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Electromagnetic duality anomaly in curved spacetimes

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The source-free Maxwell action is invariant under electric-magnetic duality rotations in arbitrary spacetimes. This leads to a conserved classical Noether charge. We show that this conservation law is broken at the quantum level in presence of a background classical gravitational field with a non-trivial Chern-Pontryagin invariant, in a parallel way to the chiral anomaly for massless Dirac fermions. Among the physical consequences, the net polarization of the quantum electromagnetic field is not conserved.

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1. Introduction. It has long been known that the *source-free* Maxwell equations in four dimensions are manifestly invariant under duality rotations of the electromagnetic field $F_{\mu\nu} \rightarrow F_{\mu\nu} \cos\theta + {}^\star F_{\mu\nu} \sin\theta$, where ${}^\star F_{\mu\nu}$ is the dual strength tensor. It was proven in [1] that this transformation is indeed a symmetry of the action—at the level of the basic dynamical variables $\vec{A}$, and for an arbitrary spacetime—and the associated conserved charge was identified. This symmetry extends to the quantum theory in Minkowski spacetime. The goal of this paper is to analyze whether the duality invariance persists in quantum field theory in curved spacetimes or, as for the chiral invariance of massless fermions, the presence of spacetime curvature induces an anomaly.

If the symmetry exists and leaves the vacuum state invariant, vacuum expectation values of operators that reverse sign under a discrete duality transformation, such as $F_{\mu\nu}F^{\mu\nu}(x) = 2\left[\vec{B}^2(x) - \vec{E}^2(x)\right]$, must vanish. However, it has been found in [2] using adiabatic renormalization that this is not the case for a spatially flat Friedmann-Lemaitre-Robertson-Walker (FLRW) spacetime

$$\langle F_{\mu\nu}F^{\mu\nu}\rangle = \frac{1}{480\pi^2}\left[-9R_{\alpha\beta}R^{\alpha\beta} + \frac{23}{6}R^2 + 4\Box R\right] , \quad (1)$$

where $R_{\alpha\beta}$ is the Ricci tensor and R its trace. This result signals a breaking of the duality symmetry. On the other hand, the same approach produces a vanishing value of $\langle F_{\mu\nu}{}^\star F^{\mu\nu}\rangle = 4\langle \vec{E}\cdot\vec{B}\rangle$ in FLRW spacetimes. Given its pseudo-scalar character, this quantity is expected to be proportional to the Chern-Pontryagin invariant density $R_{\mu\nu\lambda\sigma}{}^\star R^{\mu\nu\lambda\sigma}$, where ${}^\star R^{\mu\nu\lambda\sigma} = 1/(2\sqrt{-g})\epsilon^{\mu\nu\alpha\beta}R_{\alpha\beta}{}^{\lambda\sigma}$, which vanishes in FLRW backgrounds. This was indeed worked out in [3], finding that

$$\langle F_{\mu\nu}{}^\star F^{\mu\nu}\rangle = \frac{1}{48\pi^2}R_{\alpha\beta\lambda\sigma}{}^\star R^{\alpha\beta\lambda\sigma} . \quad (2)$$

Although these results are suggestive, to establish the existence of an anomaly one needs to go a step further and analyze the extension of the classical conservation law to the quantum theory. This was the strategy followed to prove the existence of chiral anomaly for spin 1/2 fermions interacting with an external electromagnetic [4] or a gravitational field [5] in the late 60's. The purpose of this paper is to built a similar formalism for the electromagnetic field. Even though the gauge freedom adds new difficulties, the analysis brings out an interesting formal relation between the electromagnetic and fermionic dynamics which happens to be of great utility. We show that Maxwell equations in radiation gauge can be rewritten as spin 1 Dirac-type equations $\beta^\mu\nabla_\mu\Psi = 0$, where Ψ is a two-component object made of the potentials of the self and anti-self dual parts of the electromagnetic field (these components describe right and left circularly polarized waves, respectively), and the matrices β^μ are spin 1 analogs of the familiar γ^μ matrices for spin 1/2 fermions. Duality rotations are then generated by β_5, which is defined in the standard way (see below). The extensive theoretical machinery developed to derive the fermionic chiral anomaly can then be extended to the electromagnetic case. In particular, following the well known Fujikawa method [6, 7], we show that the duality anomaly originates in the failure of the measure of the path integral to respect the symmetry of the action. In the rest of the paper we spell out the details of the analysis and summarize the interesting relation with other mathematical structures and the physical consequences of the anomaly.

