General Approach to Quantum Channel Impossibility by Local Operations and Classical Communication

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We describe a general approach to proving the impossibility of implementing a quantum channel by local operations and classical communication (LOCC), even with an infinite number of rounds, and find that this can often be demonstrated by solving a set of linear equations. The method also allows one to design an LOCC protocol to implement the channel whenever such a protocol exists in any finite number of rounds. Perhaps surprisingly, the computational expense for analyzing LOCC channels is not much greater than that for LOCC measurements. We apply the method to several examples, two of which provide numerical evidence that the set of quantum channels that are not LOCC is not closed and that there exist channels that can be implemented by LOCC either in one round or in three rounds that are on the boundary of the set of all LOCC channels.

Although every LOCC protocol must implement a separable quantum channel, it is a very difficult task to determine whether or not a given channel is separable. Fortunately, prior knowledge that the channel is separable is not required for application of our method.

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Quantum information theory is concerned with the study of the transmission, storage, and manipulation of information when that information is encoded in the form of quantum systems. As it is impossible to completely isolate these systems, any description of their evolution must take into account their interaction with an environment, suggesting an analysis utilizing the powerful techniques available for the study of open quantum systems [1–4]. Interaction with the environment is then considered to introduce noise into the system's evolution, and there are many important questions one may ask about the action of noisy quantum operations, or channels, not the least of which is to find the capacity of a noisy channel to transmit quantum information [5–7]. When the input to the channel is one part of an entangled state, this capacity also measures its ability to establish entanglement between sender and receiver [8, 9].

For the input to be part of an entangled state, it must have first undergone an entangling evolution of its own. Hence, one is led to investigate the entangling capabilities of such evolutions, be they unitary or general quantum operations. The relatively recent recognition of the practical value of entangled states—examples ranging from teleportation [10] and superdense coding [11] to quantum cryptography [12–14] and quantum computation [15–17]—has generated considerable interest in understanding the conditions under which entanglement can be created. Thus, one may ask what characterizes a (multipartite) quantum channel's ability to create and/or increase entanglement between the subsystems upon which it acts. One significant such characterization is that the entanglement cannot increase when the channel can be simulated by local quantum operations and classical communication (LOCC), which are the only operations that can actually be implemented by spatially separated parties who lack the means to bring their subsystems together in a single laboratory. This result provides an important connection between the study of LOCC and that of entanglement. We note that it is always possible to implement any channel by LOCC when enough entanglement is pre-shared between the various parties—by using teleportation [10] or by perhaps more efficient means [18–20]—indicating that entanglement is an important resource [21–23] under the restriction to LOCC.

The importance of LOCC in quantum information processing has long been recognized, it playing a key role in teleportation [10], entanglement distillation [24, 25], one-way [26] and distributed [27] quantum computing, local cloning [28], quantum secret sharing [29, 30], and beyond. While many important results have been obtained concerning LOCC [31–46], it has nonetheless proven difficult to characterize in simple terms. Recently [47], we presented a method of designing LOCC protocols for quantum measurements that succeeds for every measurement that can be implemented by finite-round LOCC, and for which failure has the immediate implication that the measurement cannot be implemented by LOCC no matter how many rounds of communication are allowed, including when the number of rounds is infinite (see also [34, 35] for a different approach to designing LOCC protocols for measurements, an approach which is, in contrast, restricted to a finite number of rounds).

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There are, of course, significant differences between channels and measurements. A noteworthy example in the context of LOCC is the finding in [46] that the set of finite-round LOCC channels has a non-empty interior, whereas the interior of the set of finite-round LOCC measurements is, in fact, empty [48]. It is also known [44] that the set of LOCC channels is not closed, even for the simplest possible case of two qubits [46], but the question of whether or not LOCC measurements is a closed set appears to be as yet unanswered.

