Uniform Additivity in Classical and Quantum Information

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Information theory quantifies the optimal rates of resource interconversions, usually in terms of entropies. However, nonadditivity often makes evaluating entropic formulas intractable. In a few auspicious cases, additivity allows a full characterization of optimal rates. We study uniform additivity of formulas, which is easily evaluated and captures all known additive quantum formulas. Our complete characterization of uniform additivity exposes an intriguing new additive quantity and identifies a remarkable coincidence—the classical and quantum uniformly additive functions with one auxillary variable are identical.

Entropies tell us how much information is stored in a system. As a result, the answers to information theoretic questions are usually found in terms of entropies evaluated on systems arising in optimal protocols. For example, the communication capacity of a classical channel $\mathcal{N}$ that maps random variable X to Y is given by the maximization $C(\mathcal{N}) = \max_X I(X;Y)$, where the mutual information $I(X;Y) = H(X) + H(Y) - H(XY)$ is a linear combination of entropies [25]. Similarly, the cost of transmitting a quantum state ρ_A on system A is its von Neumann entropy $H(A) = -\operatorname{tr} \rho_A \log \rho_A$. A noisy quantum communication channel $\mathcal{N} : A \to B$ can be mathematically extended to an isometry $U : A \to BE$ of the input with an independent and inaccessible environment. Such a channel can be applied to a state ϕ_{VA} to create a state ρ_{VBE}. More generally, V may have many subsystems, and we may use $\phi_{V_1...V_nA}$ to create $\rho_{V_1...V_nBE}$. We can use such a state to generate an *entropic formula*: $f_\alpha(U_\mathcal{N}) = \max_{\phi_{V_1...V_nA}} f_\alpha(U_\mathcal{N}, \phi_{V_1...V_nA})$ with $f_\alpha(U_\mathcal{N}, \phi_{V_1...V_nA}) = \sum_{s\in\mathcal{P}(V_1...V_nBE)} \alpha_s H(\rho_s)$, where $\mathcal{P}(V_1...V_nBE)$ ranges over all collections of subsystems from $V_1...V_nBE$, and $H(\rho_s)$ is the entropy of collection s. We call the $V_1...V_n$ systems auxiliary variables, and they can a priori have arbitrary, even infinite, dimensions. Most operationally relevant quantities in quantum information can be expressed as a regularization of such a formula:

$$f_\alpha^\infty(\mathcal{N}) = \lim_{n\to\infty} \frac{1}{n} f_\alpha\left(\mathcal{N}^{\otimes n}\right), \tag{1}$$

where $\mathcal{N}^{\otimes n}$ is the n-fold parallel use of channel $\mathcal{N}$. The auxiliary variables in an entropic formula are usually related operationally to the structure of optimal protocols; for example, the optimal distribution X that maximizes $C(\mathcal{N}) = \max_X I(X;Y)$ to give the classical capacity defines a distribution of capacity-achieving error correcting codes.

The infinite-dimensional optimization of Eq.(1), which is called a multi-letter formula, is usually intractable. In some rare cases additivity allows a substantial simplification. An entropic formula $f_\alpha(U_\mathcal{N})$ is *additive* if $f_\alpha(U_\mathcal{N}\otimes U_\mathcal{M}) = f_\alpha(U_\mathcal{N}) + f_\alpha(U_\mathcal{M})$ for all channels $\mathcal{N}$ and $\mathcal{M}$. When f_α is additive, we have $f_\alpha^\infty(U_\mathcal{N}) = f_\alpha(U_\mathcal{N})$, which is called a single-letter formula. There are single-letter formulas for the classical capacity of a classical channel [1], the entanglement-assisted capacity of a quantum channel [2], and the quantum capacity of a quantum channel with access to a special zero-capacity assistance channel[3]. Furthermore, there are single-letter formulas for the classical capacity of an entanglement breaking channel [4] and the quantum capacity of degradable channels [5]. A single-letter formula often leads to a tractable means of evaluating a quantity, allowing us to completely characterize the optimal performance for information transmission and storage.

