Error Suppression for Hamiltonian-Based Quantum Computation Using Subsystem Codes

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Error suppression for Hamiltonian-based quantum computation using subsystem codes

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We present general conditions for quantum error suppression for Hamiltonian-based quantum computation using subsystem codes. This involves encoding the Hamiltonian performing the computation using an error detecting subsystem code and the addition of a penalty term that commutes with the encoded Hamiltonian. The scheme is general and includes the stabilizer formalism of both subspace and subsystem codes as special cases. We derive performance bounds and show that complete error suppression results in the large penalty limit. To illustrate the power of subsystem-based error suppression, we introduce fully 2-local constructions for protection against local errors of the swap gate of adiabatic gate teleportation and the Ising chain in a transverse field.

A general strategy for protecting quantum information is to encode this information into a larger system in such a way that the effect of the bath is eliminated, suppressed, or corrected [1]. A promising approach for quantum error suppression in Hamiltonian quantum computation [2–4] was proposed in Ref. [5]. In this scheme one chooses a stabilizer quantum error detection code [6], encodes the Hamiltonian by replacing each of its Pauli operators by the corresponding encoded Pauli operator of the chosen code, and adds penalty terms (elements of the code's stabilizer) that suppress the errors the code is designed to detect. This results in the suppression of excitations out of the ground subspace. By indefinitely increasing the energy scale of the penalty terms this suppression can be made arbitrarily strong [7].

By construction, this encoding necessitates greater than two-body interactions, which can make its implementation challenging. An important open question is whether there exist quantum error suppression schemes that involve only two-body interactions. However, even for the special case of quantum memory, invoking penalty terms but no encoding, two-body commuting Hamiltonians cannot in general provide suppression [8]. This no-go result left open the possibility that non-commuting two-local Hamiltonians might nevertheless suffice for quantum error suppression. Examples based on (generalized) Bacon-Shor codes [9] were recently given in Ref. [10] to show that this is the case for penalty terms and encoded single-qubit operations, and for some encoded two-qubit interactions, but without general conditions or performance bounds.

Here we show how general subsystem codes can be used for quantum error suppression. Using an exact, non-perturbative approach, we find conditions that penalty Hamiltonians should satisfy to guarantee complete error suppression in the infinite energy penalty limit. We derive performance bounds for finite energy penalties. Our formulation accounts for stabilizer subspace and subsystem codes as special cases, including the examples of Refs. [5, 7, 10]. We provide several examples where our approach results in encoded Hamiltonians and penalty terms that involve purely two-body interactions [11]. These examples include the swap gate used in adiabatic gate teleportation [12], and the Ising chain in a transverse field frequently encountered in adiabatic quantum computation and quantum annealing.

Setting.—We wish to protect a quantum computation performed by a system with Hamiltonian $H_S(t)$ against the system-bath interaction $V=\sum_j E_j\otimes B_j$, to a bath with Hamiltonian H_B. We construct the encoded system Hamiltonian, $\overline{H}_S(t)$, by replacing every operator in $H_S(t)$ by the corresponding logical operators of a subsystem code [13–15]. The strategy for protecting the computation performed by $\overline{H}_S(t)$ is to add a penalty Hamiltonian $E_\mathrm{p}H_\mathrm{p}$, chosen so that $[\overline{H}_S(t),H_\mathrm{p}]=0$ in order to prevent interference with the computation [5]. As the energy penalty E_p is increased, errors should become more suppressed.

Results in the infinite penalty limit.—We now state our main results, in the form of two related theorems that give sufficient conditions for complete error suppression in the large E_p limit. These results incorporate both those for general stabilizer penalty Hamiltonians introduced in [5, 7] and the subsystem penalty Hamiltonian examples introduced in [10]. They are also related to a dynamical decoupling approach for protecting adiabatic quantum computation [16] via a formal equivalence found in Ref. [17].