Another key difference between measurements and channels is that there are many measurements that are all associated with any given channel, see Eq. (3) below. Therefore, one can demonstrate that a channel is LOCC by showing that any one of those measurements is itself LOCC. If, on the other hand, a given channel is not LOCC, *none* of the measurements associated with it can be implemented by LOCC. This latter point may lead one to expect that the characterization of LOCC channels may be a far more difficult task than that for LOCC measurements. It turns out, however that this expectation is overly pessimistic. In the present paper, we show that the method of [47] for measurements is readily extended to the case of multipartite quantum channels, allowing one to design an LOCC protocol to implement a given channel whenever this is possible in a finite number of rounds, with a computational effort for the case of channels that is never more than a quadratic increase over that needed for measurements. Our method builds a protocol round by round, starting with the first one. If at any point in this process, the method finds that no measurement at the 'next' round is possible, then this implies that no LOCC protocol exists for the desired channel, even with an infinite number of rounds [47]. As we will see below, if no measurement is possible at the very first round, then the method proves LOCC-impossibility via the solution of a set of linear equations. We will demonstrate that solving this set of linear equations is often sufficient for proving LOCC-impossibility, illustrating this approach with various examples.

The remainder of the paper is organized as follows. We begin by reviewing the Kraus representation of quantum channels and the concomitant unitary freedom in such representations. Following this, we give a more detailed explanation of what exactly LOCC means, and how it can be represented. Then, we recall a lemma from [47], and show how the method presented there can be fully adapted to the present case of quantum channels. These ideas are then used to analyze certain example channels which are, with relative ease, found to be impossible by LOCC. Finally, we offer our conclusions.

Each quantum channel may be described by a set of Kraus operators $\{K_i\}_{i=1}^N$ [3], which indicate how that channel transforms the state of the quantum system upon which it acts. If that system, described by Hilbert space $\mathcal{H}$, starts out in state ρ_0, then its final state will be given by

$$\rho_f = \sum_{i=1}^N K_i \rho_0 K_i^\dagger, \tag{1}$$

where with $I_\mathcal{H}$ the identity operator on $\mathcal{H}$,

$$\sum_{i=1}^N K_i^\dagger K_i = I_\mathcal{H} \tag{2}$$

so that the channel is trace-preserving, $\mathrm{Tr}(\rho_f) = \mathrm{Tr}(\rho_0)$. Of course as is well-known, the set of Kraus operators describing any given channel is far from unique. For any other set of Kraus operators, $\{K'_j\}_{j=1}^{N'}$, that describes the same channel as the original set, let $\hat{N} = \max(N, N')$ and pad the smaller of the two sets with additional zero operators so that the two sets have the same number of members. Then, there exists a unitary matrix V such that for each $j = 1, \cdots, \hat{N}$,

$$K'_j = \sum_{i=1}^{\hat{N}} V_{ji} K_i. \tag{3}$$

Furthermore, any other set describing this channel may be expressed in this way [17].

An LOCC protocol involves one party making a measurement, informing the other parties of her outcome, after which according to a pre-approved plan, the other parties know who is to measure next and what that measurement should be. Since each local measurement involves a number of possible outcomes, the entire process is commonly represented as a tree, the children of any given node representing the set of outcomes of the measurement made at that stage in the protocol. Consider the cumulative action of all parties up to a given node n in the tree, represented by Kraus operator K, which since the parties each make only local

measurements, must be of the product form, $A \otimes B \otimes \cdots$. We will label each such node by the positive operator $K^\dagger K$, commonly referred to as a *POVM element*, and we will say that node n is 'equal' to its label $K^\dagger K$. For any such tree, the root node represents the situation present before any party has yet measured and will therefore always be equal to the identity operator $I_\mathcal{H} = I_A \otimes I_B \otimes \cdots$, where I_α is the identity operator on the local Hilbert space $\mathcal{H}_\alpha$ for each party α. At the end of any finite branch of the tree is a leaf node (leaf nodes being those that do not themselves have children), which since it is terminal, must be labeled by a POVM element associated with a Kraus representation of the desired channel. Similarly, any infinite branch has a sequence of nodes that approach such a POVM element in the limit. When we say that a finite-round LOCC protocol implements Kraus operators $\{K_j\}$ this means that each leaf is proportional to one of the $K_j^\dagger K_j$, with positive constant of proportionality, and the sum of all leaves proportional to $K_j^\dagger K_j$ is exactly $K_j^\dagger K_j$. For infinite protocols, it means the same but in the limit as the number of rounds approaches infinity. Note that the class of infinite-round protocols can be divided into two distinct sub-classes [46]. The first subclass, which is a subset of what we will refer to as 'LOCC' [46], involves sequences of protocols more and more closely approaching a given channel simply by adding more and more rounds of communication, but without changing the local measurements implemented in earlier rounds. The second subclass, $\overline{\text{LOCC}}$ in [46], includes limits of sequences in which measurements made at the earlier rounds are changed from one protocol in the sequence to the next. As indicated by the notation, $\overline{\text{LOCC}}$ is the topological closure of LOCC, and as mentioned above when considering quantum channels, $\overline{\text{LOCC}} \neq \text{LOCC}$ [44]. We note that the results of the present paper apply to the entire class LOCC, including those that involve an infinite number of rounds, but not to $\overline{\text{LOCC}}$.

