Holography of the Dirac Fluid in Graphene with Two Currents

Yunseok Seo, Geunho Song, Philip Kim, Subir Sachdev, and Sang-Jin Sin

Holography of the Dirac Fluid in Graphene with two currents

Yunseok Seo[1], Geunho Song[1], Philip Kim[2,3], Subir Sachdev[2,4] and Sang-Jin Sin[1]

[1]*Department of Physics, Hanyang University, Seoul 133-791, Korea.*
[2]*Department of Physics, Harvard University, Cambridge, MA 02138, USA.*
[3]*Department of Physics and Astronomy, Seoul National University, Seoul 151-747, Korea*
[4]*Perimeter Institute for Theoretical Physics, Waterloo, Ontario, Canada N2L 2Y5.*
(Dated: November 29, 2016)

Recent experiments have uncovered evidence of the strongly coupled nature of the graphene: the Wiedemann-Franz law is violated by up to a factor of 20 near the charge neutral point. We describe this strongly-coupled plasma by a holographic model in which there are two distinct conserved U(1) currents. We find that our analytic results for the transport coefficients for two current model have a significantly improved match to the density dependence of the experimental data than the models with only one current. The additive structure in the transports coefficients plays an important role. We also suggest the origin of the two currents.

PACS numbers: 11.25.Tq, 71.10.Hf.

Introduction: It has been argued that graphene near charge neutrality forms a strongly interacting plasma, the Dirac fluid. It does not have well-defined quasiparticle excitations, and amenable to a hydrodynamic description [1–10]. Evidence for such a Dirac fluid has appeared in recent experiments [11] on a violation of the Wiedemann-Franz law (WFL) in extremely clean graphene near the charge neutral point: the ratio of heat conductivity and electric conductivity, $\mathcal{L} = \kappa/T\sigma$, was found to be up to 20 times the Fermi liquid value.

The simplest hydrodynamic model [12], with point-like and uncorrelated disorder and a single conserved U(1) current, agrees with the overall experimental trends, but has difficulty capturing the density dependencies of both the electrical (σ) and thermal (κ) conductivities [13]. An alternative hydrodynamic model, the "puddle" model, with long-wavelength disorder in the chemical potential and a single conserved U(1) current, led to a better agreement with observations [13], but still left a room for improvement.

In this letter, we will explore a model with two conserved U(1) currents. The idea is that introducing a new neutral current can enhance the transport of the heat relative to that of the charge. Our model will be formulated in holographic terms[14, 15], to utilise the recent progress in the development of transport calculation in gauge/gravity duality [16–27]. The Dirac fluid in our model is described by an Anti de Sitter (AdS) black hole in 3+1 dimensions, the holographic dual of 2+1 dimensional system at finite temperature. The momentum dissipation is treated using scalar fields, which corresponds to weak point-like disorder. We calculate electric, thermo-electric power and thermal conductivities analytically. We find that, under the assumption that the conserved charges Q_1, Q_2 are proportional to each other, the theoretical results for the density dependencies of the electric and heat conductivities can now satisfactorily match the the experimental data in the Dirac fluid regime.

One possible mechanism for the extra current is the kinematic constraints of energy-momentum conservation on the Dirac cone, which reduce the phase space of electron and hole scattering significantly [4], allowing electrons and holes to form independent currents as far as the relaxation time for mixing between the currents is presumed to be is much longer than the Plankian relaxation time $\hbar/k_B T$, the time required for hydrodynamic regime at work. It should be noted, however, that the estimates of electron and hole equilibration times are made in quasiparticle framework [4], whose validity in hydrodynamic regime is just assumed here. We will see that the kinematics on the Dirac cone also provide a reason why the two charge densities can be proportional.