Let U_0, U_p, U_V, and U_W be the unitary evolutions generated by $H_0=\overline{H}_S+H_B$, $E_\mathrm{p}H_\mathrm{p}$, $H_V=H_0+E_\mathrm{p}H_\mathrm{p}+V$, and $H_W=H_0+E_\mathrm{p}H_\mathrm{p}+W$, respectively. As will become clear later, W will play the role of the suppressed version of V. We assume that $\|V\|,\|W\|<\infty$, where $\|\cdot\|$ denotes any unitarily invariant norm [18]. Let P be an arbitrary projection operator and let $H_\mathrm{p}=\sum_a\lambda_a\Pi_a$ be the eigendecomposition of the penalty term.

Theorem 1. *Set $W=cI$ ($c\in\mathbb{R}$, I is the identity oper-*

ator) and assume that

$$[\overline{H}_S, P] = [\overline{H}_S, H_\mathrm{p}] = 0 \ , \tag{1a}$$
$$\sum_a \Pi_a V \Pi_a P = cP \ . \tag{1b}$$

Then

$$\lim_{E_\mathrm{p}\to\infty} \|U_V(T)P - U_W(T)P\| = 0 \ , \tag{2}$$

where $U_W(T) = e^{-icT}U_0(T)U_\mathrm{p}(T)$.

Theorem 1 states that in the infinite penalty limit and over the support of P, the evolution generated by the total system-bath Hamiltonian H_V is indistinguishable (up to a global phase) from the decoupled evolution generated by $H_0 + E_\mathrm{p}H_\mathrm{p}$. The conditions in Eq. (1a) ensure compatibility of the subspace defined by P and of the type of penalty Hamiltonian H_p with the given encoded Hamiltonian $\overline{H}_S$. The condition in Eq. (1b) ensures the absence of a term that cannot be removed by the penalty [see the Supplementary Material (SM)].

Theorem 2. *Set* $W = \sum_{a\in\mathcal{I}} \Pi_a V \Pi_a$*, where* $\mathcal{I}$ *is some index set. Assume that in addition to Eq.* (1a) *also* $P = \sum_{a\in\mathcal{I}} \Pi_a$*. Then Eq.* (2) *holds again, with*

$$U_W(T) = \mathcal{T}\exp\int_0^T (H_0(t) + E_\mathrm{p}H_\mathrm{p} + W)dt \tag{3}$$

($\mathcal{T}$ denotes time-ordering). Theorem 2 is similar to Theorem 1, except that it allows for a more general target evolution operator $U_W(T)$. As discussed below, Theorem 1 is suitable for stabilizer subsystem codes, while Theorem 2 is suitable for general subsystem codes.

Proof sketch.—Both Theorems 1 and 2 establish the desired decoupling result, and show that in principle it is possible to completely protect Hamiltonian quantum computation against coupling to the bath. To prove them we define

$$K(t) = \int_0^t U_\mathrm{p}^\dagger(\tau)(V - W)U_\mathrm{p}(\tau)d\tau P \ , \tag{4}$$

and derive the following bounds in the SM:

$$\|U_V(T)P - U_W(T)P\| \leq \|K(T)\| \tag{5a}$$
$$+ T\sup_t \|[K(t), H_0(t)]\| + T(\|V\| + \|W\|)\sup_t \|K(t)\|$$
$$\|K(t)\| \leq \frac{2}{E_\mathrm{p}} \sum_{a\neq a'} \frac{\|V - W\|}{|\lambda_a - \lambda_{a'}|} \ . \tag{5b}$$

Theorems 1 and 2 follow in the large E_p limit, since in this limit $\|K(T)\| \to 0$, and $\|[K(t), H_0(t)]\| \leq 2\|K(t)\|\|H_0\|$. An error bound for finite E_p follows directly from Eq. (5b) (for related results see Refs. [7, 19]). While a tighter bound may not be possible without introducing additional assumptions, we note that for a Markovian bath in a thermal state, it is possible to show that the excitation rate out of the code space is exponentially suppressed as a function of E_p, and E_p need only grow logarithmically in the system size to achieve a constant excitation rate, assuming the gap of H_p is constant [5, 20].