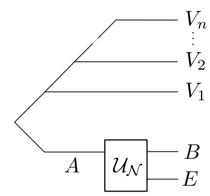

FIG. 1: Using a quantum channel to generate a quantum state. A noisy quantum channel from input A to output B can always be thought of as a unitary interaction of the input with some inaccessible environment E. We can generate a quantum state from this interaction by creating $\phi_{V_1...V_nA}$ and acting on A with $U_\mathcal{N}$, leading to the state $\rho_{V_1...V_nBE} = I \otimes U_\mathcal{N} \phi_{V_1...V_nA} I \otimes U_\mathcal{N}^\dagger$.

Many relevant entropic formulas are nonadditive, especially in the quantum setting[6–10]. Optimal performance is thus captured only by a *multi-letter* formula, which is intractable to evaluate. Even the capacities themselves can exhibit nonadditivity, displaying fundamentally quantum synergies not present classically [9–13]. As a result, many basic questions in quantum information theory remain open—the classical and quantum capacities of most channels are unknown, and even deciding if a quantum channel has nonzero quantum capacity seems insurmountable [14].

Entropy inequalities express relationships between entropies of different collections of subsystems that are satisfied for all states. Subadditivity of entropy, for example, tells us that $H(A)+H(B)-H(AB)\geq 0$, or equivalently $I(A;B)\geq 0$. Its generalization, strong subadditivity[15], tells us that conditional mutual information is also positive:

$$I(A;B|C) = H(AC) + H(BC) - H(ABC) - H(C) \geq 0. \quad (2)$$

The set of (2^n-1)-dimensional entropy vectors $\mathbf{v} = (H(X_1), ...H(X_n), ..., H(X_1...X_n))$ that can be realized by classical probability distributions on $X_1...X_n$ form a cone, whose study in terms of linear programming was formalized in [16]. The larger cone of realizable quantum entropies was studied in [17]. Entropy inequalitites are the key to proving additivity when it exists.

If f_α is an additive formula with one auxiliary variable [26], for any pair of channels $\mathcal{N},\mathcal{M}$ and any state $\phi_{VA_1A_2}$, there must be a pair of states $\tilde{\phi}_{\tilde{V}A_1}$ and $\hat{\phi}_{\hat{V}A_2}$ such that

$$f_\alpha\left(U_\mathcal{N}\otimes U_\mathcal{M}, \phi_{VA_1A_2}\right) \leq f_\alpha(U_\mathcal{N}, \tilde{\phi}_{\tilde{V}A_1}) + f_\alpha(U_\mathcal{M}, \hat{\phi}_{\hat{V}A_2}). \quad (3)$$

We call such a mapping $\phi_{VA_1A_2}\rightarrow(\tilde{\phi}_{\tilde{V}A_1}, \hat{\phi}_{\hat{V}A_2})$ a decoupling. In principle, the appropriate decoupling may depend in an arbitrary way on the channels $\mathcal{N},\mathcal{M}$ and the state $\phi_{VA_1A_2}$. In practice, useful decouplings are invariably what we call *standard* decouplings, which have a very simple form and are described in Fig. 2. Once we have fixed a decoupling and f_α, we can use entropy inequalities to determine if Eq. (3) is satisfied. When f_α does satisfy Eq. (3) with $(\tilde{\phi},\hat{\phi})$ defined by a standard decoupling D, we say f_α is uniformly subadditive with respect to D. Since we also have $f_\alpha(U_\mathcal{N}\otimes U_\mathcal{M}, \tilde{\phi}\otimes\hat{\phi}) = f_\alpha(U_\mathcal{N},\tilde{\phi}) + f_\alpha(U_\mathcal{M},\hat{\phi})$, subadditivity implies that

$$f_\alpha(U_\mathcal{N}\otimes U_\mathcal{M}) = f_\alpha(U_\mathcal{N}) + f_\alpha(U_\mathcal{M}) \quad (4)$$

and we call f_α uniformly additive with respect to D. All known proofs of quantum additivity proceed by choosing a standard decoupling and proving Eq. (3) via entropy inequalities [2, 3, 18].

