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Floquet topological order in interacting systems of bosons and fermions

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Periodically driven noninteracting systems may exhibit anomalous chiral edge modes, despite hosting bands with trivial topology. We find that these drives have surprising many-body analogs, corresponding to class A, which exhibit anomalous charge and information transport at the boundary. Drives of this form are applicable to generic systems of bosons, fermions and spins, and may be characterized by the anomalous unitary operator that acts at the edge of an open system. We find that these operators are robust to all local perturbations and may be classified by a pair of coprime integers. This defines a notion of dynamical topological order that may be applied to general time-dependent systems, including many-body localized phases or time crystals.

Introduction. Time-dependent quantum systems can support a host of novel phenomena that are impossible to realize with a static Hamiltonian. These include topological adiabatic cycles [1–6], Floquet analogs of topological insulators [7–19]; novel examples of driven symmetry-protected topological phases (SPTs) [20–24]; and phases which exhibit spontaneous symmetry breaking in the time domain, dubbed time crystals or π-spin glasses [20, 25–28]. In addition to being of theoretical interest, much progress has been made towards realizing Floquet systems in the laboratory [29–35].

Many of these unusual Floquet phases are distinguished by their anomalous edge behavior: While a periodic drive may have no overall effect on a closed system, its action at a boundary can be nontrivial. This is a kind of holography that signifies the presence of an inherently dynamical type of order. In this paper, we introduce a set of 2d drives that generate dynamical topological order of this form in generic systems of interacting bosons, fermions or spins. The topological order manifests as robust chiral edge modes at the boundary of an open system, which are stable to all perturbations and which cannot be generated by a 1d Hamiltonian.

We study these drives by considering the action of the unitary evolution restricted to the edge of the system, finding that it may be classified by a pair of coprime integers. Through homotopy arguments, this defines a robust topological invariant that may be applied very generally to classify Floquet many-body phases (for example, by incorporating many-body localization (MBL) [36]). However, our approach may also be used to provide a topological classification of more exotic unitary evolutions, including those corresponding to time crystals or those with only partial MBL.

As motivation, we recall that two fundamental examples of SPT phases are those of class D, protected by particle-hole symmetry, and class A, protected by U(1) charge conservation. In the time-dependent case, a 1d class D system has a $\mathbb{Z}_2 \times \mathbb{Z}_2$ classification [8, 10, 16], which persists in the presence of interactions [21, 23, 24]. In 2d, class A corresponds to the integer quantum Hall effect (IQHE) [37], which has a well-defined (static) integer classification that also persists in the presence of interactions [38]. A nontrivial driven system belonging to class A (without interactions) was given in Ref. 9, which shows IQHE-like chiral edge modes at the boundary of a 2d lattice, even though the bulk band has Chern number zero. This model is stable to disorder [17], and has recently been realised using photonic waveguides [35].

In the first section of this article we examine the effect of interactions on this drive by constructing its many-body analogs, taking into account the particle statistics. We consider the anomalous action of the drive at the edge, and find that charge conservation protects this against any local, charge-conserving, 1d perturbation. The structure of these particle-based models motivates a more general set of exchange models, which we introduce in the context of spin systems. We find that these also exhibit a robust edge action, and we provide a classification scheme for their anomalous behavior.

Interacting Class A Drive. We first introduce a unitary drive in class A that reduces to the drive of Ref. 9 for a single-particle system. We recall that the unitary evolution operator for a Hamiltonian $H(t)$ is $U(T) = \mathcal{T}\exp\left(-i\int_0^T H(t)\mathrm{d}t\right)$, where $\mathcal{T}$ is the time-ordering operator and T is the period of evolution. We are specifically interested in unitary loops, which we define to be an evolution which, in a closed system, satisfies $U(T) = \mathbb{I}$. In the corresponding open system, however, $U(T)$ will not necessarily be proportional to the identity. The component of $U(T)$ that acts in the vicinity of the boundary, which we call the effective edge unitary, characterizes the anomalous edge action of the evolution. Although this class of unitaries may seem somewhat restrictive, a classification of unitary loops in fact gives a very general classification of dynamical topological order in the space of unitary evolutions (see the Supplemental Material [36] and Ref. [24]).