Subsystem codes.—Before demonstrating the implications of Theorems 1 and 2 we first briefly review subsystem codes. Assume that the system's Hilbert space can be decomposed as $\mathcal{H}_S = C \oplus C^\perp$, where $C = A \otimes B$. The channel (completely positive map) $\mathcal{E} = \{E_j\}$ is *detectable* on the "information subsystem" B if (see the SM for a proof):

$$\forall E_j\ \exists G_j : P_C E_j P_C = P_C G_j \otimes I_B P_C \ , \tag{6}$$

where I_B is the identity on B and P_C denotes the projector onto C. Here A plays the role of a "gauge subsystem"; the G_j operators are arbitrary and do not affect the information stored in subsystem B.

Stabilizer subsystem codes [21] are of particular interest. Intuitively, one can think of such codes as subspace stabilizer codes [6] where some logical qubits and the corresponding logical operators are not used. A stabilizer code can be defined as the subspace stabilized by an Abelian group $\mathcal{S} = \langle S_1, ..., S_s\rangle$ of Pauli operators, with $-I \notin \mathcal{S}$, where $\{S_i\}_{i=1}^s$ are the group generators. The projector onto the codespace is $P_C = \prod_{i=1}^s \frac{I+S_i}{2}$. To induce a subsystem structure we define logical operators $\mathcal{L}$ and gauge operators $\mathcal{A}$ as Pauli operators that leave the codespace invariant, and also demand that the three sets $\mathcal{S}$, $\mathcal{L}$, and $\mathcal{A}$ mutually commute. The generators of $\mathcal{L}$ and $\mathcal{A}$ can be organized in canonical conjugate pairs: the set of bare logical operators $\mathcal{L} = \{\overline{Z}_1, \overline{X}_1, ..., \overline{Z}_k, \overline{X}_k\}$ that preserve the code space and act trivially on the gauge qubits [22], and the set of gauge operators $\mathcal{A} = \{Z_1', X_1', ..., Z_r', X_r'\}$, where for $A, B \in \{\overline{X}, \overline{Z}\}$ or $A, B \in \{X', Z'\}$ we have $[A_i, B_j] = 0$ if $i \neq j$, and $\{\overline{X}_i, \overline{Z}_i\} = 0$. The gauge group is defined as $\mathcal{G} = \langle S_1, ..., S_s, Z_1', X_1', ..., Z_r', X_r'\rangle$, and is non-Abelian. A Pauli error E_j is detectable iff it anti-commutes with at least one of the stabilizer generators [21], or equivalently iff $P_C E_j P_C = 0$ [since $(I + S_i)(I - S_i) = 0\ \forall i$].

Protection using stabilizer codes.—To satisfy the condition $[\overline{H}_S, H_\mathrm{p}] = 0$ in Eq. (1a) we may choose H_p as a linear combination of elements of the gauge group $\mathcal{G}$ (not necessarily the generators) [10, 23],

$$H_\mathrm{p} = \sum_i \alpha_i g_i \ , \quad g_i \in \mathcal{G} \ , \quad |\alpha_i| \leq 1 \ , \forall i \ . \tag{7}$$

To satisfy the condition $[\overline{H}_S, P] = 0$ we may choose $P = \sum_{a\in\mathcal{I}} \Pi_a$. Equation (1b) then becomes $\Pi_a V \Pi_a = c\Pi_a\ \forall a \in \mathcal{I}$, a condition that is already satisfied with $c = 0$ for a stabilizer error detecting code (for which $P_C V P_C = 0$) if the support of P is in the codespace (i.e., $PP_C = P_C P = P$). This is true, in particular, if $\mathcal{I}$ contains just the ground subspace of H_p. We may thus state the following corollary of Theorem 1: *For $H_{\rm p}$ chosen as in Eq. (7), the joint system-bath evolution completely decouples in the large penalty limit for initial states in the ground subspace of $H_{\rm p}$, with this subspace itself being a subspace of the codespace.*