We have found all entropic formulas f_α that are uniformly additive with respect to standard decouplings. We do this by enumerating all standard decouplings, and using the linear programming formulation of entropy inequalities to determine which f_α are uniformly subadditive for each decoupling. Our approach captures all previously known examples of additive formulas and more. This method opens a line of attack on a variety of questions, from classical multiuser information theory to finding new classes of channels with additive capacities, and clarifies when and where to expect quantum synergies like superactivation [11].

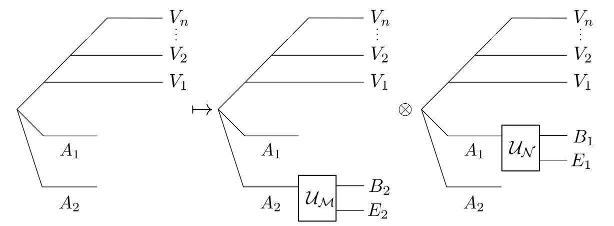

FIG. 2: Decoupling is the process of mapping one state that can be acted on by two channels into two separate states, each of which can be acted on by a single channel use. It maps a state $\phi_{V_1...V_nA_1A_2}$ to two states, $\tilde{\phi}_{\tilde{V}_1...\tilde{V}_nA_1}$ and $\hat{\phi}_{\hat{V}_1...\hat{V}_nA_2}$. Here A_1 and A_2 are the input spaces to $\mathcal{N}$ and $\mathcal{M}$, so that $U_\mathcal{N}\otimes U_\mathcal{M}$ can be applied to $\phi_{V_1...V_nA_1A_2}$ to make $\rho_{V_1...V_nB_1E_1B_2E_2}$, while $U_\mathcal{N}$ acts on $\tilde{\phi}_{\tilde{V}_1...\tilde{V}_nA_1}$ to make $\tilde{\rho}_{\tilde{V}_1...\tilde{V}_nB_1E_1}$ and $U_\mathcal{M}$ acts on $\hat{\phi}_{\hat{V}_1...\hat{V}_nA_2}$ to make $\hat{\rho}_{\hat{V}_1...\hat{V}_nB_2E_2}$. For a *standard* decoupling, the states $\tilde{\phi}_{\tilde{V}_1...\tilde{V}_nA_1}$ and $\hat{\phi}_{\hat{V}_1...\hat{V}_nA_2}$ are constructed from $\phi_{V_1...V_nA_1A_2}$ as follows. To obtain $\tilde{\phi}_{\tilde{V}_1...\tilde{V}_nA_1}$, we first apply $U_\mathcal{M}$ to make $\phi_{V_1...V_nA_1B_2E_2}$. Given $\phi_{V_1...V_nA_1B_2E_2}$, we define $\tilde{V}_i$ to contain V_i. B_2 and E_2 are each either assigned to one of the $\tilde{V}_i$ (or perhaps traced out) to generate $\tilde{\phi}_{\tilde{V}_1...\tilde{V}_nA_1}$. We define $\hat{\phi}_{\hat{V}_1...\hat{V}_nA_2}$ similarly.

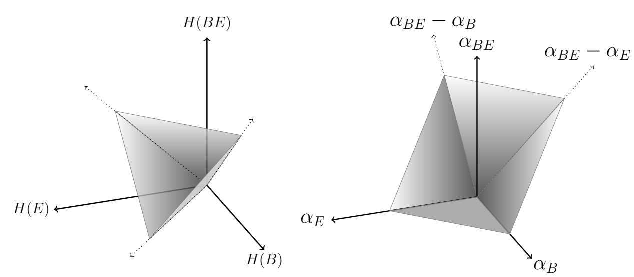

FIG. 3: (Left) Quantum Entropy Cone for two systems. The entropies of a bipartite quantum state ρ_{BE} form a vector $(H(B),H(E),H(BE))$. The vectors of entropies that can be realized by a quantum state lie in a cone. For two systems, the faces of this cone are implied by strong subadditivity. This is also true for $n=3$ systems, but for $n\geq 4$ we do not know whether the quantum entropy cone lies strictly inside the cone implied by strong subadditivity. (Right) Additivity cone. Fixing a decoupling gives an entropy inequality that implies additivity. We check whether this inequality is satisfied by using known entropy inequalites, as expressed by the quantum entropy cone. We find a cone of coefficients defining the entropy formulas that are uniformly additive with respect to the fixed decoupling. The cone above is the additive cone for zero-auxiliary variable formulas.