The drive in Ref. 9 consists of four principal steps, each of which generates hopping across a different set of neighboring bonds, as shown in Fig. 1(a). After the complete drive, a particle in the bulk returns to its initial position, but a particle located at certain positions on the edge is

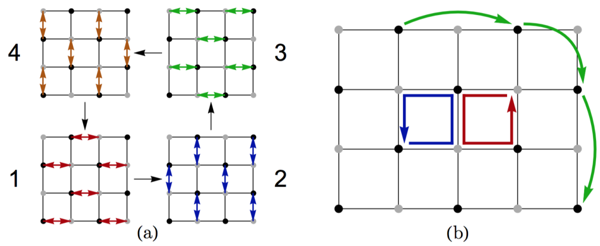

Figure 1. (a) Four steps in the anomalous Floquet drives considered in the main text, based on the model of Ref. 9. (b) Representation of the action of the complete unitary drive. See main text for details.

transported along the boundary, represented pictorially in Fig. 1(b). For simplicity, we initially work with hard-core bosons. In this case, the unitary that generates a hop between two sites is

$$U^B_{\mathbf{r}\mathbf{r}'} = 1 + b^\dagger_{\mathbf{r}'}b_{\mathbf{r}} + b^\dagger_{\mathbf{r}}b_{\mathbf{r}'} - b^\dagger_{\mathbf{r}}b_{\mathbf{r}} - b^\dagger_{\mathbf{r}'}b_{\mathbf{r}'} + 2b^\dagger_{\mathbf{r}'}b_{\mathbf{r}'}b^\dagger_{\mathbf{r}}b_{\mathbf{r}}, \quad (1)$$

where $b^\dagger_{\mathbf{r}}$ creates a boson on site $\mathbf{r}$ and satisfies $[b_{\mathbf{r}}, b^\dagger_{\mathbf{r}'}] = \delta_{\mathbf{r},\mathbf{r}'}$ and $\left(b^\dagger_{\mathbf{r}}\right)^2 = 0$. It may be verified that this operator is unitary and that it acts on a general two-site state as

$$U^B_{\mathbf{r}\mathbf{r}'}\left(b^\dagger_{\mathbf{r}}\right)^{n_{\mathbf{r}}}\left(b^\dagger_{\mathbf{r}'}\right)^{n_{\mathbf{r}'}}|0\rangle = \left(b^\dagger_{\mathbf{r}}\right)^{n_{\mathbf{r}'}}\left(b^\dagger_{\mathbf{r}'}\right)^{n_{\mathbf{r}}}|0\rangle\,, \quad (2)$$

with $n_{\mathbf{r}}, n_{\mathbf{r}'} \in \{0, 1\}$. Labeling the two sublattices as A and B (filled and open circles, respectively, in Fig. 1), and setting the intersite spacing to one, each step of the unitary drive may be written $U^B_j = \prod_{\mathbf{r}\in A} U^B_{\mathbf{r},\mathbf{r}+\mathbf{b}_j}$ with $\mathbf{b}_1 = -\mathbf{b}_3 = (1, 0)$ and $\mathbf{b}_2 = -\mathbf{b}_4 = (0, -1)$. The complete unitary drive is then $U^B = U^B_4 U^B_3 U^B_2 U^B_1$, which can be written as the product of evolutions by four local Hamiltonians.

Within each step of the drive, the two-site operators $U^B_{\mathbf{r},\mathbf{r}'}$ act on disjoint pairs of sites and commute. By tracking the position of a particular particle across all steps of the unitary, it can be verified that the action of the complete drive translates particles as in Fig. 1(b). On a many-body product state, the unitary acts as a permutation of particle occupation numbers at the edge. Since the unitary acts identically on any product state, the permutation is also well defined for superposition states.