Note that the difference between the subspace and subsystem case manifests itself in the appearance of $U_{\rm p}(T)$ in Eq. (2). If the penalty Hamiltonian consisted of only stabilizer terms [i.e., $g_i \in \mathcal{S}\ \forall i$ in Eq. (7)], the penalty Hamiltonian would at most change the overall phase of states in the codespace. But here, as the elements of penalty Hamiltonian can be any element of the gauge group, $U_{\rm p}$ can have a nontrivial effect on states in C. Nevertheless, as the gauge operators commute with the logical operators of the code, this unitary does not change the result of a measurement of the logical subsystem. In the SM we provide a formal argument using a distance measure to quantify state distinguishability using generalized measurements restricted to the logical subsystem.

Protection using general subsystem codes.—Choose a code C with projector P_C such that the error-detection condition (6) is satisfied for all the error operators $\{E_j\}$ in $V = \sum_j E_j \otimes B_j$. Assume that the penalty is chosen so that $[\overline{H}_S, H_{\rm p}] = 0$ in Eq. (1a) holds, and set $P = P_C$ in Theorem 2 (thus also the condition $[\overline{H}_S, P] = 0$ holds). Then $\Pi_a V \Pi_a = \sum_j (\Pi_a G_j \otimes I_B \Pi_a) \otimes B_j\ \forall a \in \mathcal{I}$, so that $W = \sum_{a\in\mathcal{I}} \sum_j (\Pi_a G_j \otimes I_B \Pi_a) \otimes B_j$, with trivial action ($I_B$) on the information subsystem B. The unitary $U_W(T)$ [Eq. (3)] appearing in Theorem 2 thus has a nontrivial effect on B only via the $H_0(t)$ term, as desired.

Block encoding.—A useful simplification results when the logical qubits can be partitioned into n separate blocks. In this case the total penalty Hamiltonian becomes $H_{\rm p} = \sum_{\bar{i}=1}^{n} h_{\rm p}^{\bar{i}}$, where $h_{\rm p}^{\bar{i}} = \sum_j \alpha_j^{\bar{i}} g_j^{\bar{i}}$ denotes the penalty Hamiltonian on logical qubit $\bar{i}$, with $g_j^{\bar{i}} \in \mathcal{G}$, and $[h_{\rm p}^{\bar{i}}, h_{\rm p}^{\bar{j}}] = 0$ for $i \neq j$. The code space projector becomes $P_C = \otimes_{\bar{i}=1}^{n} p^{\bar{i}}$, where $p^{\bar{i}}$ is the projector onto the code space of the ith logical qubit. We may also partition the system-bath interaction according to the logical qubits it acts on: $V = \sum_{\bar{i}=1}^{n} v^{\bar{i}}$ (note that we do not assume that $[v^{\bar{i}}, v^{\bar{j}}] = 0$). Clearly, $K(t)$ can also be expressed as a sum over blocks, as can inequality (5a). Using the eigendecomposition $h_{\rm p}^{\bar{i}} = \sum_a e_a^{\bar{i}} \pi_a^{\bar{i}}$, condition (1b) can then be replaced by

$$\pi_a^{\bar{i}} v^{\bar{i}} \pi_a^{\bar{i}} p^{\bar{i}} = c^{\bar{i}} p^{\bar{i}} \quad \forall a, \bar{i}\ . \tag{8}$$

Using the block encoding structure, in the SM we tighten the error bound resulting from Eq. (5b). We show, in particular, that the bound is extensive in the system size and depends only on the bath degrees of freedom that couple locally to the system, so that the bound is not extensive in the bath size.

A simplified sufficient condition.—To check whether Theorem 1 applies one can simply find the eigendecomposition of $h_{\rm p}^{\bar{i}}$ and check if Eq. (8) holds for a given system-bath interaction and choice of code space. Instead, we next identify conditions that are less general but are easier to check. We assume that the interaction Hamiltonian has the 1-local form $V = \sum_{\bar{i}} v^{\bar{i}}$, where $v^{\bar{i}} = \sum_j \sigma_j^{\bar{i}} \otimes B_j^{\bar{i}}$ and $\sigma_j^{\bar{i}}$ is an arbitrary non-identity Pauli operator acting on qubit j in block $\bar{i}$. From now on we drop the block superscript for notational simplicity. Furthermore, we choose a penalty term that satisfies $[h_{\rm p}, p] = 0$ given a code block projector p, which implies $[\pi_a, p] = 0\ \forall a$.