Formulas with no auxiliary variables are particularly simple:

$$f_\alpha(U_\mathcal{N},\phi_A) = \alpha_B H(B) + \alpha_E H(E) + \alpha_{BE} H(BE). \quad (5)$$

Here we have only one standard decoupling to consider: $\phi_{A_1A_2}\rightarrow(\phi_{A_1},\phi_{A_2})$. The conditions for uniform additivity in this case are

$$\alpha_B + \alpha_{BE} \geq 0,\ \alpha_E + \alpha_{BE} \geq 0, \quad (6)$$
$$\alpha_B + \alpha_E + \alpha_{BE} \geq 0,\ \alpha_{BE} \geq 0.$$

These inequalities define a cone of αs, which we refer to as a uniform additivity cone. Eq. (6) describes this cone in terms of its facets, but a cone can equally well be described in terms of extremal rays: letting

$$\alpha_0 = (1,0,0) \equiv H(B),\ \alpha_1 = (0,1,0) \equiv H(E) \quad (7)$$
$$\alpha_2 = (0,-1,1) \equiv H(B|E),\ \alpha_3 = (-1,0,1) \equiv H(E|B),$$

α satisfies Eq. (6) exactly when $\alpha = \sum_i \lambda_i \alpha_i$ with $\lambda_i \geq 0$.

We now argue that Eq (6) captures all uniformly additive formulas with no auxiliary variables. To begin, for a zero auxilliary variable f_α, we define

$$\begin{aligned}\Delta^{\varnothing}(\alpha, U_{\mathcal{N}} \otimes U_{\mathcal{M}}, \phi_{A_1A_2}) &= f_\alpha(U_{\mathcal{N}}, \phi_{A_1}) + f_\alpha(U_{\mathcal{M}}, \phi_{A_2}) - f_\alpha(U_{\mathcal{N}} \otimes U_{\mathcal{M}}, \phi_{A_1A_2}) \\ &= \alpha_B I(B_1;B_2) + \alpha_E I(E_1;E_2) + \alpha_{BE} I(B_1E_1;B_2E_2),\end{aligned}$$

so that f_α is uniformly additive with respect to the standard decoupling exactly when $\forall \mathcal{N}, \mathcal{M}, \phi_{A_1A_2}$ we have $\Delta^{\varnothing}(\alpha, U_{\mathcal{N}} \otimes U_{\mathcal{M}}, \phi_{A_1A_2}) \geq 0$. We make use of the alternate characterization of Eq.(6) in terms of extremal rays, Eq. (7). It is easy to verify that the αs associated with each of the extremal rays $H(B)$, $H(E)$, $H(E|B)$, and $H(B|E)$ lead to positive $\Delta^{\varnothing}$s. For example, $H(B)$ corresponds to $(\alpha_B, \alpha_E, \alpha_{BE}) = (1,0,0)$ and $\Delta^{\varnothing}(\alpha, U_{\mathcal{N}} \otimes U_{\mathcal{M}}, \rho_{A_1A_2}) = I(B_1;B_2) \geq 0$, while $H(B|E)$ corresponds to $(\alpha_B, \alpha_E, \alpha_{BE}) = (0,-1,1)$ and gives

$$\Delta^{\varnothing}(\alpha, U_{\mathcal{N}} \otimes U_{\mathcal{M}}, \rho_{A_1A_2}) = I(B_1E_1;B_2E_2) - I(E_1;E_2),$$

