Dynamical Response near Quantum Critical Points

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Dynamical response near quantum critical points

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We study high frequency response functions, notably the optical conductivity, in the vicinity of quantum critical points (QCPs) by allowing for both detuning from the critical coupling and finite temperature. We consider general dimensions and dynamical exponents. This leads to a unified understanding of sum rules. In systems with emergent Lorentz invariance, powerful methods from conformal field theory allow us to fix the high frequency response in terms of universal coefficients. We test our predictions analytically in the large-N O(N) model and using the gauge-gravity duality, and numerically via Quantum Monte Carlo simulations on a lattice model hosting the interacting superfluid-insulator QCP. In superfluid phases, interacting Goldstone bosons qualitatively change the high frequency optical conductivity, and the corresponding sum rule.

A quantum critical point (QCP) is a zero-temperature phase transition, driven by quantum fluctuations, reached by tuning a non-thermal parameter such as a magnetic field [1], as shown in Fig. 1. Proximity to a QCP alters many observables, even if the (detuned) ground state is otherwise conventional. Of particular importance are dynamical response functions such as the optical conductivity $\sigma(\omega)$ [1–14], where changing the frequency probes physics at different energy scales set by the non-thermal detuning and by the temperature. What often complicates the analysis of the real-time dynamics, especially on short time scales, is the destruction of quasiparticles at the QCP, and the corresponding abundance of incoherent excitations at finite but small detuning.

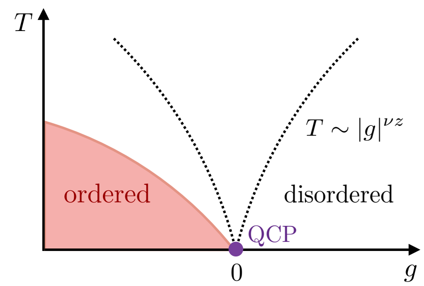

FIG. 1. Phase diagram near a canonical quantum critical point. g is the non-thermal coupling that needs to be tuned. The dotted lines roughly delimit the quantum critical "fan".

In this letter, we focus on a large family of non-metallic QCPs [1] found in magnetic insulators, Dirac semimetals, cold atomic gases in optical lattices [15–17], thin film superconductors or arrays of Josephson junctions [2]. This will serve as comparison ground for the more intricate metallic QCPs occuring in heavy fermion materials for example [18]. Specifically, we study how the detuning of the non-thermal parameter from its critical value, as well as temperature, modify the optical conductivity. In particular, our analysis at large frequencies is not restricted to the quantum critical fan. We derive sum rules for the conductivity that generalize the standard f-sum rule [19] to the scaling regime near QCPs. Our methods are not perturbative in any interaction strength. We test our predictions using large-scale quantum Monte Carlo simulations of an interacting superfluid-insulator QCP. While our focus is on the portions of the phase diagram smoothly connected to the critical fan, we also point out the qualitative changes to $\sigma(\omega)$ and the resulting sum rules which result from interacting Goldstone bosons in broken-symmetry phases.

Setup: Let us consider a system near a QCP that is reached by tuning a non-thermal parameter g to zero. We work in the universal scaling regime, at frequencies smaller than microscopic (UV) scales, and assume that hyperscaling is obeyed. Such a system is described by the following low-energy action in d spatial dimensions:

$$S = S_{\text{critical}} - g \int \mathrm{d}t\, \mathrm{d}^d\mathbf{x}\ \mathcal{O}(t,\mathbf{x}), \tag{1}$$

where $\mathcal{O}$ is the only relevant operator whose coupling g necessitates fine-tuning; it has (spatial) scaling dimension

$$\Delta = d + z - 1/\nu, \tag{2}$$

where $\nu > 0$ is the correlation length critical exponent, and z is the dynamical exponent. The equal-time 2-point function of $\mathcal{O}$ at the QCP is thus $\langle \mathcal{O}(0,\mathbf{x})\mathcal{O}(0,0)\rangle \propto 1/|\mathbf{x}|^{2\Delta}$. For example at the superfluid-insulator QCP in 2d belonging to the Wilson-Fisher universality class, $\mathcal{O} \sim \phi_a\phi_a$ is the "mass" term of the 2-component order parameter field ϕ_a. At $T=0$, the correlation length diverges as $\xi \sim g^{-\nu}$ in the insulator.