A sufficient condition for Eq. (8), and hence for Theorem 1, is then the following:

Condition 1. *$h_{\rm p}p$ and $\sigma_j h_{\rm p} \sigma_j p$ do not share an eigenvalue for any σ_j in the support of p.*

To see that this is a sufficient condition, we note that $\pi_a p$ and $\sigma_j \pi_a \sigma_j p$ are both projectors, corresponding to the same eigenvalue e_a of $h_{\rm p}$ and $\sigma_j h_{\rm p} \sigma_j$. If both projectors are nonzero then there exists at least one (nonzero) eigenvector for each of $h_{\rm p}p$ and $\sigma_j h_{\rm p} \sigma_j p$ with eigenvalue e_a, in contradiction to our condition. So, the stated condition guarantees that for any eigenvalue e_a we have either $\pi_a p = 0$ or $\sigma_j \pi_a \sigma_j p = 0$. Thus, $\forall a$: $0 = (\sigma_j \pi_a \sigma_j p)(\pi_a p) = \sigma_j \pi_a \sigma_j p p \pi_a = \sigma_j \pi_a \sigma_j p \pi_a = \sigma_j(\pi_a \sigma_j \pi_a p)$, so that, $\forall a$: $\pi_a \sigma_j \pi_a p = 0$, which implies Eq. (8) (with $c^{\bar{i}} = 0\ \forall i$). We now consider a number of interesting cases, and show that Condition 1 holds, thus guaranteeing error suppression via Theorem 1.

Stabilizer penalty Hamiltonians.— As in Ref. [5], let

$$h_{\rm p} = \sum_i \alpha_i S_i \tag{9}$$

with $S_i \in \mathcal{S}$, $\alpha_i \neq 0$ and $p = p_c$. Clearly $[h_{\rm p}, p] = 0$. Let us define $a_{ij} = 0$ or 1 if $[S_i, \sigma_j] = 0$ or $\{S_i, \sigma_j\} = 0$, respectively. In the support of p (i.e., in the code space) $h_{\rm p}p = (\sum \alpha_i)p$, so the eigenvalue of $h_{\rm p}$ there equals $\sum_i \alpha_i$, while the eigenvalue of $\sigma_j h_{\rm p} \sigma_j p$ there equals $\sum_i \alpha_i (-1)^{a_{ij}}$. Condition 1 thus requires $\forall j$: $\sum_i \alpha_i \neq \sum_i \alpha_i (-1)^{a_{ij}}$. When all α_i have the same sign this becomes the familiar error detection condition, that every σ_j anticommutes with at least one of the terms in the sum of stabilizers.

The penalty Hamiltonian considered in Ref. [7] corresponds to $h_{\rm p} = I - p$, so that $[h_{\rm p}, p] = 0$ holds. Condition 1 is also satisfied in this case since since $h_{\rm p}p = 0$, while $\sigma_j h_{\rm p} \sigma_j p = p - \sigma_j p \sigma_j p = p$ (where we used the error detection condition $p\sigma_j p = 0$), so in the support of p the eigenvalues are, respectively, 0 and 1.

Gauge group penalty Hamiltonians.—A family of generalized Bacon-Shor codes can be identified with a binary matrix A, which fully characterizes all the code properties [22]. E.g., each nonzero element of A corresponding to a qubit on a planar grid, and two ones in a row (column) of the matrix correspond to an XX (ZZ) generator acting on the corresponding qubits (see the SM for more details). As pointed out in Ref. [10], because of the locality of the generators of these codes, they are promising candidates for use in error suppression schemes. We present several examples for suppressing local errors that originate from this construction.

