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Bootstrapping rapidity anomalous dimension for transverse-momentum resummation

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Soft function relevant for transverse-momentum resummation for Drell-Yan or Higgs production at hadron colliders are computed through to three loops in the expansion of strong coupling, with the help of bootstrap technique and supersymmetric decomposition. The corresponding rapidity anomalous dimension is extracted. An intriguing relation between anomalous dimensions for transverse-momentum resummation and threshold resummation is found.

Introduction. The transverse-momentum (q_T) distribution of generic high-mass color-neutral systems (Drell-Yan lepton pair, Higgs, EW vector boson pair, etc.) produced in hadron collisions is of great interest since the early days of Quantum Chromodynamics (QCD) [1–17]. It provides a testing ground for examination and improvement of our understanding of QCD, both perturbatively and non-perturbatively. When q_T is small compared with the invariant mass Q of the system, fixed-order perturbation theory breaks down due to the appearance of large logarithms of the form $\ln^k(q_T^2/Q^2)/q_T^2$, with $k \geq 0$ at each order in strong coupling α_s . These large logarithms originate from incomplete cancellation of soft and collinear divergences between real and virtual diagrams. Fortunately, Collins, Soper, and Sterman (CSS) have shown that they can be systematically resummed to all orders in perturbation theory [5], thanks to QCD factorization.

In recent years, there have been increasing interests in applying Soft-Collinear Effective Theory (SCET) [18–22] to resum large logarithms in perturbative QCD using renormalization group (RG) method. For q_T resummation this has been done by a number of authors [23–29]. For transverse-momentum observable, the relevant momentum modes in light-cone coordinate for fields in the effective theory are soft $p_s \sim Q(\lambda, \lambda, \lambda)$, collinear $p_c \sim Q(\lambda^2, 1, \lambda)$ and anti-collinear $p_{\bar{c}} \sim Q(1, \lambda^2, \lambda)$. Here $\lambda \sim q_T/Q$ is a power counting parameter. The corresponding effective theory is SCET_{II}. An important feature of SCET_{II} is that soft and collinear modes live on the same hyperbola of virtuality, $p_s^2 \sim p_c^2 \sim p_{\bar{c}}^2 \sim \lambda^2 Q^2$. Besides the usual large logarithms of ratio between hard scale Q and soft scale λQ , there are also large rapidity separations between soft, collinear, and anti-collinear modes which need to be resummed. In this Letter we adopt the rapidity RG formalism of Chiu, Jain, Neill, and Rothstein [27, 28]. According to the rapidity RG formalism, cross section at small q_T factorizes into hard function H , Transverse-Momentum-Dependent (TMD) beam functions B , and TMD soft function S_\perp . Schematically the factorization formula reads:

$$\frac{1}{\sigma} \frac{d^3 \sigma^{\text{(res.)}}}{d^2 \vec{q}_T dY dQ^2} \sim H(\mu) \int \frac{d^2 \vec{b}_\perp}{(2\pi)^2} e^{i \vec{b}_\perp \cdot \vec{Q}_T}$$

$$\cdot [B \otimes B](\vec{b}_\perp, \mu, \nu) S_\perp(\vec{b}_\perp, \mu, \nu) \quad (1)$$

Large logarithms in virtuality is resummed by running in the renormalization scale μ , while large logarithms in rapidity is resummed by running in the rapidity scale ν . The μ evolution of the hard function can be derived from quark or gluon form factor and is well-known [30–32]. Since the physical cross section is independent of μ and ν order by order in the perturbation theory, it follows that the μ and ν evolution of $[B \otimes B]$ is fixed once the corresponding evolution for the soft function is known. The knowledge of μ and ν evolution of hard, beam, and soft function, together with the boundary conditions of these functions at initial scales, determine the all order structure of large logarithms of q_T .