We are interested in probing the properties of the nearly critical system by studying dynamical response functions such as the optical conductivity: $\sigma(\omega) = \frac{1}{\mathrm{i}\omega}\langle J_x(-\omega)J_x(\omega)\rangle_{g,T}$, where ω is the frequency, and J is the current operator that enters in the retarded correlator. Near the QCP, the conductivity will obey scaling:

$\sigma(\omega) = \omega^{(d-2)/z} f_\pm\left(\frac{\omega}{|g|^{z\nu}}, \frac{\omega}{T}\right)$, where $f_\pm$ is a dimensionless scaling function that depends on which side of the transition the system is poised. We have set $\hbar = k_B = 1$, and the charge $Q=1$ and $c=1$, where c appears in the energy scale $c|\mathbf{k}|^z$. Other response functions such as order parameter susceptibilities or the shear viscosity will have an analogous structure.

Large frequencies: We focus on the conductivity at high frequencies $\omega \gg T, |g|^{z\nu}$, which allows us to controllably study the deviations away from criticality. The resulting asymptotics will also serve as the key ingredient in the derivation of sum rules for the response functions. Our first main result is that the asymptotic behavior is

$$\sigma(\omega) = (\mathrm{i}\omega)^{(d-2)/z} \left(\sigma_\infty + c_1 \frac{g}{(\mathrm{i}\omega)^{(d+z-\Delta)/z}} + c_2 \frac{\langle \mathcal{O}\rangle_{g,T}}{(\mathrm{i}\omega)^{\Delta/z}} + \cdots\right), \quad (3)$$

where σ_∞, $c_{1,2}$ are real constants fixed by the universality class, independent of detuning and T. The σ_∞ term is the conductivity of the critical theory; the $c_{1,2}$ terms arise from deviations from the QCP due to detuning and temperature. Note that the c_1 term in brackets simply scales as $\omega^{-1/\nu}$, by virtue of (2). In odd d, the imaginary part of σ can have a non-universal logarithmic contribution, not written here. For simplicity, we consider the generic case where the $c_{1,2}$ power-laws are not equal, and more generally do not differ by $2n/z$ (n being an integer), *i.e.* $2\Delta \neq d + z + 2n$.[20]

When $z=1$, recent work has derived [11] the c_2 term in (3) at $T>0$ but zero detuning $g=0$. Here, we identify the new effects coming from detuning, and their interplay with temperature. In particular, the c_1 term purely arises from g and can have important consequences on the dynamics. Its existence was glimpsed deep in the quantum critical fan, $T \gg |g|^{z\nu}$, in a specific AdS/CFT calculation [14], and in fact holds much more broadly. For CFTs ($z=1$) we will derive (3), present a universal expression for c_1/c_2, and confirm our predictions with two independent computations in non-trivial CFTs. For $z \neq 1$, we provide a general scaling argument for the c_1 term, and confirm that Eq. (3) is satisfied by a class of strongly interacting QC theories described by the gauge-gravity duality.

Working at general z, we first explain the origin of the c_1 term by using a scaling argument. Let us imagine that the system is at $T > 0$ in the QC fan. Since there is no phase transition in the fan, the conductivity will receive a correction $\delta\sigma$ that is *analytic* in the coupling g about $g = 0$, which generally will be linear. Further, by using the scaling dimension of g, and the fact that $\omega \gg T$ is the dominant energy scale, we get $\delta\sigma \sim g/\omega^{(2+z-\Delta)/z}$. We stress that this term does *not* depend on T. A more precise and general argument can be made by first expressing the dynamical conductivity as $\sigma(\omega) = \frac{1}{\mathrm{i}\omega}\langle J_x J_x \mathrm{e}^{-\mathrm{i}g\int_x \mathcal{O}}\rangle_T \mathcal{Z}_{0,T}/\mathcal{Z}_{g,T}$, using (1), where $\mathcal{Z}_{g,T}$ is the full partition function. The expectation value is taken using the $g = 0$ action, and temperature $T \geq 0$. We expand $\mathrm{e}^{-\mathrm{i}g\int_x \mathcal{O}}$ to first order in g, and evaluate the resulting 3-point function $\langle J_x(\omega) J_x(-\omega)\mathcal{O}(\tilde\omega \to 0)\rangle_T = \omega^{(\Delta-z-d)/z}\mathcal{F}(T/\omega)$, for a scaling function $\mathcal{F}$ (note that spatial momenta are set to zero). Generically, $\mathcal{F}(0) \neq 0$ and is a property of the QCP at $T = 0$. Hence, as $\omega \gg T$, $c_1 = \mathcal{F}(0)$ and is T-independent. If there is no phase transition as we vary T at fixed $g \neq 0$, by adiabaticity c_1 must remain unchanged all the way to, and including, $T=0$.