We will need the following lemma, proved in [47] (but reworded here to conform to the present context).

Lemma 1. *Suppose the tree $\mathcal{L}$ represents an LOCC protocol (finite or infinite), which implements the set of Kraus operators $\{K_j'\}$. Then the POVM element E, representing the accumulated action of all parties up to any given node in $\mathcal{L}$, is equal to a positive linear combination of the set of operators, $\{K_j'^\dagger K_j'\}$. That is, $E = \sum_j c_j K_j'^\dagger K_j'$, with $c_j \geq 0$.*

Without loss of generality, let us assume that Alice measures first to initiate an LOCC protocol that implements quantum channel $\mathcal{E}$. More specifically, assume the protocol implements any set of Kraus operators, $\{K_j'\}$, that represents $\mathcal{E}$. Since Alice has measured first, the other parties have as yet done nothing, which implies that the outcome of Alice's initial measurement will correspond to a (multipartite) Kraus operator of the form $A \otimes I_{\bar{A}}$, where $I_{\bar{A}}$ is the identity operator on $\mathcal{H}_{\bar{A}}$, the Hilbert space describing all parties other than Alice. The associated POVM element is then $E = \mathcal{A} \otimes I_{\bar{A}}$, with $\mathcal{A} = A^\dagger A$, which according to Lemma 1 implies that with $c_j \geq 0$,

$$\mathcal{A} \otimes I_{\bar{A}} = \sum_{j=1}^{\hat{N}} c_j K_j'^\dagger K_j' = \sum_{i,i'=1}^{\hat{N}} \left[\sum_{j=1}^{\hat{N}} V_{ji}^* c_j V_{ji'} \right] K_i^\dagger K_{i'}$$

$$= \sum_{i,i'=1}^{\hat{N}} \mathcal{C}_{ii'} K_i^\dagger K_{i'}, \tag{4}$$

and $\mathcal{C}_{ii'} = \sum_{j=1}^{\hat{N}} V_{ji}^* c_j V_{ji'}$.

At this point it is convenient to choose a set of index pairs, $\mathcal{S} \subseteq \{(i,i')\}_{i,i'=1}^{\hat{N}}$, such that the set of operators $\{K_i^\dagger K_{i'}\}_{(i,i') \in \mathcal{S}}$ is linearly independent. Then with $K_j^\dagger K_{j'} = \sum_{(i,i') \in \mathcal{S}} r_{jj'}^{ii'} K_i^\dagger K_{i'}$ for $(j,j') \notin \mathcal{S}$,

$$\mathcal{A} \otimes I_{\bar{A}} = \sum_{(i,i') \in \mathcal{S}} \left(\mathcal{C}_{ii'} K_i^\dagger K_{i'} + \sum_{(j,j') \notin \mathcal{S}} \mathcal{C}_{jj'} r_{jj'}^{ii'} K_i^\dagger K_{i'} \right)$$

$$= \sum_{(i,i') \in \mathcal{S}} \tilde{\mathcal{C}}_{ii'} K_i^\dagger K_{i'}, \tag{5}$$

where $\tilde{\mathcal{C}}_{ii'} = \mathcal{C}_{ii'} + \sum_{(j,j') \notin \mathcal{S}} \mathcal{C}_{jj'} r_{jj'}^{ii'}$. This condition must hold for all sets of Kraus operators $\{K_i\}$ representing $\mathcal{E}$. Since Kraus representation $\{K_j'\}$ for $\mathcal{E}$ can also be arbitrarily chosen, we see that Eq. (5) with

$0 \leq \mathcal{A} \not\propto I_A$ (or else Alice did not actually measure) is a necessary condition for the existence of an LOCC protocol that implements $\mathcal{E}$.

