. Luttinger liquid description of the interface*** – In Fig. 1(a), we consider an array of 2D topological states in phase 1 (blue) and phase 2 (brown), surrounded by the vacuum. As shown in the Supplementary Material (SM), the most generic gappable interface for one-component states is characterized by $k_1 = pn^2$ and $k_2 = pm^2$, where $p, m, n \in \mathbb{Z}_+$. The low energy Lagrangian of each interface along the x-direction is given by

$$\mathcal{L}_x = \frac{1}{4\pi}\,\partial_t\,\Phi^T\,K\,\partial_x\,\Phi - \frac{1}{4\pi}\,\partial_x\,\Phi^T\,V\,\partial_x\,\Phi - \mathcal{H}_{\rm int}[\Phi]\,, \tag{1a}$$

$$K = \begin{pmatrix} p\,n^2 & 0 \\ 0 & -pm^2 \end{pmatrix}\,, \quad \Phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}\,, \tag{1b}$$

where $\phi_{1,2}$ represent the right- and left-moving edge modes originating from phases 1 and 2, V is a velocity matrix, and $\mathcal{H}_{int}[\Phi]$ is a local interaction discussed below. The edge fields satisfy commutation relations $[\partial_x\,\phi_i(t,x), \phi_j(t,y)] = -2\,\pi\,\mathrm{i}\,K_{ij}^{-1}\,\delta(x-y)$. To simplify our discussion we will choose $m = 1$ and provide the details for $m > 1$ in the SM. This case is also the most experimentally relevant as it includes interfaces between a $\nu = 1$ integer quantum Hall state with, e.g., a $\nu = 1/9$ fractional quantum Hall state when $p = 1, n = 3$. [31, 32]

With appropriate conventions, the quasiparticle excitations on the edge are created by the vertex operators $\exp\left(\mathrm{i}\,\ell^T\,\Phi\right)$, where $\ell$ is an integer vector. The exchange statistics associated with taking a quasiparticle $\ell_a$ adiabatically around another quasiparticle $\ell_b$ is given by the statistical phase $S_{ab} = e^{\mathrm{i}\,\theta_{ab}} = e^{\mathrm{i}\,2\pi\,\ell_a^T\,K^{-1}\,\ell_b}$, and the (topological) spin of each quasiparticle is given by the self-statistics phase $h_a = e^{\mathrm{i}\,\pi\,\ell_a^T\,K^{-1}\,\ell_a}$. Local excitations are identified with $\psi = e^{\mathrm{i}\,\Lambda^T\,K\,\Phi}$, where $\Lambda$ is an integer vector.

In Eq. (1a) $\mathcal{H}_{\rm int}[\Phi] = -J\,\cos\left(\Lambda^T\,K\,\Phi\right)$ $(J > 0)$ is a local gap opening interaction parametrized by the integer null vector $\Lambda^T = (a, b)$ satisfying: [33]

$$0 = \Lambda^T\,K\,\Lambda = p\,(a^2\,n^2 - b^2)\,. \tag{2}$$

$\Lambda = (1, n)$ is a *primitive* solution[34] of (2) representing the interaction between a single local operator $\psi_1 = e^{\mathrm{i}\,p\,n^2\,\phi_1}$ of phase 1 with $n$ local operators $\psi_2^\dagger = e^{-\mathrm{i}\,p\,\phi_2}$ of phase 2:

$$\mathcal{H}_{\rm int} = -J\cos\left(n\Theta\right) \propto -J\,\psi_1\,\underbrace{\psi_2^\dagger\,...\,\psi_2^\dagger}_{n} + \text{h.c.}\,, \tag{3}$$

where $\Theta(x) \equiv p\,n\,\phi_1(x) - p\,\phi_2(x)$. We explicitly show in the SM that one can always tune $V$ to make interactions of the form $\mathcal{H}_{int}$ relevant.

This interaction generates an AC at the interface as we will now describe. In phase 1 (phase 2), there are $p\,n^2$ $(p)$ quasiparticle-types labeled $\varepsilon_1^{a_1}$ $(\varepsilon_2^{a_2})$, $a_1 = 1, ..., p\,n^2$ $(a_2 = 1, ..., p)$. The set of anyons forms a discrete lattice[26, 35–39], whereby anyons are topologically indistinguishable upon the attachment of local quasiparticles $\psi_i = \varepsilon_i^{k_i}$, $i = 1, 2$. In the context of a Laughlin fermionic (bosonic) state, $\psi_i$ represents the local fermion (boson) of the $i$-th phase.