In contrast to the c_1-term, the c_2 term depends on both g and T through the expectation value of $\mathcal{O}$, and was previously identified at finite temperature but zero detuning $g=0$ (and $z=1$) [11]. Let us recall the main idea of that derivation, focusing on the case $z = 1$, and see how it generalizes to $g \neq 0$. The Kubo formula for the conductivity states that we need to evaluate the current-current correlation function. Since we are interested in short times (large-frequencies) we consider the operator product expansion (OPE) of $J_x(t,0)J_x(0,0)$ in the $t \to 0$ limit. Crucially, by spacetime locality the product can be replaced by a sum of local operators evaluated at $t=0$, with increasing scaling dimensions. The first non-trivial term in the sum will generally arise due to the leading relevant operator at the QCP, $\mathcal{O}$, and will be $\sim t^{\Delta-2d}\mathcal{O}(t=0)$. We can take the expectation value of the OPE at finite g and T since we work at short times, $t \ll |g|^{-\nu z}, T^{-1}$. Fourier transforming then leads to the c_2 term in Eq. (3). c_2 itself depends on neither g nor T; it is related to a coefficient in the OPE. In contrast, the $z \neq 1$ case is not as simple due to the lack of a sharp notion of spacetime locality needed to constrain the OPE. The c_2 term at $z \neq 1$ is allowed by scaling, and below we will confirm its existence in a class of interacting Lifshitz theories.

The perturbative expansion used to derive the c_1-term is different from the commonly used perturbative expansions about a free (Gaussian) theory: it uses the structure of the generally interacting QCP itself to determine the corrections at finite detuning. The expansion should hold when the detuned system has a finite correlation length, but can fail in regions separated from the "fan" by a phase transition, where potentially new gapless modes can arise. We will see an example of this failure later.

We have obtained the asymptotic expansion (3) near generic QCPs. In the context of *classical* critical phenomena, similar expansions for short-distance spatial correlators of the order parameter have been found for thermal Wilson-Fisher fixed points in 3D (where $z = 1$) [21, 22]. The coefficients in the expansion for these spatial correlators have recently been computed for the strongly-coupled Ising critical point [23]. These classical results are most similar to (3) analytically continued [24, 25] to imaginary time, when $z = 1$ and $T = 0$. In this limit, the asymptotic behavior of short-distance correlators contains both analytic and non-analytic terms in the thermal detuning parameter $(T - T_c)$, since $\langle\mathcal{O}\rangle \sim |g|^{\nu\Delta}$ where g is interpreted as $(T - T_c)$ under the quantum-to-classical mapping. This highlights that (3), just as in the classical case, cannot be derived via a single perturbative expansion. Our derivation indeed illustrates the different mechanisms behind the c_1 and c_2 terms, and is valid near QCPs at finite g and T, as well as when $z \neq 1$.

Universal ratios: For QCPs described by CFTs ($z=1$) the expansion described above to get the c_1 term is called *conformal perturbation theory*, and is very powerful because the 3-point function $\langle J(x_1)J(x_2)\mathcal{O}(x_3)\rangle_{\rm QCP}$ is fixed by conformal symmetry and operator dimensions up to a single theory-dependent constant (not the case for general z). The conformal symmetry thus allows us to show that for all CFTs the ratio c_1/c_2 is universal and only depends on Δ and the normalization of $\mathcal{O}$:

$$\frac{c_1}{c_2} = \mathcal{C}_{\mathcal{O}\mathcal{O}} \frac{-\Gamma(4-\Delta)\Gamma(\Delta-\frac{3}{2})}{2^{6-4\Delta}\Gamma(1+\Delta)\Gamma(\frac{3}{2}-\Delta)}, \quad c_2 = \mathcal{C}_{JJ\mathcal{O}}, \quad (4)$$

where we have given the answer in 2d. Γ is the gamma function, and $\mathcal{C}_{\mathcal{O}\mathcal{O}}$ appears in the correlator $\langle \mathcal{O}(-p)\mathcal{O}(p)\rangle_{\rm QCP} = \mathcal{C}_{\mathcal{O}\mathcal{O}} p^{2\Delta-3}$ expressed in frequency-momentum space. The real constant $\mathcal{C}_{JJ\mathcal{O}}$ enters in the 3-point function $\langle JJ\mathcal{O}\rangle_{\rm QCP}$. The detailed derivation of Eq. (4) and its generalization to $d \neq 2$ is given in [26].

In order to get insight about the generic z case, we employ the holographic gauge-gravity duality [37–39] to study charge transport in a class of interacting large-N matrix field theories. These are dual to gravitational theories existing in a $(d+2)$-dimensional curved spacetime whose isometries mapt to the Lifshitz symmetries of the matrix field theories at general z. This approach is useful because techniques such as conformal perturbation theory, which are non-perturbative in interaction strength and robust against the large N limit, are not known for $z \neq 1$. Details of the computation will be presented in [40]; we give the result for $\langle JJ\mathcal{O}\rangle$ for general d in [26]. We follow the logic of conformal perturbation theory to demonstrate (3) and predict $c_{1,2}$; a direct computation of the high frequency asymptotics of $\sigma(\omega)$ using gauge-gravity duality confirms our prediction [40]. In 2d, we find

$$\frac{c_1}{c_2} = \frac{-\mathcal{C}_{\mathcal{O}\mathcal{O}}\,\Gamma(2+\frac{2-\Delta}{z})\Gamma(\frac{\Delta-1}{z}-\frac{1}{2})}{2^{\frac{2}{z}(2+z-2\Delta)}\Gamma(1+\frac{\Delta}{z})\Gamma(\frac{1}{2}+\frac{1-\Delta}{z})}, \quad c_2 = \mathcal{C}_{JJ\mathcal{O}}, \quad (5)$$

for $2\Delta \neq d+z+2n$, for integer n. Results for general d can be found in [26]. We note that (5) reduces to (4) when $z=1$. Unlike (4), the holographic result for c_1/c_2 at $z\neq 1$ is unlikely generic. Indeed, $\langle J_x(\omega_1)J_x(\omega_2)\mathcal{O}(\omega_3)\rangle_{\rm QCP}$ is not sharply constrained by Lifshitz symmetry. We do expect, however, that the asymptotic form of (3) remains the same near other $z \neq 1$ QCPs. Indeed, above we have provided a general scaling argument for the c_1-term at any z.

Sum rules: We can use the high-frequency expansion Eq. (3) to derive sum rules for the conductivity. This was previously done for CFTs at finite temperature but zero detuning [7, 11, 41, 42]. At $g \neq 0$, one must take into consideration the new c_1 term in the asymptotic expansion (3), which will drastically change the result in many cases. For $d+z-2 < \Delta < 2$, the sum rule reads

$$\int_0^\infty \mathrm{d}\omega\ \mathrm{Re}\left[\sigma(\omega) - \sigma(\omega)\big|_{T=g=0}\right] = 0\,. \quad (6)$$

If $\Delta > 2$ or $\Delta < z+d-2$, the integral becomes infinite making (6) ill-defined. Thus, in contrast to $d=2$, most states in $d=3$ will not obey (6) since generally $z \geq 1$. In the special case of 1d CFTs, the conditions on Δ for the validity of (6) are trivially satisfied. For general d, $\Delta=2$ or $z+d-2$ constitute special cases since the rhs of (6) can be finite and non-zero (see the $O(N)$ model calculation below). Again, (6) holds in the same regime as the asymptotic expansion, *i.e.* for points in the (g,T) phase diagram that can be reached from the QC region without crossing phase transitions. Knowledge about the expansion is needed to ensure that $\sigma(\omega)$ decays sufficiently fast at large frequencies. The other ingredient is the analyticity of σ in the upper half-plane of complex frequencies (causality), which allows us to prove the sum rule by contour integration [26].