The naive definition of the TMD soft function is a vacuum expectation value of light-like Wilson loops with a transverse separation, which suffers from light-cone/rapidity divergence [3]. A proper definition of the TMD soft function requires the introduction of appropriate regulator for the rapidity divergence. Proposals to regularize the rapidity divergence includes non-light-like axial gauge without Wilson lines [5], tilting Wilson lines off the lightcone [33], nearly light-like Wilson lines with subtraction of soft factor [34], modifying the phase space measure [26, 27, 35], modifying the $i\epsilon$ prescription of eikonal propagator [36], etc. In this Letter, we follow the recent proposal [37] by Neill and the current authors of implementing an infinitesimal shift in the time direction to the Wilson loop correlator. Specifically, the TMD soft function with the rapidity regulator of Ref. [37] reads:

$$\begin{aligned} S_\perp(\vec{b}_\perp, \mu, \nu) &= \lim_{\nu \rightarrow +\infty} S_{\text{F.D.}}(\vec{b}_\perp, \mu, \nu) \\ &\equiv \lim_{\nu \rightarrow +\infty} \frac{1}{d_a} \langle 0 | T [S_n^\dagger(-\infty, 0) S_n(0, -\infty)] \\ &\quad \cdot \overline{T} [S_n^\dagger(-\infty, y_\nu(\vec{b}_\perp)) S_n(y_\nu(\vec{b}_\perp), -\infty)] | 0 \rangle \end{aligned} \quad (2)$$

where the two Wilson loops are separated by the distance $y_\nu(\vec{b}_\perp) = (i b_0/\nu, i b_0/\nu, \vec{b}_\perp)$, with $b_0 = 2e^{-\gamma_E}$. $S_n(\vec{n})$ are path-ordered Wilson lines on the light-cone. They carry fundamental or adjoint color indices, depending on whether the color-neutral system is produced in $q\bar{q}$ annihilation ($d_a = N_c$) or gg fusion ($d_a = N_c^2 - 1$). T is the time-ordered operator. The soft function S_\perp in eq. (2)

is closely related to the so-called fully differential soft function [25], $S_{\text{F.D.}}$. The limit $\nu \rightarrow +\infty$ means that only the non-vanishing terms of $S_{\text{F.D.}}$ are kept in that limit. The important role of $S_{\text{F.D.}}$ in our calculation will be explained in the next section. Note that our definition for the TMD soft function doesn't rely on perturbation theory. However, we restrict to the perturbatively calculable part of the soft function in this Letter.

After minimal subtraction of dimensional regularization pole $1/\epsilon^n$ in $\overline{\text{MS}}$ scheme, the soft function S_\perp depends on both the renormalization scale μ and the rapidity scale ν . The μ evolution of the TMD soft function is specified by the RG equation:

$$\frac{d \ln S_\perp(\vec{b}_\perp, \mu, \nu)}{d \ln \mu^2} = \Gamma_{\text{cusp}}[\alpha_s(\mu)] \ln \frac{\mu^2}{\nu^2} - \gamma_s[\alpha_s(\mu)] \quad (3)$$

where Γ_{cusp} is the well-known light-like cusp anomalous dimension [38, 39], which is known to three loops in QCD [40]. γ_s is the soft anomalous dimension governing the single logarithmic evolution, which can be extracted through to three loops from QCD splitting function [40] and quark and gluon form factor [30–32], as is confirmed by explicit three-loop calculation [41]. The rapidity evolution equation for the TMD soft function reads:

$$\begin{aligned} \frac{d \ln S_\perp(\vec{b}_\perp, \mu, \nu)}{d \ln \nu^2} &= \int_{\mu^2}^{b_0^2/\vec{b}_\perp^2} \frac{d\bar{\mu}^2}{\bar{\mu}^2} \Gamma_{\text{cusp}}[\alpha_s(\bar{\mu})] \\ &+ \gamma_r[\alpha_s(b_0/|\vec{b}_\perp|)] \end{aligned} \quad (4)$$

where the rapidity anomalous dimension γ_r is introduced for the single logarithmic evolution of rapidity logarithms. Thanks to the non-Abelian exponentiation theorem [42–44] which our regularization procedure [37] preserves, the perturbative soft function can be written as an exponential:

$$S_\perp(\vec{b}_\perp, \mu, \nu) = \exp \left[a_s S_1^\perp + a_s^2 S_2^\perp + a_s^3 S_3^\perp + \mathcal{O}(a_s^4) \right] \quad (5)$$

where we have defined $a_s = \alpha_s(\mu)/(4\pi)$ as our perturbative expansion parameter throughout this Letter. The one and two-loop coefficients $S_{1,2}^\perp$ can be found in Ref. [37]. In the next section we outline the procedure we used to calculate the three-loop coefficient S_3^\perp , from which the rapidity anomalous dimensions can be extracted to the same order.