DC Transport with two $U(1)$ fields : We start from the action $S = \int d^4 x \sqrt{-g}\mathcal{L}$ with two gauge fields A_μ, B_μ, a dilaton field ϕ and the scalar fields χ_1, χ_2 for momentum dissipation:

$$\mathcal{L} = R - \frac{1}{2}\left[(\partial\phi)^2 + \Phi_1(\phi)(\partial\chi_1)^2 + \Phi_2(\phi)(\partial\chi_2)^2\right]$$
$$- V(\phi) - \frac{Z(\phi)}{4}F^2 - \frac{W(\phi)}{4}G^2, \quad (1)$$

where $F = dA$, $G = dB$ and $F^2 = F_{\mu\nu}F^{\mu\nu}$ etc. We also require positivity of $\Phi_i(\phi)$, $Z(\phi)$ and $W(\phi)$. The action (1) yields equations of motion:

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\mathcal{L} = T_{\mu\nu}, \ \nabla_\mu(\sqrt{-g}\Phi_i\nabla^\mu\chi_i) = 0 = \partial_\mu(\sqrt{-g}ZF^{\mu\nu}),$$

$$\nabla^2\phi - \sum_{i=1}^{2}\frac{\Phi_i'}{2}(\phi)(\partial\chi_i)^2 - V'(\phi) - \frac{Z'(\phi)}{4}F^2 - \frac{W'(\phi)}{4}G^2 = 0,$$

$$T_{\mu\nu} = \frac{1}{2}\partial_\mu\phi\partial_\nu\phi + \sum_{i=1}^{2}\frac{\Phi_i}{2}\partial_\mu\chi_i\partial_\nu\chi_i + \frac{Z}{2}F_\mu^\lambda F_{\nu\lambda} + \frac{W}{2}G_\mu^\lambda G_{\nu\lambda}.$$
$$(2)$$

We take the ansatz for metric and the gauge fields as

$$ds^2 = -U(r)dt^2 + \frac{1}{U(r)}dr^2 + e^{v(r)}(dx^2 + dy^2)$$

$$A = A(r)dt, \quad B = B(r)dt. \quad (3)$$

The gauge field A has the chemical potential and charge density as its components of its near boundary expansion, $A(r) = \mu_1 - q_1/r + \cdots$. At the horizon at $r = r_0$, U vanishes and $A, B \to 0$ also for the regularity. If we take the following solution, $\chi_1 = kx$, $\chi_2 = ky$, it provides momentum relaxation. From now on, we set $\Phi_1 = \Phi_2 = \Phi$ for simplicity. The only non-zero components in the Maxwell equations are that for the tr-component of the field strengths whose first integral give conserved charges,

$$Q_1 = \sqrt{-g}Z(\phi)F^{tr} = Z(\phi)e^v A'(r)$$
$$Q_2 = \sqrt{-g}W(\phi)G^{tr} = W(\phi)e^v B'(r). \quad (4)$$

One can see that if $e^v \sim r^2$ in asymptotic region, Q_i corresponds to the charge density of the boundary field theory. To compute the transport coefficients, we turn on small fluctuations around the background solution:

$$\delta G_{tx} = t\delta f_1(r) + \delta g_{tx}(r), \quad \delta G_{rx} = e^{v(r)}\delta h_{rx}(r),$$
$$\delta A_x = t\delta f_2(r) + \delta a(r), \quad \delta B_x = t\delta f_3(r) + \delta b(r), \quad (5)$$

as well as $\delta\chi_i(r)$'s. We choose the functions $f_i(r)$ as

$$\delta f_1(r) = -\zeta U(r), \quad \delta f_2(r) = -E_1 + \zeta A(r),$$
$$\delta f_3(r) = -E_2 + \zeta B(r), \quad (6)$$

such that the time t does not appear in the equations of motion of the fluctuations. Here, E_1, E_2 are thmo-electric forces acting on J_1, J_2 respectively and $\zeta = -\nabla T/T$. From the A field fluctuation equations, the currents are defined by [18],

$$J_1 = \sqrt{-g}Z(\phi)F^{xr}, \qquad J_2 = \sqrt{-g}W(\phi)G^{xr}$$
$$\mathcal{Q} = U(r)^2 \frac{d}{dr}\left(\frac{\delta g_{tx}(r)}{U(r)}\right) - A(r)J_1 - B(r)J_2. \quad (7)$$