which is also positive for all $\rho_{A_1A_2}$. $H(E)$ and $H(E|B)$ follow mutatis mutandis. Eq. (6) is thus a sufficient condition for uniform additivity. To see that it is also a necessary condition, we find states (in fact, classical distributions) p^0, p^1, p^2, p^3 and channels $\mathcal{N}$, $\mathcal{M}$ such that

$$\begin{aligned}\Delta^{\varnothing}(\alpha, U_{\mathcal{N}} \otimes U_{\mathcal{M}}, p^0) &= \alpha_B + \alpha_{BE} \\ \Delta^{\varnothing}(\alpha, U_{\mathcal{N}} \otimes U_{\mathcal{M}}, p^1) &= \alpha_E + \alpha_{BE} \\ \Delta^{\varnothing}(\alpha, U_{\mathcal{N}} \otimes U_{\mathcal{M}}, p^2) &= \alpha_B + \alpha_E + \alpha_{BE} \\ \Delta^{\varnothing}(\alpha, U_{\mathcal{N}} \otimes U_{\mathcal{M}}, p^3) &= \alpha_{BE}.\end{aligned}$$

This shows that for any α that doesn't satisfy Eq. (6) there are states and channels such that $\Delta^{\varnothing}(\alpha, U_{\mathcal{N}} \otimes U_{\mathcal{M}}, p) < 0$. Thus, Eq. (6) are both necessary and sufficient for uniform additivity.

Formulas with one auxiliary variable require us to consider multiple decouplings, capturing the choice of $\tilde{V}$ and $\hat{V}$ in the decoupling map $D : \phi_{VA_1A_2} \to (\tilde{\phi}_{\tilde{V}A_1}, \hat{\phi}_{\hat{V}A_2})$. A standard decoupling always has $\tilde{V} = \tilde{M}_2 V$ with $\tilde{M}_2$ chosen from $\{\varnothing, B_2, E_2, B_2E_2\}$ and $\hat{V} = \hat{M}_1 V$ with $\hat{M}_1$ chosen from $\{\varnothing, B_1, E_1, B_1E_1\}$. We can parametrize these by (a,b), with a and b running from 0 to 3. We take advantage of two simplifications that can be made without loss of generality. First, given f_α, $\alpha = (\alpha^{\varnothing}, \alpha^V)$ with $\alpha^{\varnothing} = (\alpha_B, \alpha_E, \alpha_{BE})$ and $\alpha^V = (\alpha_V, \alpha_{BV}, \alpha_{EV}, \alpha_{BEV})$, we can define $f^{\varnothing}_{\alpha^{\varnothing}}$ and $f^V_{\alpha^V}$ such that f_α is uniformly additive with respect to decoupling (a,b) if and only if $f^{\varnothing}_{\alpha^{\varnothing}}$ is uniformly additive with respect to the decoupling $\phi_{A_1A_2} \to (\phi_{A_1}, \phi_{A_2})$ and $f^V_{\alpha^V}$ is uniformly additive with respect to (a,b). Second, these formulas have two useful symmetries that reduce the number of decouplings we must consider: 1) for any additive formula, we get a similar additive formula by exchanging B and E and 2) $f^V_{\alpha^V}$ with $\alpha^V = (\alpha_V, \alpha_{BV}, \alpha_{EV}, \alpha_{BEV})$ is equivalent via purification of the quantum state to $f^V_{\tilde{\alpha}^V}$ with $\tilde{\alpha}^V = (\alpha_{BEV}, \alpha_{EV}, \alpha_{BV}, \alpha_V)$. This leaves only 5 inequivalent decouplings to be considered.

Table I describes the functions $f^V_{\alpha^V}$ that are uniformly additive with respect to the 5 inequivalent decouplings. They are positive linear combinations[27] of the extreme rays in the corresponding row of the table. The uniformly additive functions with respect to decoupling (a,b) are the sum of any $f^{\varnothing}_{\alpha^{\varnothing}}$ satisfying Eq. (6) and such an $f^V_{\alpha^V}$ found from Table I.