Let $\{\Lambda_\mu\}$ and $\{\Gamma_\nu\}$ be orthonormal bases of the space of operators acting on $\mathcal{H}_A$ and $\mathcal{H}_{\bar{A}}$, respectively, $\mathrm{Tr}(\Lambda_\mu^\dagger \Lambda_{\mu'}) = \delta_{\mu\mu'}$ and $\mathrm{Tr}(\Gamma_\nu^\dagger \Gamma_{\nu'}) = \delta_{\nu\nu'}$, with $\Lambda_0 = I_A$ and $\Gamma_0 = I_{\bar{A}}$. Choose any μ and any $\nu \neq 0$ and multiply Eq. (5) by $\Lambda_\mu^\dagger \otimes \Gamma_\nu^\dagger$ to obtain

$$0 = \sum_{(i,i')\in\mathcal{S}} \tilde{\mathcal{C}}_{ii'} \mathrm{Tr}\left(\left[\Lambda_\mu^\dagger \otimes \Gamma_\nu^\dagger\right] K_i^\dagger K_{i'}\right) = \sum_{(i,i')\in\mathcal{S}} \tilde{\mathcal{C}}_{ii'} \mathcal{Q}_{i'i}^{(\mu\nu)}, \ \mu = 0, \cdots, d_A^2 - 1, \ \nu = 1, \cdots, d_{\bar{A}}^2 - 1 \qquad (6)$$

where d_A is the dimension of $\mathcal{H}_A$, $d_{\bar{A}}$ is that of $\mathcal{H}_{\bar{A}}$, and

$$\mathcal{Q}_{i'i}^{(\mu\nu)} := \mathrm{Tr}\left(\left[\Lambda_\mu^\dagger \otimes \Gamma_\nu^\dagger\right] K_i^\dagger K_{i'}\right). \qquad (7)$$

The next step is to form the coefficients $\tilde{\mathcal{C}}_{ii'}$ into an $|\mathcal{S}|$-dimensional column vector $\vec{c}$ (by, say, stacking each column one below the next) and also the $\mathcal{Q}_{i'i}^{(\mu\nu)}$ into a row vector for each μ and $\nu \neq 0$, collecting all the latter row vectors into a matrix $\mathbf{Q}_A$. Then, we have from Eq. (6),

$$0 = \mathbf{Q}_A \vec{c}, \qquad (8)$$

showing that any initial local measurement by Alice in an LOCC protocol exactly implementing the desired quantum channel $\mathcal{E}$ is determined by the nullspace of matrix $\mathbf{Q}_A$.[1] In similar fashion, one can use this approach to obtain all later measurements, thus designing a full LOCC protocol for $\mathcal{E}$ whenever possible. This design approach is described in detail in [47] for the case of implementing a quantum measurement rather than a channel. For measurements, the set of Kraus operators is fixed and unitary V in Eq. (4) is equal to the identity; then the free parameters are the $\hat{N}$ coefficients c_j, rather than the $|\mathcal{S}|$ coefficients $\tilde{\mathcal{C}}_{ii'}$. Thus, we see that the approach described in [47] can be directly taken over to the case of channels simply by increasing the dimension of the domain of matrix $\mathbf{Q}_A$ from $\hat{N}$ [47] to $|\mathcal{S}| \leq \hat{N}^2$, where $\hat{N}$ is the number of Kraus operators defining the measurement (in the first case) or representing the channel (in the second case).

By Eq. (2), there is always at least one solution of Eq. (8), which we will denote as $\vec{c}_I$ corresponding to $\mathcal{A} = I_A$ in Eq. (5). If this is the only solution, then Alice cannot measure first. If this is true for every party, then no LOCC protocol exists for the given channel.