Notice that near the boundary, the heat current becomes $\mathcal{Q} = T^{tx} - \mu_1 J_1 - \mu_2 J_2$. Moreover, these currents are conserved along radial direction r. Therefore their boundary values are related to that of horizon data, which can be determined from the regularity at the horizon [19]:

$$\delta a(r) \sim -\frac{E_1}{4\pi T}\ln(r - r_0), \quad \delta g_{tx}(r) \sim \delta g_{tx}^{(0)}, \cdots. \quad (8)$$

Using above horizon behavior we get the boundary current in terms of the external sources:

$$J_1 = \left(Z_0 + \frac{e^{-v_0}Q_1^2}{k^2\Phi_0}\right)E_1 + \frac{e^{-v_0}Q_1Q_2}{k^2\Phi}E_2 + \frac{4\pi TQ_1}{k^2\Phi_0}\zeta,$$
$$J_2 = \left(W_0 + \frac{e^{-v_0}Q_2^2}{k^2\Phi_0}\right)E_2 + \frac{e^{-v_0}Q_1Q_2}{k^2\Phi}E_1 + \frac{4\pi TQ_2}{k^2\Phi_0}\zeta$$
$$\mathcal{Q} = \frac{4\pi TQ_1}{k^2\Phi_0}E_1 + \frac{4\pi TQ_2}{k^2\Phi_0}E_2 + \frac{(4\pi T)^2 e^{v_0}}{k^2\Phi_0}\zeta. \quad (9)$$

The eq. (9) can be written in matrix form, $J_i = \Sigma_{ij}E_j$, with $J_3 = \mathcal{Q}$ and $E_3 = \zeta$. The transport coefficients can be read off from the eq. (9) and the definition

$$\begin{pmatrix} \sigma_1 & \delta & \alpha_1 T \\ \bar\delta & \sigma_2 & \alpha_2 T \\ \bar\alpha_1 T & \bar\alpha_2 T & \bar\kappa T \end{pmatrix} := \Sigma. \quad (10)$$

Notice that the matrix is real and symmetric, so that the Onsager relations hold:

$$\bar\alpha_i = \alpha_i, \quad \bar\delta = \delta. \quad (11)$$

The heat conductivity κ is defined by the response of the temperature gradient to the heat current in the absence of other currents: setting J_1 and J_2 to be zero in (9), we can express E_1 and E_2 in terms of ζ. Substituting these to the first line of (9), we get

$$\kappa = \bar\kappa - \frac{T\bar\alpha_1(\alpha_1\sigma_2 - \alpha_2\delta)}{\sigma_1\sigma_2 - \delta\bar\delta} - \frac{T\bar\alpha_2(\alpha_2\sigma_1 - \alpha_1\bar\delta)}{\sigma_1\sigma_2 - \delta\bar\delta}. \quad (12)$$

To discuss more explicitly, we consider a black hole solution with two charges:

$$U(r) = r^2 - \frac{m_0}{r} - \frac{k^2}{2} + \frac{1}{4r^2}\left(\frac{Q_1^2}{Z_0} + \frac{Q_2^2}{W_0}\right), \quad (13)$$

where m_0 is given by $U(r_0) = 0$ and the temperature is

$$T = \frac{r_0}{4\pi}\left(3 - \frac{k^2}{2r_0^2} - \frac{Q_1^2}{4Z_0 r_0^4} - \frac{Q_2^2}{4W_0 r_0^4}\right). \quad (14)$$

The solutions of $U(1)$ gauge fields are $a(r) = \mu_1 - \frac{q_1}{r}$, $b(r) = \mu_2 - \frac{q_2}{r}$. Notice $q_i = Q_i/Z_i$ with Z_1, Z_2 being Z_0, W_0 respectively. For the finite vector norm $g^{\mu\nu}A_\mu A_\nu$ at the horizon $r = r_0$, we need $\mu_i = q_i/r_0$.