Method. To obtain the TMD soft function S_\perp through to three loops, we first calculate the fully differential soft function to the same order. $S_{\text{F.D.}}$ obeys a RG equation identical to eq. (3) [25]:

$$\frac{d \ln S_{\text{F.D.}}(\vec{b}_\perp, \mu, \nu)}{d \ln \mu^2} = \Gamma_{\text{cusp}}[\alpha_s(\mu)] \ln \frac{\mu^2}{\nu^2} - \gamma_s[\alpha_s(\mu)] \quad (6)$$

In $S_{\text{F.D.}}$, ν is a parameter of the theory, not a regulator. Therefore the ν dependence of $S_{\text{F.D.}}$ is in general

complicated. The perturbative solution to $S_{\text{F.D.}}$ is then determined by eq. (6) and the boundary condition at initial scale, $S_{\text{F.D.}}(\vec{b}_\perp, \mu = \nu, \nu)$. Similar to S_\perp , $S_{\text{F.D.}}$ can also be written as an exponential, as in eq. (5). The one and two-loop coefficients $S_{1,2}^{\text{F.D.}}$ were first computed in Ref. [45], and reproduced in Ref. [37].

By dimension analysis, $S_{\text{F.D.}}(\vec{b}_\perp, \nu, \nu)$ is a function of $x = -\vec{b}_\perp^2 \nu^2 / b_0^2$. A strategy based on the bootstrap program for scattering amplitudes [46] is proposed in Ref. [37] to compute $S_{\text{F.D.}}(\vec{b}_\perp, \nu, \nu)$, which we briefly recall below. In Ref. [45], the one and two-loop coefficients $S_{1,2}^{\text{F.D.}}$ are written in terms of classical and Nielsens polylogarithms with argument x . A crucial observation made in Ref. [37] is that the same results can be written in terms of harmonic polylogarithms (HPL) $H_{\vec{w}}(x)$ [47], with weight indices drawn from the set $\{0, 1\}$. Furthermore, for the available one and two-loop data, the leftmost and the rightmost index of the weight vectors were found to be 0 and 1, respectively. The rightmost index has to be 1, because the two cusp points of the Wilson loops are separated by Euclidean distance for $x < 0$, and no branch cut is expected. On the other hand, the condition on the leftmost-index comes empirically from the observation of the one- and two-loop results; as we will show below, this condition breaks down at three loops in QCD. Nevertheless, for now we proceed with the empirical ansatz for L -loop fully differential soft function proposed in Ref. [37], which is a linear combination of HPLs with undetermined rational coefficients, and whose weight vectors obey the leftmost- and rightmost-index conditions. The undetermined coefficients of the HPLs can then be fixed by performing an expansion around $x \sim 0$, together with the constraint that rapidity divergence is only a single logarithmic divergence at each order for the expansion coefficients in eq. (5). It turns out that the $x \rightarrow 0$ limit of $S_{\text{F.D.}}$ is smooth, and the expansion is simply a Taylor series in x . As explained in Ref. [37], the leading x^0 term of the expansion reproduces the threshold soft function [41], while the coefficient of x^n can be obtained by inserting a numerator $(l^+ l^- - l^2)^n$ into the integrand of the threshold soft function, where l is the total momenta of real radiation from the time-ordered Wilson loop. Furthermore, using Integration-By-Parts (IBP) identities [49, 50], integrals with high rank numerator insertion can be reduced to a small number of master integrals, which have been computed for other purpose recently [51–56].