(i) The $[[4, 1, 2]]$ code was proposed in Ref. [10] to overcome the aforementioned no-go theorem for error suppression using 2-local commuting Hamiltonians [8]. Each qubit is encoded into four qubits using this code (block encoding), so the entire code corresponds to a block diagonal A matrix, with 2×2 blocks of all ones. The stabilizer, gauge and bare logical generators are:

$$\mathcal{S} = \langle S_1 = X^{\otimes 4}, S_2 = Z^{\otimes 4} \rangle \tag{10a}$$
$$\mathcal{A} = \{X' = X_1X_2, Z' = Z_1Z_3\} \tag{10b}$$
$$\mathcal{L} = \{\overline{X} = X_1X_3, \overline{Z} = Z_1Z_2\} \ . \tag{10c}$$

Thus $\mathcal{G} = \langle S_1, S_2, X', Z' \rangle = \langle S_1X', S_2Z', X', Z' \rangle = \langle X_3X_4, Z_2Z_4, X_1X_2, Z_1Z_3 \rangle \equiv \langle \{g_i\}_{i=1}^4 \rangle$, i.e., the generators are 2-local. The penalty Hamiltonian is $h_{\mathrm{p}} = E_{\mathrm{p}} \sum_{i=1}^4 g_i$ and again, clearly $[h_{\mathrm{p}}, p] = 0$. One may check that the eigenvalues of $h_{\mathrm{p}}p$ and $\sigma_j h_{\mathrm{p}} \sigma_j p$ are $0, \pm 2E_{\mathrm{p}}$ and $\pm 2\sqrt{2}E_{\mathrm{p}}$, respectively (see the SM). Thus Condition 1 is satisfied. While the penalty Hamiltonian is 2-local, unfortunately the encoding of a 2-local interaction (which is necessary for universal quantum computation), still requires 4-local interactions.

(ii) We show how to encode and protect the adiabatic swap gate introduced in [12] using purely 2-local interactions. This Hamiltonian is one of the key building blocks of a proposal for universal quantum computation using adiabatic gate teleportation. The Hamiltonian is: $H(s) = (1-s)(X_bX_c + Z_bZ_c) + s(X_aX_b + Z_aZ_b)$. By slowly increasing s from 0 to 1 any state initially prepared on qubit a transfers onto qubit c. To encode and protect this Hamiltonian, we use the following $[[8, 3, 2]]$ subsystem code:

$$\mathcal{S} = \langle S_1 = X^{\otimes 8}, S_2 = Z^{\otimes 8} \rangle \tag{11a}$$
$$\begin{aligned}\mathcal{L} = \{&\overline{X}_1 = X_1X_8, \overline{X}_2 = X_1X_2X_3X_8, \overline{X}_3 = X_4X_5, \\ &\overline{Z}_1 = Z_1Z_2, \overline{Z}_2 = Z_3Z_4Z_5Z_6, \overline{Z}_3 = Z_5Z_6\}\end{aligned} \tag{11b}$$
$$\mathcal{G} = \langle X_1X_2, X_3X_4, X_5X_6, X_7X_8, Z_2Z_3, Z_4Z_5, Z_6Z_7, Z_8Z_1 \rangle$$

The penalty Hamiltonian is the sum of all the gauge group generators $g_i \in \mathcal{G}$, which is manifestly 2-local. One can check that Condition 1 is satisfied for this Hamiltonian (see the SM), and so we obtain the desired protection. The encoded Hamiltonian becomes:

$$\begin{aligned}\overline{H}(s) &= (1-s)(\overline{X}_2\overline{X}_3 + \overline{Z}_2\overline{Z}_3) + s(\overline{X}_1\overline{X}_2 + \overline{Z}_1\overline{Z}_2) \\ &= (1-s)(X_6X_7 + Z_3Z_4) + s(X_2X_3 + Z_7Z_8),\end{aligned} \tag{12}$$

where in the second line we used the fact that $\overline{X}_2\overline{X}_3 = S_1X_6X_7$ and $\overline{Z}_1\overline{Z}_2 = S_2Z_7Z_8$ are equivalent logical operators. Thus, the encoded Hamiltonian remains 2-local.