The conductivities for any number of conserved currents can be calculated explicitly:

$$\sigma_i = Z_i + \frac{Q_i^2}{r_0^2 k^2}, \quad \sigma_{ij} = \frac{Q_iQ_j}{r_0^2 k^2}, \quad \kappa = \frac{\bar\kappa}{1 + \sum_i 4\pi Q_i^2/sk^2 Z_i},$$

with $\bar\kappa = 4\pi sT/k^2$, $s = 4\pi r_0^2$ and Z_i is the coupling of A_i. If we identify the total electric current as $J = \sum_i J_i$ and thermo-electric force as $E_i = E - T\nabla(\mu_i/T)$, we can calculate the electric conductivity to give

$$\sigma = \frac{\partial J}{\partial E} = \sum_i \sigma_i + \sum_{i,j}\sigma_{ij} = Z + 4\pi Q^2/sk^2, \quad (15)$$

where $Q = \sum_i Q_i$ and $Z = \sum_i Z_i$, showing the additivity of charge-conjugation-invariant part [26] of the electric conductivity. If we define the heat conductivity due to the i-th current by $1/\kappa_i = 1/\bar\kappa + Q_i^2/Z_i s^2 T$, then the heat conductivity formula leads us to *additivity of dissipative part of the inverse heat conductivity*. Therefore

$$D[1/\kappa] = \sum_i D[1/\kappa_i], \qquad \bar D[\sigma] = \sum_i \bar D[\sigma_i], \quad (16)$$

where $D[f]$ denotes the dissipative part of f and $\bar D[f] = f - D[f]$.

Finally we claim that the experimental data of graphene will be well fit with two current theory if we assume the proportionality of two charges

$$Q_2 = gQ_1, \qquad (17)$$

whose justification will be discussed later. This assumption together with the additivities in eq.(16) are what makes our two current model work.

Origin of two Currents in Graphene: What is the nature and the origin of the extra current in the graphene. There are a few attractive candidates. The first idea is the effect of imbalance [4] between the electrons and holes due to the kinematical constraints of the Dirac cone. When there is such an deviation of electron and hole density from their equilibrium value, then the system has tendency to reduce the difference by creating/absorbing electron-hole pair:

$$e^- \leftrightarrow e^- + h^+ + e^-, \quad h^+ \leftrightarrow h^+ + h^+ + e^- \qquad (18)$$

In such processes, energy and momemtum conservations must hold. The point is that, for the graphene, the linear dispersion relation severely reduces the kinematically available states [4]: If we define $\vec{q}$ as a momentum measured from a Dirac point,

$$\vec{q_1} = \vec{q_2} + \vec{q_3} + \vec{q_4}, \quad |\vec{q_1}| = |\vec{q_2}| + |\vec{q_2}| + |\vec{q_3}|, \qquad (19)$$

which request the co-linearity of all momentum vectors $\vec{q_1}, \cdots, \vec{q_4}$. Therefore available phase space is greatly reduced. Such kinematical constraints maintains the non-equilibrium states and as a consequence, the two currents J_e, J_h behave independently for a long time compared with the Planck time $\sim \hbar/kT$, which is the time for hydrodynamics to work.

The net electric current J and total number current J_n which become neutral at Dirac point, are defined by $J = J_e + J_h$, $J_n = J_e - J_h$, respectively and their electric charge densities and number densities are related by $Q_1 = en_1$ and $Q_2 = -en_2$. The total electric conductivity $\sigma = \frac{\partial J}{\partial E}$ and κ can be expressed in terms of $Q = Q_1 + Q_2$ and $Q_n = Q_1 - Q_2$ together with the proportionality constant g_n of $Q_n = g_n Q$:

$$\sigma = \sigma_0(1 + (Q/Q_0)^2), \; \kappa = \frac{\bar{\kappa}}{1 + (1 + g_n^2)(Q/Q_0)^2}, \quad (20)$$

where

$$\sigma_0 = \frac{e^2}{\hbar} 2Z_0, \quad \bar{\kappa} = \frac{4\pi k_B}{\hbar} \frac{sT}{k^2}, \quad Q_0^2 = \frac{\hbar \sigma_0}{4\pi k_B} sk^2. \quad (21)$$