$O(N)$ **model:** We now examine the physics described above in the context of the interacting QCPs in the $O(N)$ model in 2d, which have $z=1$ and are CFTs. We focus on 2 cases: $N=\infty$ (which is solvable), and $N=2$ which describes an interacting superfluid-insulator QCP. These QCPs are described by a relativistic ϕ^4-theory for an order parameter field ϕ_a with N real components [43]:

$$S = -\int \mathrm{d}^3x \left(\frac{1}{2}\partial_\mu\phi_a\partial^\mu\phi_a + r\phi_a\phi_a + \frac{u}{2N}(\phi_a\phi_a)^2\right). \quad (7)$$

This action is written in real time. When r is large, this model yields a gapped phase with unbroken O(N) symmetry; when r is small, O(N) is spontaneously broken and the low energy effective theory contains Goldstone bosons if $N > 1$. There are conserved currents $J^\mu_{ab} = \phi_a\partial^\mu\phi_b - \phi_b\partial^\mu\phi_a$, and our goal is to compute the corresponding conductivity. When $1 < d < 3$, dimensional analysis suggests that this QCP has a relevant operator $\phi_a\phi_a$ with detuning parameter $g \sim r$. This is qualitatively correct; in [26]we precisely identify $\mathcal{O}$ and g in terms of slightly different variables.

When $N=\infty$, this model is exactly solvable through large-N techniques [43]. The resulting QCP has $\nu=1$ and is thus distinct from the Gaussian fixed point at $u=0$. Let us begin by studying the disordered phase, which occupies the entire phase diagram except the broken symmetry state at $T=0$ and $g<0$. We obtain the following asymptotic expansion via an explicit computation of the conductivity [26].

$$\sigma(\omega) = \frac{1}{16} + \frac{4g}{\mathrm{i}\omega} - \frac{\langle\mathcal{O}\rangle_{g,T}}{4N\omega^2} + \cdots, \quad (8)$$

where $\langle\mathcal{O}\rangle_{g,T} = Nm^2$, with $m(g,T)$ being the detuning and temperature induced mass [26]. Using the previously derived values $\sigma_\infty = \frac{1}{16}$, $\Delta = 2$, $\mathcal{C}_{JJ\mathcal{O}} = \frac{1}{4N}$ and $\mathcal{C}_{\mathcal{O}\mathcal{O}} =$

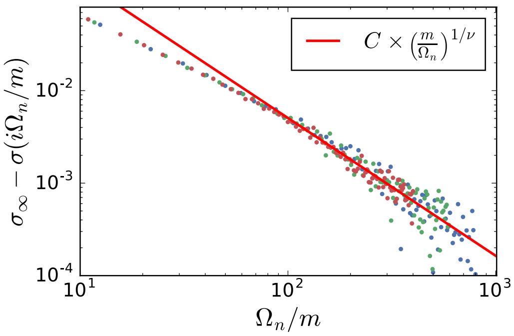

FIG. 2. Log-log plot of the asymptotic behavior of $\sigma(\mathrm{i}\Omega_n)$ at imaginary frequencies, in the disordered phase of the O(2) model, computed using QMC in the limit $T\to 0$. Each set of colored dots represents a different detuning g. $m \propto g^\nu$ is the single particle gap. The line is the field theory prediction (3) at large Ω_n, with $\nu = 0.67$.

$-16N$ [11], we find exact agreement with (4). Now, the g-linear term, although purely imaginary, alters the sum rule Eq. (6) from its $g = 0$ form because we have the special situation $\Delta = 2$. Indeed, we find that the rhs of Eq. (6) becomes finite, $-2\pi g$, which is *independent of temperature*, and changes sign across $g=0$, see [26].