Let us augment $\mathbf{Q}_A$ by adding the single row $\vec{c}_I^\dagger$ (and, hoping not to cause confusion, we continue to call this $\mathbf{Q}_A$). Then $\vec{c}_I$ is no longer in the nullspace of $\mathbf{Q}_A$, and we have the following theorem.

Theorem 2. *For a given multipartite quantum channel $\mathcal{E}$, if for every party α, the nullspace of $\mathbf{Q}_\alpha$ is empty, then $\mathcal{E}$ cannot be implemented by LOCC, even when using an infinite number of rounds.*

Note that this theorem provides a way to easily prove certain channels are not LOCC, requiring only the solution of a set of linear equations. Below is a list of examples for which this method provides such a proof. For each example, we give the ratio of the smallest eigenvalue of $\mathbf{Q}_\alpha^\dagger \mathbf{Q}_\alpha \geq 0$ to the largest, minimized over parties α, denoting this ratio as $\hat{\lambda}$.[2] When $\hat{\lambda} \neq 0$, the nullspace of $\mathbf{Q}_\alpha$ is empty for all parties α, and then Theorem 2 shows that the given channel cannot be implemented by LOCC. If $\hat{\lambda}$ is small, then $\mathbf{Q}_\alpha$ is close to singular, implying there is a first measurement that introduces only a small error into the protocol. For example, by a small change in the set of Kraus operators one could alter $\mathbf{Q}_\alpha$ so that it becomes singular. However, it should be remembered that this is just the first measurement of a protocol, so that small $\hat{\lambda}$ does not by itself indicate there is a protocol that closely approximates the desired channel. In addition, even when $\hat{\lambda}$ is large, it may still be that the channel lies in the set closure, $\overline{\mathrm{LOCC}}$ (at least, we as yet have no argument to the contrary).

Let us now look at the examples.

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[1] Note that once the index set $\mathcal{S}$ and the orthonormal bases $\{\Lambda_\mu\}$ and $\{\Gamma_\nu\}$ are chosen, matrix $\mathbf{Q}_A$ is completely defined by the channel itself through the Kraus representation $\{K_i\}$. In addition, we claim that the choice of index set and bases does not change the results of our approach. The reason why the basis choices change nothing has been discussed in footnote 4 of [47]. Choosing index set $\mathcal{S}'$ instead of $\mathcal{S}$, corresponds to a choice of linearly independent operators, $K_j^\dagger K_{j'} = \sum_{(i,i')\in\mathcal{S}} T_{jj'}^{ii'} K_i^\dagger K_{i'}$, for $(j,j') \in \mathcal{S}'$, where T is a full-rank matrix whose columns are indexed by (j,j') and rows by (i,i'). This changes $\mathbf{Q}_A \to \mathbf{Q}'_A = \mathbf{Q}_A T$, so while the nullspace of $\mathbf{Q}'_A$ differs from that of $\mathbf{Q}_A$, the dimensions of the two nullspaces are identical. Since our results depend only on this dimension, see Theorem 2, we see that our claim is justified.

[2] The computational complexity of finding all eigenvalues of an $n \times n$ matrix in the large-n limit is $\mathcal{O}(n^3)$ by Householder reduction to tridiagonal form followed by either QR factorization or direct solution of the resulting characteristic equation [49]. We believe this calculation is the limiting step in our procedure of checking LOCC-impossibility. See Section 4 of [50] for our Matlab code implementing this procedure, where we use Matlab's 'eig' function to find the eigenvalues, it being plenty fast enough for the problems we have addressed here.

170   1. The bipartite channel determined by Kraus operators equal to projectors onto the four maximally
171   entangled Bell states [51]. We find that $\hat{\lambda} = 1$, implying that neither party can make a first measurement
172   and showing that this channel is not LOCC.

173   2. The channel defined by Kraus operators equal to the nine projectors onto the (product) domino states
174   of [52], which first demonstrated nonlocality without entanglement. We find that $\hat{\lambda} = 1/6$ for this case,
175   implying that neither party can make a first measurement and showing (the well-known result) that
176   this channel is not LOCC.