(iii) Our next example, an open Ising chain in a transverse field, does not involve block encoding:

$$H_S(s) = (1-s)\sum_{i=1}^{N} X_i + s\sum_{i=1}^{N-1} J_iZ_iZ_{i+1} \ . \tag{13}$$

This Hamiltonian appears frequently in adiabatic quantum optimization. The goal is again to provide encoding and error suppression using only 2-local Hamiltonians.

Using an A-matrix derived $[[2N+2, N, 2]]$ code (see the SM for details), we obtain:

$$\begin{aligned}H_{\mathrm{p}} &= -\sum_{i=1}^{N+1} X_{2i-1}X_{2i} + \sum_{i=1}^{N} Z_{2i}Z_{2i+1} + Z_1Z_{2N+2} \\ \overline{H}_S(s) &= (1-s)\sum_{i=1}^{N} X_{2i}X_{2i+1} + s\sum_{i=1}^{N-1} J_iZ_{2i+1}Z_{2i+2} \ .\end{aligned} \tag{14}$$

We have verified numerically that the ground subspace of H_{p} is a subspace of the codespace, which as we showed above is sufficient for error suppression in the stabilizer case. We also find numerically that the minimum gap of H_{p} decreases as $1/(N+1)$ (see the SM), so that E_{p} should grow with N to maintain the protection obtained in this case as the system size increases, since this gap separates the logical ground subspace from the undecodable excited states. While in general this is undesirable, it is compatible with examples where H_S (and hence also $\overline{H}_S$) exhibits more rapidly closing gaps for certain choices of the couplings $\{J_i\}$ (e.g., an exponentially small gap [24]).

Non-additive codes.—Theorems 1 and 2 allow us to go beyond the framework of Ref. [7] and examples of Ref. [10], and employ non-additive codes (also known as non-stabilizer codes) to encode and protect evolutions [25]. Non-additive codes can achieve higher rates (ratio of the number of encoded to physical qubits) than stabilizer codes [26–29]. For example, using 5 physical qubits to detect any single-qubit error stabilizer codes can encode at most 2 qubits, but using a non-additive code one can encode up to $\log_2 6$ qubits [26]. The encoding procedure is straightforward. Choosing a subspace code C, one can expand the system Hamiltonian in a basis $\{|i\rangle\}$ and then replace each basis vector in the expansion with the corresponding code state $\{|\bar{i}\rangle\}$. One possible choice of a penalty Hamiltonian is $E_{\mathrm{p}}H_{\mathrm{p}}$, where $H_{\mathrm{p}} = -P_C$ and $P_C = \sum_{i \in C} |\bar{i}\rangle\langle\bar{i}|$. Theorem 1 guarantees that with this choice, starting from an initial state in the codespace, leakage out of the codespace is suppressed in the large E_{p} limit, and the desired system Hamiltonian is implemented in the codespace with a higher rate than what could be achieved using stabilizer codes. Moreover, Theorem 2 allows using non-additive subsystem codes such as the codes introduced in Ref. [29].

Conclusions.—We have presented conditions guaranteeing error suppression for Hamiltonian quantum computation using general subsystem error detecting codes, along with conditions that the corresponding penalty Hamiltonians should satisfy, and performance bounds that improve monotonically with increasing energy penalty. Stabilizer subsystem codes are more flexible than stabilizer subspace codes when there are constraints on the spatial locality of the generators of the code [22]. This allowed us to use these codes to present examples of fully 2-local encoded Hamiltonian quantum information processing with error suppression. This should hopefully pave the way towards a similar result for protected universal Hamiltonian quantum computation.

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[25] Ref. [7] proved a less general version of Theorem 1, where Eq. (1b) is replaced by $P_C V P_C = 0$; this excludes non-additive codes. Ref. [10] used certain stabilizer subsystem codes but did not consider non-additive codes.
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