We follow the convention $\epsilon_{0123} = 1$ and metric signature $(+,-,-,-)$.

2. Duality symmetry and Noether charge. A detailed analysis of the duality symmetry of the classical, source-free Maxwell theory was presented in [1] (see [8] for an earlier work), and the reader is referred to these references for details. At the level of the electromagnetic potential, duality rotations are implemented by the transformation $\delta A_\mu = \theta\, Z_\mu$, with θ and infinitesimal parameter. Z_μ is a vector field that, on shell, must satisfy

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$\nabla_\mu Z_\nu - \nabla_\nu Z_\mu = {}^\star F_{\mu\nu}$, and can be understood as a non-local function of the basic variables A_μ [9]. By taking exterior derivative, one can see that the transformation above reduces to the more familiar form $\delta F_{\mu\nu} = \theta\ {}^\star F_{\mu\nu}$ on shell. The associated conserved current can be easily obtained from the Lagrangian density, and reads

$$j^\mu_D = \frac{1}{2}(A_\nu {}^\star F^{\mu\nu} - I^{\mu\nu} Z_\nu) \ , \tag{3}$$

where $I_{\mu\nu} = 2F_{\mu\nu} + \sqrt{-g}\epsilon_{\mu\nu\alpha\beta}\nabla^\alpha Z^\beta$, which on-shell $I^{\mu\nu} = F^{\mu\nu}$. This current is gauge dependent and non-local in space. However, the integral on a spatial Cauchy hypersurface of its "0-component" produces a well defined, gauge invariant *conserved charge* Q_D, which physically accounts for the difference in amplitude between left and right polarized components of the electromagnetic radiation. This quantity is often called the optical helicity, and it is the Noether generator of duality transformations.

Note that the first term in (3) is proportional to the Pauli-Ljubanski vector $K^\mu \equiv -A_\nu {}^\star F^{\mu\nu}$ used in [3] to compute $\langle F_{\mu\nu} {}^\star F^{\mu\nu}\rangle = -2\langle \nabla_\mu K^\mu\rangle$. This vector is not conserved, already at the classical level. Moreover, the spatial integral of K^0 is related to the so-called magnetic helicity [10], which does not generate electromagnetic duality transformations. The second term in (3), $-\frac{1}{2}I^{\mu\nu}Z_\nu$, is needed for the current to be conserved in the classical theory and the associated charge to generate duality rotations. The evaluation of $\langle \nabla_\mu j^\mu_D\rangle$, which turns out to be equivalent to $-\langle(\nabla_\mu F^{\mu\nu})Z_\nu\rangle$, requires to deal with an operator different from $F_{\mu\nu}{}^\star F^{\mu\nu}$ and $F_{\mu\nu}F^{\mu\nu}$. Consequently, a different calculation needs to be elaborated in order to analyze the electromagnetic duality in the quantum theory.

3. Weyl-type representation of Maxwell's equations. Before moving to the analysis of the above conservation law in the quantum theory, we rewrite Maxwell's equations in a convenient form for our purposes. We first describe the formalism in Minkowski spacetime and then generalized it to other geometries.

Maxwell equations in absence of charges and currents in Minkowski spacetime *decouple* when written in terms of the complex fields $\vec{H}_\pm \equiv \frac{1}{2}[\vec{E} \pm i\vec{B}]$, with $\vec{E}$ and $\vec{B}$ the electric and magnetic fields, and take the form

$$\vec{\nabla} \times \vec{H}_\pm = \pm\, i\frac{\partial}{\partial t}\vec{H}_\pm \ , \quad \vec{\nabla}\cdot\vec{H}_\pm = 0. \tag{4}$$