177   3. The channel defined by the rotated domino states [37, 52]. We find that for these channels, neither
178   party can measure first, other than in the exceptional cases where at least one pair of the dominos is
179   not rotated at all, in which case there exists a straightforward LOCC protocol for implementing these
180   channels. As seen in Fig. 1 (see [50] for further explanation), $\hat{\lambda}$ is nonzero, but smoothly approaches
181   zero as the degree of rotation (measured here by $\theta_{\min}$) vanishes. Thus, we see a smooth approach to the
182   existence of a first local measurement, and indeed to a fully LOCC channel that can be implemented
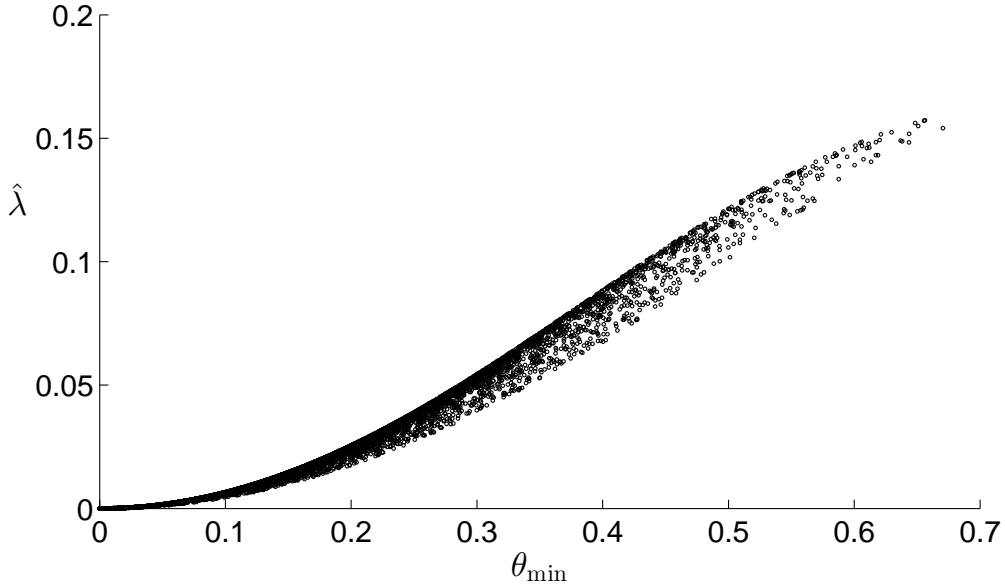183   using three rounds of communication in this limit [50].

184   FIG. 1. Impossibility of a first measurement for rotated domino states.
185

186   4. Random unitary channels, where the Kraus operators are proportional to (randomly chosen) unitaries.
187   The results in this case depend strongly on the number of random unitaries, N_u. For each number of
188   parties, each set of local dimensions, each N_u, and equal *a priori* probabilities of the random unitaries,
189   we have numerically checked 100 different sets of randomly chosen unitaries for each of the following
190   cases: two qubits, three qubits, two qutrits, and a bipartite qubit-qutrit system. In all cases, we have
191   found that no party can ever measure first when $N_u < D$, where D is the dimension of the global
192   Hilbert space $\mathcal{H}$, while a first measurement is not excluded for any party when $N_u > D$. In these
193   cases, $\hat{\lambda}$ decreases from no smaller than about 0.01 when $N_u = 2$ to no smaller than 3×10^{-4} when
194       $N_u = D - 1$. When $N_u > D$, $\hat{\lambda}$ is always less than about 10^{-16}. Thus we see a sharp transition
195   around $N_u = D$. When $N_u = D$, the situation is more diverse. For three qubits none of the parties
196   can measure first when $N_u = D = 8$, $\hat{\lambda}$ never being less than 10^{-4}. The case $N_u = D$ yields $\hat{\lambda}$ ranging
197   between about 10^{-5} to 10^{-9} for two qutrits and between about 10^{-3} to 10^{-8} for two qubits, showing
198   that neither party can measure first for these channels, but it is sometimes possible they are close to
199   channels where one or the other party can. For the qubit-qutrit case and $N_u = D = 6$, the party with
200   the qubit can never measure first ($\hat{\lambda}$ is never less than 10^{-5}), but it may be possible that the party

201 holding the qutrit can ($\hat{\lambda}$ is always less than about 10^{-16}). This provides evidence that 'almost' all
202 random unitary channels having small N_u are not LOCC.