To fix the parameters, we used four measured values of ref. [11] at 75K, $\sigma_0 = 0.338/k\Omega$, $\bar{\kappa} = 7.7nW/K$, $Q_0 = e \cdot 320/(\mu m)^2$, together with the curvature of density plot of κ to fix $g_n = 3.2$ and assumed charge conjugation symmetry to set $W_0 = Z_0$. Using these, the basic parameters of the theory as well as the entropy density can be determined: $2Z_0 = 1.387, k^2 = \frac{454}{(\mu m)^2}, s = 2044\frac{k_B}{(\mu m)^2}$.
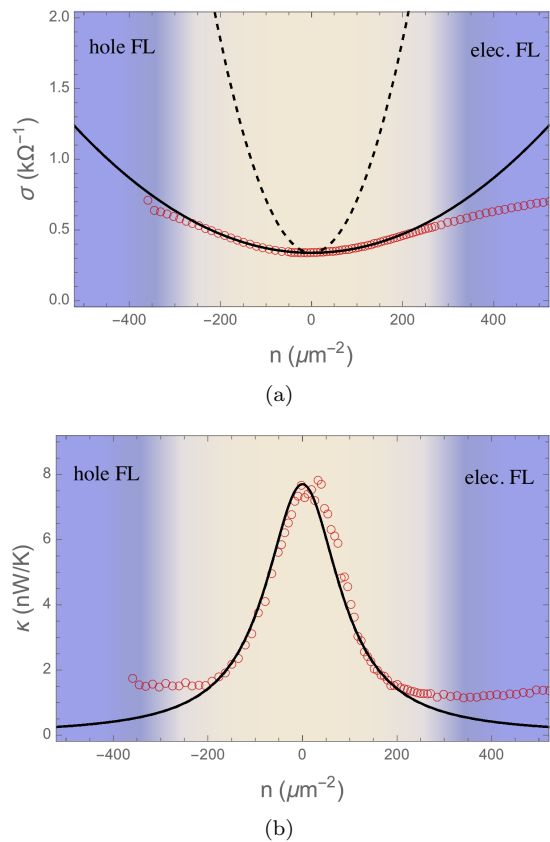

(a)

(b)

FIG. 1. Theory vs. Data: (a)density plot of σ, (b)that of κ. Red circles are for data used in [11, 13], dashed lines are for one current model and real lines are for two current model. The blue color is for the Fermi Liquid regime and near charge neutral point is the Dirac fluid regime, where our theory works. Color grading is given to guide the eye.

We replace all r_0 dependence by s, the entropy density by $s = 4\pi k_B r_0^2$. Cosmological constant is not determined due to the inherent scale symmetry. The resulting fit to data is given in Fig.1.

Now, why we can set the *proportionality of the two charges* as given in eq. (17). To avoid the issues involved in the transport by puddle, we simply assume that the system is homogeneous. Then the number densities of electrons and holes created by thermal excitation is proportional to the net charge density: for the fermi liquid case, out of total degree of freedom (d.o.f) $n \sim k_F^2 \sim \mu^2$, excitable d.o.f is $\sim kT \cdot \mu$, because the excitable shell width is kT. But in hydrodynamic regime, $kT >> \mu$, therefore entire non-degenerate charge distribution region is excitable. In fact this is a typical situation of fermion dynamics described by AdS black hole [28, 29]. In summary, in case of the hydrodynamic regime, the charge carrier density created is proportional to total degree of freedom, Q, which is the volume of the Dirac cone above the Dirac point.

We remark that due to strong Coulomb interaction,

the created electron hole pairs can form the bound state, exciton. Such excitons in homogeneous graphene satisfies the linear relations between the electric charge and the exciton number. Although exciton in graphene has been discussed extensively [30, 31], mosts are only for bi-layer graphene. However, we expect that strong coulomb interaction in Dirac Fluid regime of single layer graphene also should be able to make bound state.