Although the strategy outlined above is straightforward, it has two caveats. First, the maximal weight of HPLs at three loops for massless perturbation theory is 6. It follows that the number of coefficients need to be fixed is $\sum_{i=0}^4 2^i = 31$. In other words, one needs to insert a high-rank numerator $(l^+ l^- - l^2)^{31}$ into the integrand of threshold soft function in order to have enough data to fix the coefficients, which is unfortunately beyond the

ability of the tools for IBP reduction [57–60]. Second, it is not clear whether the conjectured sets of function in Ref. [37] is sufficient to describe the three-loop soft function. To circumvent the above difficulties, we first perform the calculation for soft Wilson loops whose matter content [41, 52, 54] resembles those of $\mathcal{N} = 4$ Supersymmetric Yang-Mills theory (SYM). This has a number of advantages: 1) it has been observed that for soft Wilson loops in SCET [41], the results in $\mathcal{N} = 4$ SYM has uniform degrees of transcendentality with transcendental weight $2L$ at L loops. Furthermore, the $\mathcal{N} = 4$ results match the maximal-weight part of the corresponding QCD results. Similar phenomenon was first observed for anomalous dimension of twist-two operator for Wilson lines [61]. It also holds for some other quantities, e.g., perturbative form factor [30, 62, 63]. Assuming that this is also true in our current calculation, by calculating $S_{\text{F.D.}}$ in $\mathcal{N} = 4$ SYM first, we should automatically obtain the maximal-weight part of $S_{\text{F.D.}}$ in QCD; 2) since the $\mathcal{N} = 4$ SYM results have uniform degrees of transcendentality, there are only 16 coefficients to be fixed at three

loops, which can be achieved within the current computation power; 3) the remaining parts of the QCD result have transcendental weight lower than 6, therefore only requires 15 coefficients to be fixed. Alternatively, since the Feynman diagrams corresponding to the lower-weight part have less complicated analytical structure, they can be computed by brute force. Direct calculation can also test the completeness of the ansatz. And it turns out that although the ansatz remain complete for the three-loop $\mathcal{N} = 4$ SYM result, it fails for the three-loop QCD one. Fortunately, for QCD result, a brute-force calculation for the terms proportional to n_f is possible using the method of Ref. [55]. More importantly, the result for n_f terms indicates which set of functions we should add to the existing ansatz. The full results, for both $\mathcal{N} = 4$ SYM and QCD, are presented in the next section.

Results. We first present the results for $S_{\text{F.D.}}$ in $\mathcal{N} = 4$ SYM. We only give the results at the initial scale, $\mu = \nu$. The full scale dependence can be inferred from eq. (6). The one and two-loop coefficients can be found in Ref. [37]. The three-loop coefficient in the four-dimensional-helicity scheme [64] reads

$$\begin{aligned} S_{3,\mathcal{N}=4}^{\text{F.D.}} \Big|_{\mu=\nu} = & c_{3,\mathcal{N}=4}^s + N_c^3 \left(16\zeta_2 H_4 + 48\zeta_2 H_{2,2} + 64\zeta_2 H_{3,1} + 96\zeta_2 H_{2,1,1} + 120\zeta_4 H_2 + 48H_6 + 24H_{2,4} + 40H_{3,3} \right. \\ & + 72H_{4,2} + 128H_{5,1} + 16H_{2,1,3} + 56H_{2,2,2} + 80H_{2,3,1} + 80H_{3,1,2} + 144H_{3,2,1} + 224H_{4,1,1} \\ & \left. + 64H_{2,1,1,2} + 96H_{2,1,2,1} + 160H_{2,2,1,1} + 256H_{3,1,1,1} + 192H_{2,1,1,1,1} \right) \end{aligned} \quad (7)$$

where $c_{3,\mathcal{N}=4}^s = 492.609N_c^3$ is the three-loop constant for threshold soft function in $\mathcal{N} = 4$ SYM [41]. We have used the shorthand notation for the HPLs [47] and neglected the argument x . It is interesting to note that each term in eq. (7) has uniform sign and integer coefficient. Further-

more, overall sign is alternating at each order in α_s [37]. Similar behavior of alternating uniform signs in perturbative expansion with increasing loop order for certain observable was known before, see Ref. [48]. The corresponding results for QCD in 't Hooft-Veltman scheme reads:

$$\begin{aligned} S_3^{\text{F.D.}} \Big|_{\mu=\nu} = & c_3^s + \frac{C_A C_A^2}{N_c^3} \left(S_{3,\mathcal{N}=4}^{\text{F.D.}}(x) \Big|_{\mu=\nu} - c_{3,\mathcal{N}=4}^s \right) + C_A C_A^2 \left[-\frac{1072}{9}\zeta_2 H_2 - 176\zeta_3 H_2 - \frac{88}{3}\zeta_2 H_3 + 88\zeta_2 H_{2,1} \right. \\ & + \frac{30790}{81}H_2 + \frac{7120}{27}H_3 - \frac{104}{9}H_4 - \frac{440}{3}H_5 - \frac{8}{3} \left(H_{1,1} - \frac{H_{1,1}}{x} \right) - \frac{7120}{27}H_{2,1} - \frac{1072}{9}H_{2,2} - \frac{88}{3}H_{2,3} \\ & \left. - \frac{3112}{9}H_{3,1} - 88H_{3,2} - \frac{352}{3}H_{4,1} - \frac{392}{3}H_{2,1,1} + \frac{88}{3}H_{2,1,2} + \frac{352}{3}H_{2,2,1} + \frac{352}{3}H_{3,1,1} + 352H_{2,1,1,1} \right] \\ & + C_A C_A n_f \left[\frac{160}{9}\zeta_2 H_2 + \frac{16}{3}\zeta_2 H_3 - 16\zeta_2 H_{2,1} - \frac{7988}{81}H_2 - \frac{2312}{27}H_3 - \frac{64}{3}H_4 + \frac{80}{3}H_5 + \frac{8}{3} \left(H_{1,1} - \frac{H_{1,1}}{x} \right) \right. \\ & + \frac{2312}{27}H_{2,1} + \frac{160}{9}H_{2,2} + \frac{16}{3}H_{2,3} + \frac{224}{3}H_{3,1} + 16H_{3,2} + \frac{64}{3}H_{4,1} - \frac{32}{9}H_{2,1,1} - \frac{16}{3}H_{2,1,2} - \frac{64}{3}H_{2,2,1} \\ & \left. - \frac{64}{3}H_{3,1,1} - 64H_{2,1,1,1} \right] + C_A n_f^2 \left(\frac{400}{81}H_2 + \frac{160}{27}H_3 + \frac{32}{9}H_4 - \frac{160}{27}H_{2,1} - \frac{32}{9}H_{3,1} + \frac{32}{9}H_{2,1,1} \right) \end{aligned}$$

$$+ C_a C_F n_f \left(32\zeta_3 H_2 - \frac{110}{3} H_2 - 8H_3 + 8H_{2,1} \right) \quad (8)$$

where $C_a = C_F$ for Drell-Yan process, and $C_a = C_A$ for Higgs production. c_3^s is the three-loop scale independent part of the threshold soft function in QCD, $c_3^s = S_3^{\text{thr.}}(\tau, \mu = \tau^{-1})$, see for example Refs. [37, 41, 65]. It can be found in eq. (3.2) of Ref. [41] by multiplying a casimir rescaling factor C_a/C_A . We note that the only term that goes beyond the empirical ansatz [37] is $(H_{1,1} - H_{1,1}/x)^1$, which can be inferred from the direct calculation of the n_f -dependent part using Feynman diagram method. Specifically, if all the relevant integrals are known, the result for $\mathcal{N} = 4$ SYM in eq. (7) can also be obtained using Feynman diagram method, in a gauge theory with $n_f = 4$ adjoint fermions, $n_s = 6$ adjoint real scalars, and with proper Yukawa interaction between the fermions and scalars. While the integrals for the pure gluon contribution are challenging, we manage to compute the n_f - and n_s -dependent terms by brute-force Feynman diagram calculation. We observe that for both the fermion and scalar contributions, the only addition needed to correct the empirical ansatz at three loops is the combination $(H_{1,1} - H_{1,1}/x)$. From there we can readily extract the gluon contribution, which is the same in $\mathcal{N} = 4$ SYM and QCD, by subtracting from eq. (7) the corresponding fermion and scalar contributions. We can also conclude that the only addition to the ansatz of the gluon contribution is the combination $(H_{1,1} - H_{1,1}/x)$.