The effective edge unitary of the drive may be read off directly from the complete time evolution operator. Writing a generic many-body product state as $|n_1, n_2, \ldots\rangle$, where n_j gives the boson occupation number on site j, the unitary drive maps product states onto product states through the relation $|n'_1, n'_2, \ldots\rangle = U^{B,\{n'\}}_{\{n\}}|n_1, n_2, \ldots\rangle$. From the discussion above, the matrix elements are

$$U^{B,\{n'\}}_{\{n\}} = \prod_{j\in\text{bulk}} \delta_{n_j,n'_j} \prod_{j\in\text{edge}} \delta_{n_j,n'_{j+1}}, \quad (3)$$

where the sites at the edge have been indexed appropriately [39]. The effective edge unitary, U^B_{eff}, is characterized by the matrix elements of the second factor.

It is natural to ask whether this many-body generalization applies also to fermions. We define fermionic unitary operators, $U^F_{\mathbf{r}\mathbf{r}'}$, U^F_j and U^F by replacing $b^\dagger_{\mathbf{r}}$ with $f^\dagger_{\mathbf{r}}$ in the bosonic definitions above. The operators $f^\dagger_{\mathbf{r}}$ satisfy $\{f_{\mathbf{r}}, f^\dagger_{\mathbf{r}'}\} = \delta_{\mathbf{r},\mathbf{r}'}$ and have the occupation number-exchanging property

$$U^F_{\mathbf{r}\mathbf{r}'}\left(f^\dagger_{\mathbf{r}}\right)^{n_{\mathbf{r}}}\left(f^\dagger_{\mathbf{r}'}\right)^{n_{\mathbf{r}'}}|0\rangle = \left(f^\dagger_{\mathbf{r}}\right)^{n_{\mathbf{r}'}}\left(f^\dagger_{\mathbf{r}'}\right)^{n_{\mathbf{r}}}|0\rangle\,, \quad (4)$$

with $n_{\mathbf{r}}, n_{\mathbf{r}'} \in \{0, 1\}$. In this case, the presence of the vacuum state $|0\rangle$ is important. For a many-body Slater determinant, anticommuting the relevant fermion operators so that they are adjacent to the vacuum will introduce an overall sign, which depends on the occupation of other lattice sites. In this way, the fermionic drive U^F acting on a closed system may return a Slater determinant state to itself, or to minus itself; in an open system, the unitary translates particles at the edge only up to a sign. The fermionic matrix elements are related to their bosonic counterparts through $U^{F,\{n'\}}_{\{n\}} = (-1)^s\, U^{B,\{n'\}}_{\{n\}}$, where s is an integer that depends non-locally on $\{n'\}$ and $\{n\}$. For a superposition state, the unitary may introduce different signs for different components.

Nevertheless, the fermionic Floquet drive U^F has many interesting properties, and also exhibits anomalous edge behavior. Any charge distribution at the edge of a many-body state will be translated around the boundary. Furthermore, if the drive is run twice (which we call the 'doubled fermion drive'), then its action in the bulk is exactly the identity, since the sign factors square to one. This is reminiscent of fermionic Hamiltonians that avoid the sign problem. In this case, the bulk and edge behavior can be disentangled and an effective edge unitary can be defined [36].

The drives described above have been constructed to give the desired edge behavior, and one might ask whether they are truly representative of a finite parameter space. We now argue that this is the case, and that the anomalous action is stable to local unitaries at the edge. We initially consider the bosonic version of the drive. To proceed, we consider the action of the effective unitary restricted to the 1d edge, which we take to have length $2L$. Acting on a product state, we find $U^B_{\text{eff}}|n_{-L}, n_{-L+1}, \ldots, n_L\rangle = |n_L, n_{-L}, \ldots, n_{L-1}\rangle$, shown pictorially in Fig. 2(a). We will now assume that this unitary may be generated by a local, 1d charge-conserving Hamiltonian $H(t)$ that acts for a finite time T at the edge, and we will show that this leads to a contradiction [40].

Since $H(t)$ is local, the complete drive has a maximum Lieb-Robinson velocity for the speed of information propagation, v_{LR} [41]. We can obtain an open version of the drive, $\tilde{U}^B_{\text{op}}$, by cutting open the 1d edge and consistently

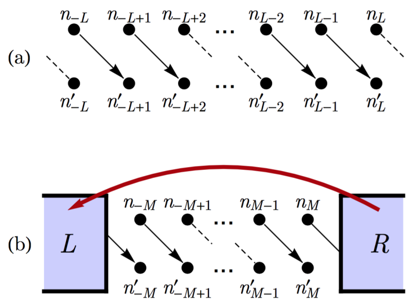

Figure 2. (a) Permutation of boson occupation numbers at the edge under the anomalous Floquet drive. (b) Action of the putative open 1d unitary $\tilde{U}^B_{\rm op}$. The regions marked L and R are within a distance $v_{LR}T$ of the cut. See main text for details.