Discussions: In the presence of an extra current that carries mainly heat, the violation of WFL is not direct evidence of a Dirac fluid. However, the fact that such a phenomenon is quantitatively well described by hydrodynamics and gauge/gravity duality, indicates that the system is strongly correlated.

Disorder and the nature of the scalar field: The scalar field provides momentum dissipation only when both its gradient and the vacuum expectation value of its dual operator, $\langle \mathcal{O}_I \rangle$, are nonzero. The latter is the analogue of charge density in electric field as one can see from the Ward identity,

$$\nabla_\nu T^{\mu\nu} = \langle \mathcal{O}_I \rangle \nabla_\mu \chi_I^0 + F_{\mu\nu}^0 \langle J^\nu \rangle. \qquad (22)$$

The role of the source field $\chi_I^0 = k x_I$ is the chemical potential of impurity and that of $\langle \mathcal{O}_I \rangle$ is the density of impurity, whose presence gives the momentum dissipation. It is identified as the subleading order term of the fluctuation of the scalar field near the boundary and nonzero due to the presence of curvature in AdS spacetime. k^2 can be understood as the density of the uniform impurity.

Other origins of the second current: We suggested imbalacnce and excitons as possible sources for the extra current. Here we discuss other candidates. *i) Spin charge separation:* This is the simplest to explain the phenomena if such separation could be experimentally confirmed: the spinons are obviously the chargeless heat carrier and densities of spinnons and holons must be the same and equal to the original electron density. *ii) valley currents:* Graphene consists of two sublattices A and B and such electrons in each sublattice do not scatter, hence they form two conserved currents. However, they do not necessarily satisfy the linearity condition eq.(17). *iii) Phonon:* At high temperature, the phonons are the main heat carriers in carbon materials. However, there are good reasons that phonon is not the main player in the regime we are discussing [11].

Future directions: It would be interesting if we can extend our method to multilayered graphene and graphite. Some holographic analysis for the latter was already reported [27]. The thermo-electric power and magneto-transport are also very interesting observable for the Dirac Fluid regime. We note that some of the early data began to be produced[32]. From the experimental side, the abundance of excitons in single layer graphene is remained to be verified experimentally. The strong correlation was measured only in the limited temperature window $45K < T < 90K$ and in the density regime near the charge neutral point[11], outside of which graphene has well been described as weekly interacting system whose gravity dual is hard to find if exist. Nevertheless presence of such strongly interacting regime can give us extraordinary guide in constructing the general theory of strongly interacting many body system.

[1] M. Müller and S. Sachdev, Phys. Rev. B **78**, 115419 (2008), arXiv:0801.2970 [cond-mat.str-el].

[2] L. Fritz, J. Schmalian, M. Müller, and S. Sachdev, Phys. Rev. B **78**, 085416 (2008), arXiv:0802.4289.

[3] M. Müller, L. Fritz, and S. Sachdev, Phys. Rev. B **78**, 115406 (2008), arXiv:0805.1413 [cond-mat.str-el].

[4] M. S. Foster and I. L. Aleiner, Physical Review B **79**, 085415 (2009), arXiv:0810.4342 [cond-mat.mes-hall].

[5] M. Müller, J. Schmalian, and L. Fritz, Phys. Rev. Lett. **103**, 025301 (2009), arXiv:0903.4178 [cond-mat.mes-hall].

[6] M. Mendoza, H. J. Herrmann, and S. Succi, Phys. Rev. Lett. **106**, 156601 (2011), arXiv:1201.6590 [cond-mat.mes-hall].

[7] A. Tomadin, G. Vignale, and M. Polini, Phys. Rev. Lett. **113**, 235901 (2014), arXiv:1401.0938 [cond-mat.mes-hall].

[8] A. Principi and G. Vignale, Phys. Rev. Lett. **115**, 056603 (2015), arXiv:1406.2940 [cond-mat.mes-hall].