We briefly describe the available checks on our results in eqs. (7) and (8). Firstly, as mentioned above, due to the relative simplicity in the resulting integrals, we have been able to compute all the n_f -dependent part in eq. (8) by directly calculating the Feynman diagrams. We find that our ansatz, even including the $(1 - 1/x)H_{1,1}$ term, is insufficient to express the result in the intermediate step of the direct calculation. The additional terms needed are $(1 - 1/x)H_1$, H_2/x , $\zeta_2 H_1 - H_{1,2}$. Interestingly, they all cancel out in the sum of real and virtual contributions. Secondly, our ansatz can be uniquely fixed at three loops using the data from Taylor expansion over x through to x^{10} . However, we have obtained the expansion data through to x^{17} , leading to an over constrained system of equations. We found that the solution exist and is unique for the system, thus providing a strong check of our calculation. See, e.g. Ref. [66] for similar discussion on using over constrained system of equations to fix ansatz.

With the fully differential soft function at hand, it is straightforward to obtain S_\perp by taking the limit $\nu \rightarrow +\infty$ using the package HPL [67]. The soft anomalous dimension γ_s through to three loops can be found, e.g., in eq. (A.4-6) of Ref. [41] by an rescaling factor C_a/C_A . The rapidity anomalous dimensions are given by:

$$\begin{aligned} \gamma_0^r &= 0 \\ \gamma_1^r &= C_a C_A \left(28\zeta_3 - \frac{808}{27} \right) + \frac{112C_a n_f}{27} \\ \gamma_2^r &= C_a C_A^2 \left(-\frac{176}{3}\zeta_3\zeta_2 + \frac{6392\zeta_2}{81} + \frac{12328\zeta_3}{27} + \frac{154\zeta_4}{3} \right. \\ &\quad \left. - 192\zeta_5 - \frac{297029}{729} \right) + C_a C_A n_f \left(-\frac{824\zeta_2}{81} - \frac{904\zeta_3}{27} \right. \\ &\quad \left. + \frac{20\zeta_4}{3} + \frac{62626}{729} \right) + C_a n_f^2 \left(-\frac{32\zeta_3}{9} - \frac{1856}{729} \right) \\ &\quad + C_a C_F n_f \left(-\frac{304\zeta_3}{9} - 16\zeta_4 + \frac{1711}{27} \right) \end{aligned} \quad (9)$$

Note that γ_0^r and γ_1^r can be obtained from QCD anomalous dimension known long time ago [68–70]. They have also been reproduced in SCET recently [37, 71–73]. The three-loop coefficient γ_2^r is new and is one of the main results of this Letter. It is also straightforward to obtain the boundary condition of S_\perp at the initial scale, $c_3^\perp \equiv S_3^\perp(\vec{b}_\perp, \mu = b_0/|\vec{b}_\perp|, \nu = b_0/|\vec{b}_\perp|)$:

$$\begin{aligned} c_3^\perp &= C_a C_A^2 \left(\frac{928\zeta_3^2}{9} + \frac{1100}{9}\zeta_2\zeta_3 - \frac{151132\zeta_3}{243} - \frac{297481\zeta_2}{729} \right. \\ &\quad \left. + \frac{3649\zeta_4}{27} + \frac{1804\zeta_5}{9} - \frac{3086\zeta_6}{27} + \frac{5211949}{13122} \right) \\ &\quad + C_a C_A n_f \left(\frac{40}{9}\zeta_3\zeta_2 + \frac{74530\zeta_2}{729} + \frac{8152\zeta_3}{81} - \frac{416\zeta_4}{27} \right. \\ &\quad \left. - \frac{184\zeta_5}{3} - \frac{412765}{6561} \right) + C_a C_F n_f \left(-\frac{80}{3}\zeta_3\zeta_2 \right. \\ &\quad \left. + \frac{275\zeta_2}{9} + \frac{3488\zeta_3}{81} + \frac{152\zeta_4}{9} + \frac{224\zeta_5}{9} - \frac{42727}{486} \right) \\ &\quad + C_a n_f^2 \left(-\frac{136\zeta_2}{27} - \frac{560\zeta_3}{243} - \frac{44\zeta_4}{27} - \frac{256}{6561} \right) \end{aligned} \quad (10)$$