Using the familiar transformation properties of $\vec{E}$ and $\vec{B}$ under the Lorentz group, it is straightforward to show that $\vec{H}_+$ and $\vec{H}_-$ transform according to the $(1,0)$ and $(0,1)$ representations, respectively. Under a duality transformation $\vec{E} \to \cos\theta\vec{E} + \sin\theta\vec{B}$; $\vec{B} \to -\sin\theta\vec{E} + \cos\theta\vec{B}$, we have $\vec{H}_\pm \to e^{\mp i\theta}\vec{H}_\pm$. Hence, $\vec{H}_+$ and $\vec{H}_-$ are the self and anti-self dual parts of the electromagnetic field, respectively. Interestingly, duality rotations on $\vec{H}_\pm$ resemble conventional chiral rotations in the Dirac theory. Moreover, equations (4) can be rewritten as

$$(\alpha^a)^b{}_i \partial_a H^i_+ = 0 \ , \qquad (\bar{\alpha}^a)^b{}_i \partial_a H^i_- = 0 \ , \tag{5}$$

(bar denotes complex conjugation) where spacetime indices a, b run from 0 to 3, and the internal index i runs from 1 to 3 (note that $\vec{H}_+$ belongs to a three-dimensional complex space associated to the $(1,0)$ Lorentz representation; and analogously $\vec{H}_-$ to the $(0,1)$ one). Equations similar to (5) were also written in [11, 12]. The components of the $(\alpha^a)^b{}_i$ matrices can be extracted from (4), and it can be checked that they satisfy the following properties

$$\alpha^{(a}\alpha^{b)} \equiv \frac{1}{2}\left[(\alpha^a)^c{}_i(\alpha^b)_c{}^j + (\alpha^b)^c{}_i(\alpha^a)_c{}^j\right] = \eta^{ab}\delta^j_i \ , \tag{6}$$

$$\alpha^{[a}\alpha^{b]} \equiv \frac{1}{2}\left[(\alpha^a)^c{}_i(\alpha^b)_c{}^j - (\alpha^b)^c{}_i(\alpha^a)_c{}^j\right] = -2\left[{}^+\Sigma^{ab}\right]_i{}^j \ ,$$

where ${}^+\Sigma^{ab}$ is the generator of the $(1,0)$ representation of the Lorentz group, and η_{ab} is the Minkowski metric. Note the analogy with the properties of the $\sigma^\mu = (I, \vec{\sigma})$ matrices that appear in the Weyl equations ($\vec{\sigma}$ are the Pauli matrices) for massless spin 1/2 fermions.

Equations (5) are equivalent to the more conventional, manifestly Lorentz-invariant equations $\partial^\pm_a F^{ab} = 0$, due to the fact that the matrices $(\alpha^a)^b{}_i$ provide an isomorphism between $\vec{H}_+$ and the self dual part of F^{ab}, ${}^+F^{ab} = (\alpha^a)^b{}_i H^i_+$, where ${}^\pm F^{ab} \equiv \frac{1}{2}(F^{ab} \pm i\,{}^\star F^{ab})$. Similarly, ${}^-F^{ab} = (\bar{\alpha}^a)^b{}_i H^i_-$. Therefore, the $(\alpha^a)^b{}_i$ matrices can also be thought as the analog of the $\sigma^a_{AA'}$ or $\alpha^a_{\alpha\dot{\alpha}}$ maps that relate spinors and spacetime vectors [13].

Given the divergenless condition in (4), we can now introduce potentials $\vec{A}_\pm$ for $\vec{H}_\pm$: $\vec{H}_\pm \equiv i\,\vec{\nabla}\times\vec{A}_\pm$. In order to isolate the dynamical degrees of freedom we work in the radiation gauge, $\vec{\nabla}\cdot\vec{A}_\pm = 0$. With this choice, equations (4) translate to first-order differential equations for $\vec{A}_\pm$,

$$\vec{\nabla}\times\vec{A}_\pm = \pm i\frac{\partial}{\partial t}\vec{A}_\pm \ , \quad \vec{\nabla}\cdot\vec{A}_\pm = 0, \tag{7}$$

which turn out to have the same form as (4). Therefore, they can also be written as Weyl-type equations

$$(\alpha^a)^b{}_i \partial_a A^i_+ = 0 \ , \qquad (\bar{\alpha}^a)^b{}_i \partial_a A^i_- = 0 \ . \tag{8}$$

Notice that first order differential equations are obtained at the expense of working with complex fields $\vec{A}_\pm$, and therefore duplicating the number of independent variables. It is not difficult to see that equations (7) are equivalent to Hamilton's equations for the canonical formulation of Maxwell's theory if we split them into real and imaginary parts [14]. The familiar second order differential equations $\Box\vec{A}_\pm = 0$ arise from (8) by acting with the operator $(\alpha^c)_b{}^j\partial_c$ in the first equation and with $(\bar{\alpha}^c)_b{}^j\partial_c$ in the second one, and then using the properties written in (6).