203 5. The two-qubit channel defined by the five Kraus operators $|n\rangle\langle\pi_n|$, where $|\pi_1\rangle \propto |0\rangle (\alpha_1|0\rangle + \beta_1|1\rangle)$,
204 $|\pi_2\rangle \propto |0\rangle (\alpha_1|0\rangle - \beta_1|1\rangle)$, $|\pi_3\rangle \propto (\alpha_3|0\rangle + \beta_3|1\rangle)|0\rangle$, $|\pi_4\rangle = |1\rangle|1\rangle$, and $|\pi_5\rangle = (\mu|0\rangle + \nu|1\rangle)|0\rangle$. These
205 operators provide the optimal (global and separable) measurement for unambiguous state discrimina-
206 tion of the states given in Eq. (12) of [53]. Coefficients $\alpha_j, \beta_j, \mu, \nu$ are constrained as described in [53]
207 and summarized in Section 2 of the supplementary materials [50], but these constraints leave a contin-
208 uous class of example channels. It was shown in that paper that this optimal measurement cannot be
209 achieved by finite-round LOCC, but it was not previously known whether or not it could be achieved
210 by LOCC using an infinite number of rounds. We have generated several thousand examples with the
211 coefficients chosen at random, finding that no party can measure first except in the limiting cases of
212 $|\alpha_1| \to 0$, $|\alpha_1/\beta_1| \to 1$, and $\alpha_3 \to 0$, where LOCC protocols, requiring only a single round of communi-
213 cation, do exist. In all other cases, $\hat{\lambda}$ is nonzero, but approaches zero as any one of these limiting cases
214 is approached (see Figs. 2 and 3 in the supplementary materials [50]). This provides evidence that
215 almost all channels (also measurements) in this class are not LOCC, even with an infinite number of
216 rounds, and the possibility remains that none of them are. Thus, we have a sequence of channels (also
217 measurements) where no member of that sequence is LOCC, even with an infinite number of rounds,
218 but the limiting channel (measurement) is one-round LOCC. This provides numerical evidence that
219 the set of channels (measurements) that are not LOCC is not closed and that these one-round LOCC
220 channels are on the boundary of LOCC.

221 In conclusion, we have adapted the method of [47] for designing LOCC protocols implementing quantum
222 measurements so that it can be used for implementation of quantum channels. We have shown that one can
223 often prove LOCC-impossibility of a quantum channel by solving a set of linear equations and have provided
224 several examples. These examples include a new result suggesting that the class of optimal measurements
225 for unambiguous state discrimination given in [53] are all impossible by LOCC, including with an infinite
226 number of rounds, and also a result suggesting that almost all random unitary channels with few unitaries
227 are not LOCC. Additionally, we have obtained numerical evidence that the set of quantum channels that
228 are not LOCC is not closed, and that there exist one-round and three-round LOCC channels that are on the
229 boundary of the set of all LOCC channels. While our results do apply to infinite-round LOCC protocols,
230 they do not apply to the closure of LOCC, and we are presently at work studying ways to include the entire
231 boundary of LOCC in the analysis.

232 Finally, we would like to point out that our approach to LOCC-impossibility of quantum channels can
233 be recast in the form of semidefinite programs [54], see Section 3 in [50]. This and other extensions of the
234 results presented here will be discussed elsewhere.