Discussion. The explicit results for the rapidity anomalous dimension in eq. (9) can be rewritten in a remarkable form:

$$\gamma_0^r = \gamma_0^s$$

¹ This term cancels out in the $\mathcal{N} = 4$ combination, as is clear from eq. (7). It also cancels out in the pure $\mathcal{N} = 1$ SYM with adjoint gluino, in which one simply sets $n_f \rightarrow C_A$ and $C_F \rightarrow C_A$. We thank Mingxing Luo and Lance Dixon for pointing out this.

$$\begin{aligned}\gamma_1^r &= \gamma_1^s - \beta_0 c_1^s \\ \gamma_2^r &= \gamma_2^s - 2\beta_0 c_2^s - \beta_1 c_1^s + 2C_a C_A \beta_0 \zeta_4\end{aligned}\quad (11)$$

Eq. (11) is interesting because it connects between very different objects: the rapidity anomalous dimension γ_r , the soft anomalous dimension γ_s , the threshold constant c_s , and the QCD beta function. Similar relation also holds in $\mathcal{N} = 4$ SYM by dropping the beta function terms in eq. (11).

In the CSS formalism, the resummation of large q_T logarithms is controlled by two anomalous dimension, $A[\alpha_s(\mu)] = \sum_{i=1} a_s^i A_i$ and $B[\alpha_s(\mu)] = \sum_{i=1} a_s^i B_i$. It is straightforward to express these anomalous dimension in terms of the anomalous dimension in SCET, see e.g. Ref. [26, 74]. In particular, we obtain the B anomalous dimension in the original CSS scheme through to three loops:

$$\begin{aligned}B_1 &= \gamma_0^V - \gamma_0^r \\ B_2 &= \gamma_1^V - \gamma_1^r + \beta_0 c_1^V \\ B_3 &= \gamma_2^V - \gamma_2^r + \beta_1 c_1^V + 2\beta_0 \left(c_2^V - \frac{1}{2} (c_1^V)^2 \right)\end{aligned}\quad (12)$$

where γ_V is the anomalous dimension of hard function results from matching QCD onto SCET. c_V is the scale independent terms of the hard matching. For Drell-Yan production they can be extracted from quark form factor [30–32], while for Higgs production from gluon form factor [30–32], and additionally from effective coupling of the Higgs boson to gluons [75]. Eq. (12) partially explains the close connection between γ_r and γ_s , because the combination $\gamma_V - \gamma_s$ is given by the $\delta(1-x)$ part of the single pole in the QCD splitting function [40]. Substituting the actual numbers in eq. (12), we find

$$\begin{aligned}B_1^{DY} &= -8, \quad B_2^{DY} = 13.3447 + 3.4138 n_f, \\ B_3^{DY} &= 7358.86 - 721.516 n_f + 20.5951 n_f^2\end{aligned}\quad (13)$$

for Drell-Yan production. For Higgs production, the results are

$$\begin{aligned}B_1^H &= -22 + 1.33333 n_f, \quad B_2^H = 658.881 - 45.9712 n_f, \\ B_3^H &= 35134.6 - 7311.10 n_f + 293.017 n_f^2 \\ &\quad - (836 + 184 n_f - 14.2222 n_f^2) \ln \frac{m_t^2}{m_H^2}\end{aligned}\quad (14)$$

The one and two-loop results are known for a long time [68–70]. The three-loop results are new. We note that numerically B_3^{DY} is quite large for $n_f = 5$.

In summary, we have presented the first calculation of soft function for transverse-momentum resummation in rapidity RG formalism through to three loops, using the rapidity regulator recently introduced in Ref. [37]. As a by product, we have also obtained the fully differential soft function to the same order. Our calculation combine the use of bootstrap technique and supersymmetric decomposition in transcendental weight. We

found a surprising relation between the anomalous dimensions for the transverse-momentum resummation and the threshold resummation, whose explanation calls for further investigation. Our three-loop results pave the way for transverse-momentum resummation for production of color neutral system at hadron colliders at N³LL + NNLO accuracy. The method and results of our calculation also make generalizing q_T -subtraction method [76] to N³LO promising.

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