The generalization to curved spacetimes follows the same procedure as for the Dirac case. Namely, equations (5) for the fields translate to

$$(\alpha^\mu)^\nu{}_i \nabla_\mu H^i_+ = 0 \ , \qquad (\bar{\alpha}^\mu)^\nu{}_i \nabla_\mu H^i_- = 0 \ , \qquad (9)$$

and similarly for the potentials

$$(\alpha^\mu)^\nu{}_i \nabla_\mu A^i_+ = 0 \ , \qquad (\bar{\alpha}^\mu)^\nu{}_i \nabla_\mu A^i_- = 0 \ , \qquad (10)$$

where the α-matrices in curved spacetime are obtained from the flat space ones by using the vierbein formalism

$$(\alpha^\mu)^\nu{}_i(x) = e^\mu_a(x)\, e^\nu_b(x)\, (\alpha^a)^b{}_i \, . \qquad (11)$$

The equivalence with the familiar Maxwell equations in curved spacetimes, $\nabla_\mu {}^{\pm}F^{\mu\nu} = 0$, is easily shown from (9) by taking into account that the covariant derivative in the above equations satisfies $\nabla_\beta(\alpha^\mu)^\nu{}_i = 0$.

An even closer analogy with the Dirac equation can be achieved by combining together the two sets of equations in (10)

$$\beta^\mu \nabla_\mu \Psi(x) = 0 \ , \qquad (12)$$

where we have defined [15]

$$\Psi \equiv \begin{pmatrix} A_+^{\ i} \\ A_{-\ i} \end{pmatrix} \ , \qquad \beta^\mu \equiv i \begin{pmatrix} 0 & (\bar{\alpha}^\mu)_\nu{}^i \\ -(\alpha^\mu)^\nu{}_i & 0 \end{pmatrix} . (13)$$

The β^μ-matrices inherit from the α^ν-matrices the following properties

$$\bar{\beta}^{(\mu}\beta^{\nu)} = -g^{\mu\nu}\, \mathbb{I} \, , \qquad (14)$$

$$\bar{\beta}^{[\mu}\beta^{\nu]} = 2 \begin{pmatrix} {}^+\Sigma^{\mu\nu} & 0 \\ 0 & -\Sigma^{\mu\nu} \end{pmatrix} , \qquad (15)$$

where round (square) brackets denote symmetrization (anti-symmetrization), and $\mathbb{I}$ is the identity matrix when acting on Ψ. Furthermore, we can construct the chiral matrix in the standard way

$$\beta_5 \equiv i \frac{\sqrt{-g}}{32} \epsilon_{\mu\nu\sigma\rho} \beta^\mu \bar{\beta}^\nu \beta^\sigma \bar{\beta}^\rho = \begin{pmatrix} -\mathbb{I}_{3\times 3} & 0 \\ 0 & \mathbb{I}_{3\times 3} \end{pmatrix} ,$$

which can be used to write the duality transformation in the form of a conventional chiral rotation

$$\begin{pmatrix} A^i_+ \\ A_{-\ i} \end{pmatrix} \to e^{i\theta\beta_5} \begin{pmatrix} A^i_+ \\ A_{-\ i} \end{pmatrix} = \begin{pmatrix} e^{-i\theta} A^i_+ \\ e^{i\theta} A_{-\ i} \end{pmatrix} \ . \qquad (16)$$

In analogy with the terminology used for fermions, $A^i_\pm$ describe right and left-handed (circularly polarized) radiation.