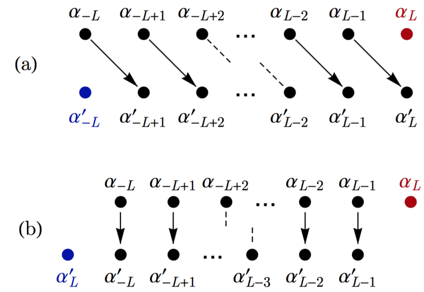

Figure 3. (a) Permutation of on-site states under the action of $\tilde{U}^{\leftrightarrow}_{\rm op}$, assuming the 1d chain is cut between sites L and $-L$. (b) Relabeling of lattice sites so that the permutation is diagonal away from the cut.

excluding all terms that connect sites across the cut. Due to the Lieb-Robinson bound, this cannot affect the unitary evolution outside of a region $\Delta j \approx v_{LR}T$ near the left and right ends. For a long chain, with $L \gg v_{LR}T$, we expect the unitary far from the cut to act as before.

Now, assume that in the large length of chain from site $j=-M$ to site $j=M$, the action of the unitary is unaffected by the cut, as shown in Fig. 2(b). Since the charge in the bulk is transported uniformly by one lattice site through the action of the unitary, it follows from charge conservation that the charge initially in sites $\{-L,-L+1,\ldots,-M-1\}\cup\{M,M+1,\ldots L\}$ must equal the final charge in sites $\{-L,-L+1,\ldots,-M\}\cup\{M+1,M+2,\ldots L\}$. However, the available space for charge on the right is reduced by this evolution, while the space for charge on the left is increased. The only way that total charge can be conserved for *any* initial charge configuration is if particles are transferred from the right edge to the left edge to address any imbalance. This distance can be made arbitrarily large by increasing the system size, which shows that in general $\tilde{U}^B_{\rm op}$ must be nonlocal (or that T must be infinite). We conclude that the anomalous action of $U^B_{\rm eff}$ cannot arise as result of a local, 1d Hamiltonian $H(t)$ acting for a finite time.

For fermionic models, this argument shows that there is no local 1d unitary which brings the action of the open system to that of the closed system. Furthermore, for the doubled fermion drive, the above bosonic argument can be straightforwardly applied to demonstrate the anomalous nature of the edge unitary.

Exchange Models. The models described above may be generalized straightforwardly to spin models, or indeed any system where the on-site Hilbert spaces are equivalent. Instead of particle hops, the building blocks are now pairwise exchanges of local states. The exchange version of Eq. (1) is

$$U^{\leftrightarrow}_{\mathbf{r},\mathbf{r}'} = \sum_{\alpha\neq\beta} |\mathbf{r},\beta\rangle\otimes|\mathbf{r}',\alpha\rangle\,\langle\mathbf{r},\alpha|\otimes\langle\mathbf{r}',\beta| + \delta_{\alpha\beta}\mathbb{I}_{\mathbf{r}\mathbf{r}'}, \quad (5)$$

where $\alpha,\beta\in\mathcal{H}_{\mathbf{r}}$ take values in the on-site Hilbert space. In the above, $|\mathbf{r},\alpha\rangle$ indicates that the state at site $\mathbf{r}$ is α.