Now we note that in phase 1 the anyon subset $\{\varepsilon_1^{p\,n\,x}, x = 1, ..., n\}$ contains mutual bosons or fermions with spin $h(\varepsilon_1^{p\,n\,x}) = e^{i\pi p x^2}$. Furthermore, the quasiparticle $\chi_1 \equiv \varepsilon_1^{p\,n}$ has the same spin as the local excitation $\psi_2$ of phase 2, i.e., they are both bosons or fermions depending on the parity of $p$. Physically this implies that the composite quasiparticle $\sigma \equiv \chi_1\,\psi_2^\dagger$ is a boson that can condense, and generate a fully GI between phases 1 and 2. This condensation process, mathematically, is a consequence of the relation $k_1/k_2 = n^2 \in \mathbb{Z}^2$, which allows for the existence of a $pn$-dimensional Lagrangian subgroup $\mathcal{M}$ containing $\sigma$.

Importantly, the interaction (3), which involves one local operator of phase 1 and $n$ of phase 2, breaks the $U(1) \times U(1)$ particle conservation symmetries of each phase down to $\mathbb{Z}_1 \times \mathbb{Z}_n$, where $\mathbb{Z}_1$ means no symmetry. Hence (3) is invariant under $S_\beta$: $\psi_1 \to \psi_1$, $\psi_2 \to \psi_2 e^{i 2\pi\beta/n}$, $\beta \in \mathbb{Z}$. If the phases began with a $U(1)_{EM}$ electromagnetic charge conservation symmetry, then this interaction breaks (preserves) the symmetry when the charge vector is $t^T = (1, 1)$ ($t^T = (n, 1)$). This discrete symmetry, it turns out, plays a fundamental role in the identification of the GI as a topological parafermion wire similar to those studied in Refs. 40–46.

The topological properties of the GI can be more transparently revealed by a description in the zero correlation length limit $J \to \infty$, where the interface Hamiltonian *density* is given solely by Eq. (3), thus leading to a GI as depicted in Fig. 1(c). In this limit there are $n$ degenerate ground states $[\Theta_q = 2\pi q/n,\ q = 1, ..., n]$ associated with the vacuum expectation value of the composite bosonic operator $\sigma(x) = \chi_1(x)\,\psi_2^\dagger(x) = e^{i\Theta(x)}$, which represents a bound state of $\chi_1 = e^{i\,pn\phi_1}$ with $\psi_2^\dagger = e^{-i\,p\phi_2}$:

$$\forall x: \quad \sigma(x)\,|\,\Psi_q\,\rangle = \omega^q\,|\,\Psi_q\,\rangle\,, \quad \omega \equiv e^{i\,\frac{2\pi}{n}}\,, \quad q = 1, ..., n\,. \tag{4}$$

The eigenstates (4) are in direct correspondence with symmetry broken ground states of the ferromagnetic, zero correlation length limit of an $n$-state clock model, where $\sigma$ naturally acquires the interpretation of a clock operator satisfying $\sigma^n = 1$ and $\sigma^\dagger = \sigma^{n-1}$. However, while it would seem possible to distinguish among the degenerate states by a measurement of $\sigma(x)$, $\langle\,\Psi_q\,|\sigma(x)|\,\Psi_{q'}\,\rangle = \omega^q\delta_{q,q'}$ [which is equivalent to adding a perturbation $\delta\,\mathcal{H} = \delta\,\cos(\Theta)$ to the Hamiltonian (3)], the fact that $\sigma(x)$ is a *non-local* operator does not permit such a local distinction, and is a hallmark of the topological nature of the system. With this in mind, the eigenstates (4) indicate a degenerate symmetry breaking manifold associated with the global symmetry $\mathcal{S} \equiv S_{(\beta=-1)} = e^{-\frac{i}{n}\int_{x_L}^{x_R} dx\,\partial_x\phi_2(x)}$ whereby $\mathcal{S}^\dagger\sigma(x)\mathcal{S} = \omega\,\sigma(x)$, for $x_L \le x \le x_R$.