4. The quantum anomaly. To explore whether the classical conservation law extends to the quantum theory, we rely on the well known Fujikawa's path integral approach. Transition amplitudes for the quantized free electromagnetic field in the radiation gauge can be extracted from the following path integral [7, 16]

$$\langle A_f, t_f | A_i, t_i \rangle = \int \mathcal{D}X\, \mathcal{D}A^{(1)}_k \mathcal{D}A^{(2)}_k e^{iS_M[A]} \ , \qquad (17)$$

(sum over k is understood) where $A^{(1,2)}_k$ represents the two transverse (linear) polarizations of the potential field, and $S_M[A] = -\frac{1}{4}\int d^4x\sqrt{-g}F_{\mu\nu}F^{\mu\nu}$ is the Maxwell action. On the other hand, $\mathcal{D}X \equiv \mathcal{D}A_0 \det^{1/2}[-\nabla_\mu D^\mu \delta^{(3)}(x-y)]$, where D_μ is the spatial covariant derivative. The radiation gauge fixing is implicitly included in the measure [7]. We now rewrite this expression in terms of Ψ. To do this, first we recall the relation between linear and circular polarization $A^{(1)}_k = \frac{1}{\sqrt{2}}\left[A^+_k + A^-_k\right]$ and $A^{(2)}_k = \frac{i}{\sqrt{2}}\left[A^+_k - A^-_k\right]$; and then notice that the functional measure can be rewritten in a suggestive form, $\mathcal{D}A^{(1)}_k \mathcal{D}A^{(2)}_k = -\frac{1}{2}\mathcal{D}\bar{\Psi}\mathcal{D}\Psi$, where we have defined $\bar{\Psi} \equiv \Psi^\dagger\beta^0$. Finally, we arrive at

$$\langle A_f, t_f | A_i, t_i \rangle = \frac{1}{2} \int \mathcal{D}X\, \mathcal{D}\bar{\Psi}[A]\mathcal{D}\Psi[A] e^{iS_M[A]} \, . (18)$$

Recall that, despite the notation, the variables Ψ and $\bar{\Psi}$ are not Grassmann numbers.

To evaluate the impact of a duality transformation in the path integral (18) we use again Noether's theorem. In quantum field theory and particularly in gauge theories, the second version of the theorem—in which the infinitesimal parameter θ is promoted to an arbitrary function of space and time subject to appropriate fall-off conditions—happens to be more convenient (see e.g. [7, 17]). The variation of the Maxwell action under a transformation of the basic dynamical variables $\delta A_\mu = \theta(x)\, Z_\mu$, with $\delta A_0 = 0$, is

$$\delta S_M = -\int d^4x\sqrt{-g}\, \theta(x) \nabla_\mu j^\mu_D, \qquad (19)$$

where the resulting j^μ_D agrees with (3). Note that at the level of the field strength, the transformation implies $\delta F_{\mu\nu} = \theta(x)\, {}^*F_{\mu\nu} - Z_\mu \nabla_\nu\theta(x) + Z_\nu\nabla_\mu\theta(x)$. This differs from the transformation used in [18], where it is assumed that $\delta F_{\mu\nu} = \theta(x)\, {}^*F_{\mu\nu}$.

Quantum anomalies arise from the non-invariance of the measure in the path integral [6, 7]. The transformation properties of the measure are given by the Jacobian J, $\mathcal{D}\bar{\Psi}'\mathcal{D}\Psi' = J\mathcal{D}\bar{\Psi}\mathcal{D}\Psi$. Note that the duality rotation leaves $\mathcal{D}X$ invariant. To evaluate J it is more convenient to move to the Euclidean regime. Now, the fact that the operator $D = \beta^\mu\nabla_\mu$ is hermitian, guarantees the existence of an orthonormal basis Ψ_n of eigenstates ($D\Psi_n = \lambda_n\Psi_n$) under the inner product $(\Psi_n, \Psi_m) \equiv \int d^4x\sqrt{-g}\Psi^\dagger_n\Psi_m = \ell^2\delta_{nm}$, where ℓ is an arbitrary constant with dimensions of length; physical observables are insensitive to its value, so we fix $\ell = 1$. With this, the expression for the Jacobian can be derived by expanding the fields Ψ and $\bar{\Psi}$ in terms of this complete basis, and reads

$$J = e^{+i2\sum_{n=0}^\infty \int d^4x\sqrt{-g}\theta(x)(\Psi^\dagger_n\beta_5\Psi_n)} \ . \qquad (20)$$