We find many familar additive quantities in this way. For example, maximum output entropy ($\max_{\phi_A} H(B)$) satisfies Eq. (6). The quantity $-H(B|V)$ was shown to be additive in [18], and later refered to as reverse coherent information[19]. Since $H(B)$ satisfies Eq. (6) and $-H(B|V)$ is uniformly additive with respect to multiple decouplings, so is their sum $H(B) - H(B|V) = I(B;V)$, whose maximization gives the entanglement assisted capacity.

One extreme ray of the $(1,2)$ decoupling's additive

case	(a,b)	$\hat{M}_1$	$\tilde{M}_2$	equivalents	Additive Cone	Extreme Rays
1.	(3,3)	B_1E_1	B_2E_2	(0,0)	$\alpha_V+\alpha_{BV}+\alpha_{EV}\geq 0, \alpha_V+\alpha_{BV}\geq 0,$ $\alpha_V+\alpha_{EV}\geq 0, \alpha_V\geq 0$	$-H(E\|BV), -H(E\|V)$ $-H(B\|EV), -H(B\|V)$
2.	(3,2)	B_1E_1	E_2	$(2,3),(3,1),(1,3),(1,0)$ $(0,1),(2,0),(0,2)$	$\alpha_{BV}\leq 0, \alpha_V+\alpha_{BV}\geq 0$	$-H(BE\|V), \pm H(B\|EV)$ $-H(B\|V)$
3.	(3,0)	B_1E_1	$\varnothing$	(0,3)	$\alpha_{EV}\leq 0, \alpha_{BV}\leq 0$	$H(E\|BV), -H(E\|V), \pm H(BE\|V)$
4.	(1,1)	B_1	B_2	(2,2)	$\alpha_{EV}=0, \alpha_V\geq 0, \alpha_{BEV}\geq 0$	$-H(B\|V), H(E\|BV)$
5.	(1,2)	B_1	E_2	(2,1)	$\alpha_{BEV}\geq 0, \alpha_V\geq 0$	$\pm[H(EV)-H(BV)]$ $H(E\|BV), -H(E\|V)$

TABLE I: Functions $f^V_{\alpha^V}$ that are uniformly subadditive with respect to the 5 inequivalent standard decouplings. Fixing a decoupling D, a single-auxiliary variable f_α is uniformly subadditive with respect to D exactly when it can be written as a sum of $f^{\varnothing}_{\alpha^\varnothing}$ satisfying Eq.6 and $f^V_{\alpha^V}$ that is a positive linear combination of the extreme rays in the row corresponding to D. Multiple auxiliary variables are all found similarly.

cone is particularly intriguing:

$$I^{cc}(\mathcal{N})=\max_{\phi_{VA}}[H(VB)-H(VE)]. \tag{8}$$

We call this quantity the completely coherent information, since its relationship to the coherent information $I^{\rm coh}(\mathcal{N})=\max_A[H(B)-H(E)]$ is similar to the relationship between completely positive and positive maps. The version of this quantity evaluated on states was shown in [20] to be a lower bound on the communication cost of exchanging the B and E systems, but it was not realized that it is additive. We also show that I^{cc} is also an *upper* bound for the jointly achievable quantum communication rate from A to either B or E. We have not found a clear operational interpretation of this quantity.

We now consider formulas with multiple auxiliary variables. For concreteness, suppose we have some formula depending on two auxiliary variables V_1 and V_2. A standard decoupling is a mapping from a state $\phi_{V_1V_2A_1A_2}$ to two states $\tilde{\phi}_{\tilde{V}_1\tilde{V}_2A_1}$ and $\hat{\phi}_{\hat{V}_1\hat{V}_2A_2}$ that we get by choosing to incorporate (or not) B_2 and E_2 into one of $\tilde{V}_1$ and $\tilde{V}_2$ (and similarly for B_1, E_1 in $\hat{V}_1$ and $\hat{V}_2$). Since $\tilde{V}_1$ and $\tilde{V}_2$ should be non-overlapping, it is necessary to impose some consistency on the decouplings (a_1,b_1) and (a_2,b_2). These also give rise to a third decoupling, which we call $(a^\star,b^\star)$, that tells us which systems get included in the joint systems $\tilde{V}_1\tilde{V}_2$ and $\hat{V}_1\hat{V}_2$.