The topological nature of this system can be made explicit by changing from the clock to the parafermionic representation: [47]

$$\alpha(x) \equiv \sigma(x)\,e^{-\frac{i}{n}\int_{x_L}^{x} dz\,\partial_z\phi_2(z)} \equiv \sigma(x)\,\xi(x)\,, \tag{5a}$$

$$\alpha(x)\,\alpha(y) = \alpha(y)\,\alpha(x)\,e^{i\frac{2\pi}{n}\operatorname{sgn}(y-x)}\,, \tag{5b}$$

whereby $\alpha(x)$ is a product of the "order", $\sigma$, and the "disorder", $\xi$, operators. Importantly, the boundary parafermion operators $\alpha(x_L) = \sigma(x_L)$, $\alpha(x_R) = \sigma(x_R)\,e^{-\frac{i}{n}\int_{x_L}^{x_R} dx'\,\partial_{x'}\phi_2(x')}$ commute with the Hamiltonian (3), and the degenerate ground state manifold is given by the eigenstates of the non-local operator $\mathcal{A} = \alpha^\dagger(x_L)\,\alpha(x_R)$: $\mathcal{A}|\,\Omega_a\,\rangle = \omega^a|\,\Omega_a\,\rangle$, $a = 1, \ldots, n$, where the $|\,\Omega_a\,\rangle$ are linear combinations of the $|\,\Psi_q\,\rangle$.

***2. Edge transitions*** – As indicated in Figs. 1(b) and 1(c), the formation of the GI prevents the propagation of the edge modes in the x-direction. While any point $x \in (x_L, x_R)$ establishes a domain wall between distinct gapped bulk TPs, the end states located at $x = x_{L,R}$ correspond to domain walls between distinct *gapless* edge states. In fact we shall explicitly demonstrate the existence of parafermion operators situated at the edge transitions. These parafermions are non-trivial operators with quantum dimensions $\sqrt{n}$, which is a direct manifestation of the $n$-fold degeneracy of the GI. Similar physics was first explored in Ref. [30], which focuses on transitions between distinct edge terminations of the *same* bulk phase; our focus instead is on the interface between *different* bulk phases, which will have an accompanying transition on the edge.

An important feature of the gappable topological interface is that the bulk phases 1 and 2 can be related to each other by the confinement (or deconfinement) of a 2D $\mathbb{Z}_n$ gauge theory. In order to see this, imagine phase 2 is coupled to a $\mathbb{Z}_n$ gauge theory in its deconfined phase. Let the gauge field $\alpha_\mu$ describe the excitations of phase 2, and $(a_\mu, b_\mu)$ the excitations of the $\mathbb{Z}_n$ gauge theory. Hence, the coupled system is described by the Abelian Chern-Simons theory:

$$\mathcal{L}_{2D} = \frac{1}{4\pi}\,\varepsilon^{\mu\nu\lambda}\,c_\mu^I\,\bar{K}_{IJ}(p, n)\,\partial_\nu\,c_\lambda^J\,, \tag{6a}$$

$$\bar{K}(p, n) = \begin{pmatrix} p & -1 & 0 \\ -1 & 0 & n \\ 0 & n & 0 \end{pmatrix} \qquad c_\mu = \begin{pmatrix} \alpha_\mu \\ a_\mu \\ b_\mu \end{pmatrix}\,, \tag{6b}$$

where $\mu, \nu, \lambda \in \{0, 1, 2\}$. In this basis $e = (0, 1, 0)$ and $m = (0, 0, 1)$ represent the original charge and flux excitations of the gauge theory.

A $W \in \mathrm{GL}(3, \mathbb{Z})$ change of basis yields [54]

$$K_g \equiv W^T\,\bar{K}(p, n)\,W = p\,n^2 \oplus \Sigma\,, \tag{7}$$

where $\Sigma$ represents a Pauli matrix, i.e., a trivial sector that can always be gapped out. Thus, Eq. (7) explicitly illustrates that phase 1 can be obtained from phase 2 by a gauging mechanism; reversely, phase 2 descends from phase 1 by confining the $\mathbb{Z}_n$ gauge theory. This kind of gauging mechanism has proven useful in understanding the classification of symmetry enriched topological states [48] and hidden anyonic symmetries[27].