In each step of the drive, the tensor product of the exchange operation is taken over one of the four sets of neighboring bonds shown in Fig. 1(a), $U^{\leftrightarrow}_j = \bigotimes_{\mathbf{r}\in A} U^{\leftrightarrow}_{\mathbf{r},\mathbf{r}+\mathbf{b}_j}$, with the complete drive given by $U^{\leftrightarrow} = U^{\leftrightarrow}_4 U^{\leftrightarrow}_3 U^{\leftrightarrow}_2 U^{\leftrightarrow}_1$. Each step consists of a product of local, commuting terms, and so can be generated by a local Hamiltonian. The complete action of the drive may again be represented as in Fig. 1(b), where the arrows now indicate the trajectory of a particular on-site state through the lattice. Acting on a product state, $U^{\leftrightarrow}$ permutes the on-site states through a cyclic permutation at the edge, an action that is also well-defined for superposition states. This may be encapsulated in an effective edge unitary, $U^{\leftrightarrow}_{\rm eff}$.

A natural setting for this type of anomalous drive is a lattice of spins. If the on-site Hilbert space corresponds to $\mathbb{Z}_2$, then the model maps formally onto the hardcore boson model given previously. More general spin models may be mapped onto bosonic models that allow a different (but finite) number of particles per site. The Hamiltonians that generate these drives conserve total boson number, and their edge action can be shown to be anomalous using the arguments given previously.

Stability of Edge Unitaries. We now allow for the possibility of perturbations that do not conserve charge. In these cases we can appeal to more general information theoretic ideas to show that the effective edge action of the Floquet drive is still anomalous. Roughly speaking, the anomalous edge drives have a chiral flow of information (and not just charge) which we will show cannot occur through a local 1d unitary evolution.

As before, we begin by assuming that $U^{\leftrightarrow}_{\rm eff}$ may be generated by a local Hamiltonian $H(t)$, and so there is a maximum velocity v_{LR} at which information can flow. We cut open the chain to obtain the putative open system unitary $\tilde{U}^{\leftrightarrow}_{\rm op}$, which should reproduce the permutation action in the bulk of the chain away from the cut.

For simplicity, we will assume that the edge region (of size $v_{LR}T$) consists of a single site either side of the cut (for the more general case, see the Supplemental Material [36]). With this setup, the action of $\tilde{U}_{\rm op}^{\leftrightarrow}$ on a many-body state is to translate the on-site states to the right by one lattice site, as shown in Fig. 3(a). The unitary $\tilde{U}_{\rm op}^{\leftrightarrow}$ maps product states onto product states through the matrix elements $\left|\alpha'_{-L},\alpha'_{-L+1},\ldots,\alpha'_{L}\right\rangle = U_{\{\alpha\}}^{\{\alpha'\}}\left|\alpha_{-L},\alpha_{-L+1},\ldots,\alpha_{L}\right\rangle$, where $|\{\alpha\}\rangle$ and $|\{\alpha'\}\rangle$ are initial and final states, respectively. These matrix elements have the form

$$U_{\{\alpha\}}^{\{\alpha'\}} = \prod_{j=-L}^{L-1} \delta_{\alpha_j,\alpha'_{j+1}} f\left(\{\alpha\},\{\alpha'\}\right), \tag{6}$$

where the final factor describes the relation between states α_L and α'_{-L}.

We now relabel the site indices in the final state through $j \to j' = j+1$ in the range $-L \le j' \le L$, which makes the permutation diagonal (see Fig. 3(b)). In this new basis, the matrix elements of the unitary are

$$U'^{\{\alpha'\}}_{\{\alpha\}} = \delta_{\alpha_{\mathbf{v}},\alpha'_{\mathbf{v}}} f\left(\alpha_{\mathbf{v}} = \alpha'_{\mathbf{v}}, \alpha_L, \alpha'_L\right), \tag{7}$$

where we have used the shorthand notation $\mathbf{v}$ to represent the sites from $-L$ to $L-1$. From the unitarity of $U'^{\{\alpha'\}}_{\{\alpha\}}$, it may be shown that the factor $f\left(\alpha_{\mathbf{v}},\alpha_L,\alpha'_L\right)$ is also unitary. The unitary evolution $\tilde{U}_{\rm op}^{\leftrightarrow}$ therefore relates state α'_L to α_L through an (unspecified) unitary operation.