Form this, the expression for the vacuum expectation value $\langle\nabla_\mu j^\mu_D\rangle$ can be obtained by recalling that the path integral is independent of the name of variables

$$\int D\bar{\Psi}[A]D\Psi[A]e^{iS_M[A]} = \int D\bar{\Psi}[A']D\Psi[A']e^{iS_M[A']}$$
$$= \int D\bar{\Psi}[A]D\Psi[A]\, J\, e^{iS_M[A]-i\int d^4x\sqrt{-g}\theta(x)\langle\nabla_\mu j^\mu_D\rangle}, \quad (21)$$

and therefore $\langle\nabla_\mu j^\mu_D\rangle = 2\sum_{n=0}^{\infty}(\Psi_n^\dagger\beta_5\Psi_n)$. The right-hand side of this expression is not well-defined (is ultraviolet divergent) and must be renormalized. We follow a regularization based on the well known heat kernel expansion (see e.g. [17] for details). The kernel of the quadratic operator $\bar{\beta}^\mu\beta^\nu\nabla_\mu\nabla_\nu$, whose eigenvalues are λ_n^2, can be written as

$$K(\tau; x, x') \equiv \sum_{n=0}^{\infty} e^{-i\tau\lambda_n^2}\,\Psi_n(x)\,\Psi_n^\dagger(x'). \qquad (22)$$

where τ plays the role of regularization cut-off. With this, we can formally write

$$\langle\nabla_\mu j^\mu_D\rangle = 2\lim_{\tau\to 0}\mathrm{Tr}[\beta_5 K(\tau; x, x)]. \qquad (23)$$

where the trace refers to Ψ-indices. The importance of the heat kernel regularization method relies in the asypmptotic expansion of $K(\tau; x, x)$ in the limit $\tau \to 0$

$$K(\tau; x, x) \sim -\frac{i}{16\pi^2\tau^2}\sum_{k=0}^{\infty}(i\tau)^k E_k(x)\,. \qquad (24)$$

The functions $E_k(x)$ are local geometric quantities, constructed from the quadratic operator $\bar{\beta}^\mu\beta^\nu\nabla_\mu\nabla_\nu$; they depend on the metric and its first $2k^{th}$ derivatives. The first few coefficients of the asymptotic kernel expansion are: $E_0(x) = \mathbb{I}$, $E_1(x) = \frac{1}{6}R\,\mathbb{I} - \mathcal{Q}$, and

$$E_2(x) = \left[\frac{1}{72}R^2 - \frac{1}{180}R_{\mu\nu}R^{\mu\nu} + \frac{1}{180}R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu}\right]\mathbb{I}$$
$$-\frac{1}{30}\Box R + \frac{1}{12}W_{\mu\nu}W^{\mu\nu} + \frac{1}{2}\mathcal{Q}^2 - \frac{1}{6}R\mathcal{Q} + \frac{1}{6}\Box\mathcal{Q},$$

where $W_{\mu\nu} = [\nabla_\mu, \nabla_\nu]$, and $\mathcal{Q}$ is defined by writing the wave equation as $-\bar{\beta}^\mu\beta^\nu\nabla_\mu\nabla_\nu\Psi \equiv (\Box + \mathcal{Q})\Psi = 0$. Explicit computations produce

$$\mathcal{Q} \equiv \begin{pmatrix} Q^i{}_j & 0 \\ 0 & \bar{Q}^i{}_j \end{pmatrix}, \qquad (25)$$

with $Q^i{}_j = -[{}^+\Sigma^{\mu\nu}]^i{}_k\, R_{\mu\nu\alpha\beta}\, [{}^+\Sigma^{\alpha\beta}]^k{}_j$. Now, bringing the expansion (24) to (23) one finds that all terms in the sum (24) with $k > 2$ clearly give a vanishing contribution in the $\tau \to 0$ limit. One can also check that for $k < 2$ the terms in the sum vanish because the trace with β_5 selects the imaginary part, and $\mathrm{Im}\{\mathrm{Tr}\, Q\} = 0$. Henceforth,

$$\langle\nabla_\mu j^\mu_D\rangle = \frac{i}{8\pi^2}\mathrm{Tr}[\beta_5 E_2(x)]\,. \qquad (26)$$