We now explicitly prove the existence of domain-wall parafermions by analyzing the transitions between edge phases 1 and 2. The transitions can be analyzed starting from the bulk theory in Eq. (6), and using the standard bulk-boundary correspondence for Abelian topological phases [49]. Hence, we model gapless edge states propagating along one of the edges, say $x = x_L$, with the effective theory

$$\mathcal{L}_{x_L,\mathrm{y}} = \frac{1}{4\pi}\partial_t \Phi^{'T} K_g \partial_y \Phi' + \sum_{a=1}^{2} J_a(y)\, H_{int,a} \quad (8)$$

where $\Phi^{'T}(t, x_L, y) = (\phi_1', \phi_2', \phi_3')(t, x_L, y)$ are the edge fields. The interactions $H_{int,1}$ and $H_{int,2}$ will be chosen to stabilize the edge phases 1 and 2, respectively, in different spatial regions, i.e., the interaction $H_{int,1}$ ($H_{int,2}$) partially gaps out two of the three edge modes to leave the single-component edge mode of phase 1 (phase 2). To carry this out we use position-dependent coupling constants $J_1(y)$ and $J_2(y)$ such that, $J_1 \to \infty$ and $J_2 = 0$ in phase 1, while $J_1 = 0$ and $J_2 \to \infty$ in phase 2. For concreteness, we take $p = 2\,q + 1$ and $\Sigma = \sigma_z$ in (7), although similar results can be obtained for the $p = 2\,q$ case with $\Sigma = \sigma_x$.

The interaction choice

$$H_{int,1} = \cos\left(L_1^T\, K_g\, \Phi'\right), \quad L_1^T = (0, 1, 1) \quad (9)$$

will gap the trivial modes in $\Sigma$ yielding the edge states of phase 1. Alternatively, the interaction

$$H_{int,2} = \cos\left(L_2^T\, K_g\, \Phi'\right), \quad L_2^T = (1, q\,n, (q+1)\,n) \quad (10)$$

gives rise to the edge state of phase 2, that is, it effectively leads to the confinement of the $\mathbb{Z}_n$ gauge theory. To see this, notice that the edge excitations that remain deconfined in the presence of the interaction (10) are described by vertex operators $\exp(\mathrm{i}\,\ell^T \Phi')$, with $\ell^T = (\ell_1, \ell_2, \ell_3)$, such that $\ell^T \Phi'$ commutes with the argument of the interaction (10). From this condition, which is satisfied when $\ell_1 = -n\,[\ell_2\, q + (q+1)\ell_3]$, we find that the deconfined edge excitations are those of the phase 2 described by $k_2 = p = (2\,q+1)$. More intuitively, upon rewriting $H_{int,2} = \cos\left(L_2^T\, K_g\, \Phi'\right) = \cos\left(\bar{L}_2^T\, \bar{K}\, \bar{\Phi}'\right) = \cos\left(n\, \bar{\phi}_2'\right)$, with $\bar{L} = W L$ and $\bar{\Phi}' = W \Phi'$, (10) is seen as the expected "electric"-mass interaction that confines the excitations of the $\mathbb{Z}_n$ gauge theory.

Defining the segments $R^{\pm}_{1,i} = (y_{2i-1} \pm \varepsilon, y_{2i} \mp \varepsilon)$ and $R^{\pm}_{2,i} = \cup_i\,(y_{2i} \pm \varepsilon, y_{2i+1} \mp \varepsilon)$, $\varepsilon = 0^+$, we let the regions $R^+_a = \cup_i\, R^+_{a,i}$, with $a = 1, 2$, denote the edge phases 1 and 2 along the $x = x_L$ edge. The operators $\mathcal{O}_i^{(a)} = \exp\left[\frac{\mathrm{i}}{n}\int_{R^-_{a,i}} dy\, \partial_y \left(L_{\bar{a}}^T\, K_g\, \Phi'\right)\right]$, where $\bar{a} \equiv a + (-1)^{a+1}$, are seen to commute with the edge Hamiltonian and satisfy the non-trivial commutation relations

$$\mathcal{O}_i^{(1)}\, \mathcal{O}_k^{(2)} = \mathcal{O}_k^{(2)}\, \mathcal{O}_i^{(1)}\, e^{\frac{2\pi \mathrm{i}}{n}(\delta_{k,i-1} - \delta_{k,i})}\,. \quad (11)$$