In this case it is possible to separate the variables much as we did in the single-variable case. Indeed, any f_α with $\alpha=(\alpha^\varnothing,\alpha^{V_1},\alpha^{V_2},\alpha^{V_1V_2})$ [28] is uniformly additive with respect to decoupling $\{(a_1,b_1),(a_2,b_2)\}$ exactly when $f^{\varnothing}_{\alpha^\varnothing}$, $f^{V_1}_{\alpha^{V_1}}$, $f^{V_2}_{\alpha^{V_2}}$, and $f^{V_1V_2}_{\alpha^{V_1V_2}}$ are uniformly additive with respect to their respective decouplings. The same is true for more auxiliary variables. For any number of auxiliary variables, all f_α uniformly additive with respect to standard decouplings can be constructed from Table I and Eq. (6) [21].

Surprisingly, carrying out the same analysis as above for classical states and channels yields *exactly* the same set of uniformly additive functions for one auxillary variable. This is in spite of the fact that the classical and quantum entropy cones do not coincide. This coincidence of uniformly additive functions may explain a well-known phenomenon: Formulas that solve classical information theory problems often tend to have corresponding quantum formulas that solve an appropriately coherified version of the problem [29]. In these cases, the classical and quantum problems have a solution for the same reason: the existence of an appropriately additive formula whose additivity proofs are formally equivalent. It would be very nice to formalize this apparent correspondence and explore its limits.

In some cases, nonadditive formulas can become additive when evaluated on special classes of channels. For example, while both the holevo information and minimum output entropy are nonadditive [8], for entanglement breaking channels they become additive. Similarly, while coherent information is nonadditive [6], it is additive on degradable channels [5]. Understanding such *conditional* additivities is an important open questions, and we are currently exploring the application of our techniques to finding special classes of channels that have additive capacities. We have identified a new criterion for the additivity of coherent information: informational degradability. We say a channel is informationally degradable if for any input state ϕ_{VA} we have $I(V;B)\geq I(V;E)$. This class includes degradable channels. We suspect informational degradability is the *only* single-letter entropic constraint on a channel that implies this additivity. We have also found a set of entropic constraints that imply a state is of the c-q form, which should be useful for studying classical and private capacities of quantum channels.

We have identified the limits of the techniques used in all known instances of quantum additivity. There *are* some classical formulas that are additive but not uniformly additive (e.g., minimum output entropy of a classical channel). Proving additivity in these cases requires knowledge of the optimizing state (in the case of minimum output entropy of a quantum channel, the optimal state is a pure state, which for classical channels is also a product state.). One potential path to new quantum additive formulas beyond what we have found is to better understand the optimizing state in an entropic formula. At this point we know of no examples where this can be done, but they may well exist.

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[25] The entropy of a random variable X is $H(X) = -\sum_x p_x \log p_x$
[26] We focus on 1 auxiliary variable for simplicity. Multiple variables can be handled similarly.
[27] i.e., linear combinations with positive coefficients
[28] Here $\alpha^{\varnothing} = (\alpha_B, \alpha_E, \alpha_{BE})$,

$$\alpha^{V_1} = (\alpha_{V_1}, \alpha_{BV_1}, \alpha_{EV_1}, \alpha_{BEV_1}),$$

$$\alpha^{V_2} = (\alpha_{V_2}, \alpha_{BV_2}, \alpha_{EV_2}, \alpha_{BEV_2}),$$

and $\alpha^{V_1V_2} = (\alpha_{V_1V_2}, \alpha_{BV_1V_2}, \alpha_{EV_1V_2}, \alpha_{BEV_1V_2})$.
[29] Examples of this include 1) the correspondence between classical capacity of a classical channel and the entanglement assisted capacity of a quantum channel, 2) the connection between Slepian-Wolf and state merging, and 3) the correspondence between Csiszar-Korner solution to the broadcast channel with confidential messages and the recent analysis of the quantum one-time pad.