The crucial point is then to evaluate this quantity. Using (25) one has $\mathrm{Tr}[\beta_5 E_2] = \frac{1}{12}\mathrm{Tr}[\beta_5 W^{\mu\nu}W_{\mu\nu}] + \frac{1}{2}\mathrm{Tr}[\beta_5\mathcal{Q}^2]$. Notice that the values of $W_{\mu\nu}$ and $\mathcal{Q}$ are related to the representations of the Lorentz group associated to the physical degrees of freedom of the electromagnetic theory, namely $(1, 0)$ and $(0, 1)$. This is in close analogy with the chiral anomaly for spin-1/2 fermions, where the $(1/2, 0)$ and $(0, 1/2)$ representations play an important role. After a long calculation, one arrives at $\mathrm{Tr}[\beta_5 W^{\mu\nu}W_{\mu\nu}] = 2iR^{\mu\nu\alpha\beta}\,{}^\star R_{\mu\nu\alpha\beta}$ and $\mathrm{Tr}(\beta_5\mathcal{Q}^2) = -iR^{\mu\nu\alpha\beta}\,{}^\star R_{\mu\nu\alpha\beta}$ (details will be published elsewhere). Taking all factors into account, one gets

$$\langle\nabla_\mu j^\mu_D\rangle = \frac{1}{24\pi^2}R_{\mu\nu\lambda\sigma}\,{}^\star R^{\mu\nu\lambda\sigma}. \qquad (27)$$

Since the heat kernel asymptotic series (24) does not depend on the vacuum state chosen, this expectation value is (vacuum) state independent.

5. Conclusions and final comments. The above result implies that the charge Q_D associated to the duality symmetry of the Maxwell action is no longer conserved in the quantum theory in a general spacetime; its time derivative is given by the spatial integral of (27). Since in flat spacetime Q_D represents the difference in number between photons of opposite helicity [8], this result can be interpreted as a non-conservation of the helicity of the quantum electromagnetic field in curved spacetimes.

A physical background where this anomaly may lead to observational consequences are rotating astrophysical objects, described approximately by a Kerr metric ($R_{\mu\nu\lambda\sigma}\,{}^\star R^{\mu\nu\lambda\sigma}$ is proportional to the angular momentum of the source [19]). Light-rays coming from different sides of a rotating object such as a black hole, galaxy or cluster, not only would bend around, but an effective difference in polarization could also be induced between them. In particular, this effect would affect the polarization of the cosmic microwave background photons. The quantitative details for phenomenological implications will be analyzed in a future work.

Interestingly, the anomaly (27) can be understood as a physical realization of the Hirzebruch signature (index) theorem [20]. The anomaly arises as the difference in the number of right-handed and left-handed zero-eigenvalue solutions of the operator $\beta^\mu\nabla_\mu$, $\int d^4x\sqrt{-g}\,\langle\nabla_\mu j^\mu_D\rangle = 2[n_L - n_R]$. n_L and n_R can be computed from the $(0, 1)$ and $(1, 0)$ irreducible representations of the Lorentz group [21], respectively, and one obtains agreement with (27). This is also in analogy with the fermionic chiral anomaly, which can also be obtained from an index associated to the $(1/2, 0)$ and $(0, 1/2)$ irreducible representations of the Lorentz group.

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[9] Notice that the "0-component" Z_0 remains an arbitrary gauge function, and can be chosen to be zero in such a way that A_0 does not transform under duality [1]. This is convenient since in Maxwell theory A_0 is a non-dynamical variable, i.e. another gauge function.

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[14] There is another way to understand why Maxwell's equations can be written as first-order differential equations for the potentials. Maxwell equations can be recovered from the following two conditions: *i)* $d\,{}^{+}F = 0$, and *ii)* ${}^{+}F_{ab}$ be self-dual, i.e. ${}^{+}F_{ab} = \frac{1}{2}\left[F_{ab} + i^{\star}F_{ab}\right]$. The equation $d\,{}^{+}F = 0$ allows one to define the potential A_+; while the self-duality property ${}^{+}F_{ab} = \frac{i}{2}\epsilon_{abcd}{}^{+}F^{cd}$, when written in terms of A_+, gives the dynamical equations (8) for A_+.

[15] The lower component in this field Ψ is related to the upper one by charge conjugation. In contrast to the Dirac case, this operation just consists in complex conjugation. This simplicity in the structure of Ψ can be heuristically regarded as a manifestation of the fact that the photon is its own antiparticle.

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