The ground state manifold forms a representation of the algebra (11), which implies a ground state degeneracy of $n^{k-1}$ in the presence of $2\,k$ domain walls on the boundary, i.e., $k$ GIs. The operators

$$\alpha_{x_L,\,\ell} = e^{\frac{\mathrm{i}}{n}\left[L^T_{a_\ell} K_g \Phi'(y_\ell+\varepsilon) - L^T_{\bar{a}_\ell} K_g \Phi'(y_\ell-\varepsilon)\right]}\,, \quad (12)$$

$[a_{2i\,(2i+1)} \equiv 2\,(1)]$ with support on the domain walls along the $x = x_L$ edge satisfy, as expected, parafermionic algebra $\alpha_{x_L,k}\, \alpha_{x_L,\ell} = \alpha_{x_L,\ell}\, \alpha_{x_L,k}\, \omega^{\mathrm{sgn}(k-\ell)\,(-1)^{k+\ell}}$. For a generic GI between one-component states we have the constraint $k_1 = \frac{n^2}{m^2} k_2$ which implies that the phases must be related by the confinement of a $\mathbb{Z}_m$ gauge theory, and the subsequent gauging and deconfinement of a $\mathbb{Z}_n$ symmetry. In these cases one would find $\mathbb{Z}_{mn}$ parafermions (see SM for more detail).

A realization of the algebra (11) has been studied in Ref. [30], for the transitions between chiral bosonic edge states with $k_1 = 2\,n^2$ and $k_2 = 2$. While their approach focused solely on the edge transitions of a homogenous bulk phase, our formulation shows that the existence of non-trivial parafermionic modes (12) is a direct consequence of the formation of a GI between different chiral topological states. Hence, we have generalized their result to arbitrary one-component edge transitions, and have shown that such transitions can originate from a bulk phenomenon associated with confinement-deconfinement transitions of discrete gauge theories. Additionally, since these parafermions appear at a "T-junction" between two chiral gapless states and the termination of their GI, they represent a completely new physical phenomenon when compared with the cases studied in Refs. [20–24, 26, 27].

We note that the GI acts like an anyonic Andreev reflector in the bulk. Anyons from, say, phase 1 will hit the interface and be transformed into a combination of outgoing anyons in phase 2 as well as reflected anyons that remain in phase 1. Take $p = 1, m = 1$ for simplicity. Then as, for example, quasiparticle $\chi_1 = \varepsilon_1^n$ approaches the interface, a vacuum fluctuation can create a $(\psi_2, \bar{\psi}_2)$ pair in the region of phase 2 immediately adjacent to the interface; subsequently, the condensation of $(\chi_1 \bar{\psi}_2)$ leaves behind the quasiparticle $\psi_2$ in phase 2, as shown in Fig. 1(a). The quasiparticles $\{\varepsilon_1^{n\,x}, x \in \mathbb{Z}\}$ belonging to phase 1 can be absorbed by the GI and fully transmuted into multiples of the local excitation $\psi_2$ of phase 2. Other anyons hitting the interface will be partially transmuted and partially reflected by the condensate. For example, if $\varepsilon_1$ hits the surface it could generate a $\psi_2$ in phase 2 as well as a reflected $\varepsilon_1^{(-n+1)}$.

In summary, we have shown that a gapped interface between different topologically ordered phases cannot be topologically trivial itself. The interpolation between the topological orders generates a quasi-1D topological parafermion phase which exhibits characteristic non-Abelian defect modes where the interface intersects the boundary of the system. Although we have only shown this for one-component interfaces, we expect the generalizations to more complicated interfaces to provide a rich set of phenomena. Furthermore, our result may aid in the interpretation of the topological entanglement entropy arising at heterointerfaces of topologically ordered phases as recently calculated in Ref. 50. We leave this to future work.

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