The conductivity in the ordered phase at $N=\infty$, which occurs when $T = 0$ and $g < 0$, is qualitatively distinct. When the condensate is along the 1-direction $\langle\phi_1\rangle \neq 0$, the asymptotic conductivity for J^μ_{12} reads

$$\sigma(\omega) = \frac{1}{16} + \frac{64}{3\pi^2}\frac{|g|}{\mathrm{i}\omega}\ln\frac{\omega}{\mathrm{i}|g|} + \mathrm{O}\Big(\frac{1}{\omega}\Big). \qquad (9)$$

We find disagreement with (3), which can be understood as follows: conformal perturbation theory was based around the convergence of the g-expansion of $\langle JJ\mathrm{e}^{-\mathrm{i}g\int\mathcal{O}}\rangle_{\mathrm{QCP}}$. When $g<0$, this expansion can lead to IR divergences associated with the instability of the symmetric vacuum: ϕ_a has obtained an expectation value in the true vacuum. At $N = \infty$, logarithmic corrections to σ are a consequence of the coupling to Goldstone bosons, as we show in [26]. Deviations from Eq. (3) hence follow from the superfluid instability of the symmetric vacuum when $g<0$. We also note that the new logarithmic enhancement in (9) makes the sum rule (6) ill-defined because the integral diverges. Further, the logarithmic contribution in Eq. (9) is present when $2<d<3$, for all temperatures at which long range order exists, with a proportionality coefficient related to the superfluid density [26].

When $N = 2$, the model Eq. (7) describes a strongly interacting superfluid-insulator QCP, where quasiparticle excitations have been destroyed by fluctuations. We analyze its imaginary time conductivity numerically using large-scale lattice quantum Monte Carlo (QMC) simulations. We work with the action Eq. (7) in Euclidean spacetime (devoid of the sign problem), discretized on a $512 \times 512 \times 512$ cubic lattice. Details of the numerical methods are in [26]. Fig. 2 shows the universal part of the imaginary frequency conductivity in the disordered phase at different values of the detuning, near the QCP. We plot the conductivity relative to its groundstate value σ_∞ as a function of the frequency rescaled by the single-particle gap $m \propto g^\nu$. In order to do so, we must subtract off a non-universal lattice correction to σ, and employ $\sigma_\infty = 0.355(5)$, found with recent conformal bootstrap calculations [44] along with numerical simulations [8–12]. The resulting data collapses to a single universal curve. The large-ω field theory prediction (solid line) for the subleading term, which scales as $c_1\omega^{-1/\nu}$, with $\nu = 0.67$, is also shown. At $N = 2$, in contrast to the $N = \infty$ case Eq. (8), the next subleading term $\propto c_2\omega^{1/\nu-3}$ comes with nearly the same exponent, so that in practice we combine both the $c_{1,2}$ terms into a single one. By looking at the high frequency limit, we see that c_1 is negative, in agreement with our result at $N = \infty$, Eq. (8). The numerical data is also consistent with our predicted scaling $\sigma - \sigma_\infty \propto \omega^{-1/\nu}$, but due to the need to subtract off a large background conductivity to extract c_1 and ν, we presently cannot perform a more quantitative analysis.

In the superfluid phase, both the numerical and field theory analyses become complicated by the presence of the broken symmetry and the associated strongly coupled Goldstone boson(s) (at finite T, the order becomes algebraic). In order to analytically understand the asymptotic behavior of $\sigma(\omega)$, and the associated sum rule (6), one would need methods beyond what we have discussed so far. It will be interesting to see whether the result will be similar to the $N=\infty$ case, Eq. (9), with the associated breakdown of the sum rule. We leave this important question for the future.

Outlook: We have determined the large-frequency optical conductivity near a QCP for a wide class of theories, Eq. (3), in general dimensions. Our analysis incorporates non-thermal detuning and temperature, and thus extends beyond the QC fan which facilitates comparison with experiments. This has led to a unified understanding of sum rules in the phase diagram near such QCPs. Interestingly, we have found that in certain superfluid phases, interacting Goldstone bosons can qualitatively change the results. It will be of interest to analyze such effects more broadly. Our findings can potentially be tested at QCPs in superconductor-insulator systems or Josephson junction arrays [2], and in ultra-cold atomic gases. In the latter case, the physics of the superfluid-insulator QCP has already been realized [15–17], and proposals for measuring the optical conductivity exist, *e.g.* by periodic phase-modulation of the optical lattice [45]. Finally, although we focused on the optical conductivity, our general techniques apply to other correlation functions.

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