[9] I. Torre, A. Tomadin, A. K. Geim, and M. Polini, Phys. Rev. B **92**, 165433 (2015), arXiv:1508.00363 [cond-mat.mes-hall].

[10] L. Levitov and G. Falkovich, Nat Phys **12**, 672 (2016), arXiv:1508.00836 [cond-mat.mes-hall].

[11] J. Crossno, J. K. Shi, K. Wang, X. Liu, A. Harzheim, A. Lucas, S. Sachdev, P. Kim, T. Taniguchi, K. Watanabe, T. A. Ohki, and K. C. Fong, Science **351**, 1058 (2016), arXiv:1509.04713 [cond-mat.mes-hall].

[12] S. A. Hartnoll, P. K. Kovtun, M. Müller, and S. Sachdev, Phys. Rev. B **76**, 144502 (2007), arXiv:0706.3215 [cond-mat.str-el].

[13] A. Lucas, J. Crossno, K. C. Fong, P. Kim, and S. Sachdev, Phys. Rev. **B93**, 075426 (2016), arXiv:1510.01738 [cond-mat.str-el].

[14] J. M. Maldacena, Int.J.Theor.Phys. **38**, 1113 (1999), arXiv:hep-th/9711200 [hep-th].

[15] E. Witten, Adv. Theor. Math. Phys. **2**, 253 (1998), arXiv:hep-th/9802150.

[16] M. Blake, D. Tong, and D. Vegh, Phys. Rev. Lett. **112**, 071602 (2014), arXiv:1310.3832 [hep-th].

[17] T. Andrade and B. Withers, JHEP **1405**, 101 (2014), arXiv:1311.5157 [hep-th].

[18] A. Donos and J. P. Gauntlett, JHEP **1411**, 081 (2014), arXiv:1406.4742 [hep-th].

[19] A. Donos and J. P. Gauntlett, JHEP **01**, 035 (2015), arXiv:1409.6875 [hep-th].

[20] A. Donos, B. Goutraux, and E. Kiritsis, JHEP **09**, 038 (2014), arXiv:1406.6351 [hep-th].

[21] K.-Y. Kim, K. K. Kim, Y. Seo, and S.-J. Sin, JHEP **1412**, 170 (2014), arXiv:1409.8346 [hep-th].

[22] X.-H. Ge, Y. Ling, C. Niu, and S.-J. Sin, Phys. Rev. **D92**, 106005 (2015), arXiv:1412.8346 [hep-th].

[23] K.-Y. Kim, K. K. Kim, Y. Seo, and S.-J. Sin, Phys. Lett. **B749**, 108 (2015), arXiv:1502.02100 [hep-th].

[24] K.-Y. Kim, K. K. Kim, Y. Seo, and S.-J. Sin, JHEP **07**, 027 (2015), arXiv:1502.05386 [hep-th].

[25] M. Blake, A. Donos, and N. Lohitsiri, JHEP **08**, 124 (2015), arXiv:1502.03789 [hep-th].

[26] M. Blake and A. Donos, Phys. Rev. Lett. **114**, 021601 (2015), arXiv:1406.1659 [hep-th].

[27] Y. Seo, K.-Y. Kim, K. K. Kim, and S.-J. Sin, Phys. Lett. **B759**, 104 (2016), arXiv:1512.08916 [hep-th].

[28] S.-S. Lee, Phys. Rev. **D79**, 086006 (2009), arXiv:0809.3402 [hep-th].

[29] H. Liu, Physical Review D **83** (2011), 10.1103/PhysRevD.83.065029.

[30] J. P. Eisenstein and A. H. MacDonald, Nature **432**, 691 (2004).

[31] H. Min, R. Bistritzer, J.-J. Su, and A. H. MacDonald, Phys. Rev. B **78**, 121401 (2008).

[32] F. Ghahari, Physical Review Letters **116** (2016), 10.1103/PhysRevLett.116.136802.