Now, the unitary evolution as a whole preserves information. The diagonal factor in $U'^{\{\alpha'\}}_{\{\alpha\}}$ shows that information in the bulk is translated, while the remaining factor $f\left(\alpha_{\mathbf{v}},\alpha_L,\alpha'_L\right)$ shows that the information in state α_L is transferred to state α'_L unitarily. However, in the original basis, these states were separated by an arbitrarily large distance, and transferring information across this distance in a finite time would violate the Lieb-Robinson bound. In this way, the anomalous edge action of a general exchange model is robust, and cannot be created or destroyed by a 1d unitary drive of the form $\tilde{U}_{\rm op}^{\leftrightarrow}$.

We can construct more general drives by stacking together several systems (and thereby acting on a tensor product Hilbert space) or by running several drives in sequence. Drives generated in this way are not necessarily independent, as we now show.

The action of a general exchange drive can be characterized by a permutation of the form $|\alpha_1,\alpha_2,\ldots,\alpha_L\rangle \to |\alpha_{L-p+1},\alpha_{L-p+2},\ldots,\alpha_{L-p}\rangle$, which moves each state on the edge to the right by p lattice sites. If the on-site Hilbert space has dimension k, then we write this right-moving permutation as $R(p,k)$. Left-moving permutations may similarly be written $L(p,k)$.

We note that running the drive $R(p,k)$ q times is equivalent to running the drive $R(qp,k)$ once. Secondly, we note that by grouping together the first p lattice sites into a single effective site, the drive $R(p,k)$ is equivalent to the drive $R(1,k^p)$ [42]. This regrouping of sites is equivalent to the stacking together of p drives on different Hilbert spaces with dimension k. Stacking more general drives leads to the equivalence

$$R(p,k)\otimes R(p',k') \equiv R(1,k^p\,(k')^{p'}). \tag{8}$$

In this way, any right-moving drive is equivalent to a drive $R(1,n)$, where n is a positive integer. Using the same methods as above, it is straightforward to show that drives corresponding to different n are inequivalent (i.e. $R(1,n)$ cannot be obtained from $R(1,n')$ through a local 1d unitary evolution for $n \neq n'$).

In the Supplemental Material [36], we show that by also including left-moving permutations, a generic permutation can be brought into the form $L(1,n')\otimes R(1,n)$, where n and n' are coprime integers. A general exchange drive may therefore be characterized by a pair of integers, describing left and right-moving components of the permutation. A trivial drive can be reduced to the form $n=1, n'=1$. Again using the methods above, all of these drives can be shown to be inequivalent.

In general, an effective edge unitary, $U_{\rm eff}$, will not correspond to a pure exchange drive. From our results it follows that any effective edge unitary that is equivalent to an exchange effective edge unitary can be characterized by coprime integers n,n'. We conjecture that this classification is also complete, i.e. that every effective edge unitary belongs to one of these equivalence classes.

Conclusions. In summary, we have presented a many-body version of the anomalous Floquet drive of Ref. 9, which is applicable to both bosonic and fermionic systems. The action of the drive leads to the robust chiral propagation of charge at the boundary of an open system. Anomalous edge behavior arises more generally in exchange models, where spin states, for example, are swapped between Hilbert spaces on neighboring sites. Finite bounds on the propagation of information mean that classes of anomalous edge behavior are stable to all local perturbations.

We showed that exchange drives may be uniquely characterized (up to equivalence) by a pair of coprime integers, and we conjectured that all effective edge unitaries are equivalent to one of these exchange drives. Using homotopy arguments [24, 36], these integers provide a topological classification of Floquet systems, including MBL phases and time crystals. Our work raises a number of interesting questions which we hope will encourage further theoretical and experimental efforts. For instance, it would be interesting to study the interacting analogs and stability of other single-particle Floquet topological insulators.

We thank D. Reiss, X. Chen and the authors of Ref. [43] for useful discussions. R. R. and F. H. acknowledge support from the NSF under CAREER DMR-1455368 and the Alfred P. Sloan foundation.

Note — In the process of preparing this manuscript we became aware of Ref. [43], which considers chiral Floquet phases in the context of MBL systems. Our results, while framed in a different setting, seem consistent with this work. Some differences are in the precise definition of effective edge unitaries and in the use of MBL for setting up the discussion. Ref. [43] also suggests an experimental realisation for these systems and includes an explicit topological index, which we believe could also be used to classify our effective edge unitaries.

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