237 [1] R. P. Feynman and Jr. F. L. Vernon, "The theory of a general quantum system interacting with a linear dissipative
238 system," Ann. Phys. (NY) **24**, 118 (1963).
239 [2] W. F. Stinespring, "Positive functions on C*-algebras," Proc. Amer. Math. Soc. **6**, 211 (1955).
240 [3] K. Kraus, *States, Effects, and Operations* (Spring-Verlag, Berlin, 1983).
241 [4] M-D Choi, "Completely positive linear maps on complex matrices," Lin. Alg. Appl. **10**, 285 (1975).
242 [5] I. Devetak, "The private classical capacity and quantum capacity of a quantum channel," IEEE Trans. Inf.
243 Theory **51**, 44 (2005).
244 [6] S. Lloyd, "Capacity of the noisy quantum channel," Phys. Rev. A **55**, 1613 (1997).
245 [7] Peter W. Shor, "The quantum channel capacity and coherent information," lec-
246 ture notes, MSRI Workshop on Quantum Computation , available online at
247 http://www.msri.org/publications/ln/msri/2002/quantumcrypto/shor/1/ (2002).
248 [8] Benjamin Schumacher, "Sending entanglement through noisy quantum channels," Phys. Rev. A **54**, 2614 (1996).
249 [9] Charles H. Bennett, David P. DiVincenzo, and John A. Smolin, "Teleporting an unknown quantum state via
250 dual classical and Einstein-Podolsky-Rosen channels," Phys. Rev. Lett. **78**, 3217 (1997).
251 [10] C. H. Bennett, G. Brassard, C. Crepeau, R. Jozsa, A. Peres, and W. K. Wootters, "Teleporting an unknown
252 quantum state via dual classical and Einstein-Podolsky-Rosen channels," Phys. Rev. Lett. **70**, 1895 (1993).

[11] C. H. Bennett and S. J. Wiesner, "Communication via one- and two-particle operators using Einstein-Podolsky-Rosen states," Phys. Rev. Lett. **69**, 2881 (1992).

[12] S. Wiesner, "Conjugate coding," Sigact News **15**, 78 (1983).

[13] C. H. Bennett and G. Brassard, "Quantum cryptography: Public key distribution and coin tossing," in *Proceedings of the IEEE International Conference on Computers, Systems, and Signal Processing* (Bangalore, 1984) p. 175.

[14] A. K. Ekert, "Quantum cryptography based on Bells theorem," Phys. Rev. Lett. **67**, 661 (1991).

[15] R. P. Feynman, "Simulating physics with computers," International Journal of Theoretical Physics **21**, 467 (1982).

[16] David Deutsch, "Quantum theory, the Church-Turing principle and the universal quantum computer," Proceedings of the Royal Society A **400**, 97 (1985).

[17] M.A. Nielsen and I.L. Chuang, *Quantum Computation and Quantum Information* (Cambridge University Press, Cambridge, UK, 2000).

[18] L. Yu, R. B. Griffiths, and S. M. Cohen, "Efficient implementation of bipartite nonlocal unitary gates using prior entanglement and classical communication," Phys. Rev. A **81**, 062315 (2010).

[19] S. M. Cohen, "Optimizing local protocols for implementing bipartite nonlocal unitary gates using prior entanglement and classical communication," Phys. Rev. A **81**, 062316 (2010).

[20] Dan Stahlke and Robert B. Griffiths, "Entanglement requirements for implementing bipartite unitary operations," Phys. Rev. A **84**, 032316 (2011).

[21] R. Horodecki, P. Horodecki, M. Horodecki, and K. Horodecki, "Quantum entanglement," Rev. Mod. Phys. 81, 865 **81**, 865 (2009).

[22] J. Åberg, "Quantifying superposition," arXiv:quantph , 0612146 (2006).

[23] T. Baumgratz, M. Cramer, and M. B. Plenio, "Quantifying coherence," Phys. Rev. Lett. **113**, 140401 (2014).

[24] C. H. Bennett, D. P. DiVincenzo, J. A. Smolin, and W. K. Wootters, "Mixed-state entanglement and quantum error correction," Phys. Rev. A **54**, 3824 (1996).

[25] C. H. Bennett, G. Brassard, S. Popescu, B. Schumacher, J. A. Smolin, and W. K. Wootters, "Purification of noisy entanglement and faithful teleportation via noisy channels," Phys. Rev. Lett. **76**, 722 (1996).

[26] Robert Raussendorf and Hans J. Briegel, "A one-way quantum computer," Phys. Rev. Lett. **86**, 5188 (2001).

[27] J. I. Cirac, A. K. Ekert, S. F. Huelga, and C. Macchiavello, "Distributed quantum computation over noisy channels," Phys. Rev. A **59**, 4249 (1999).

[28] V. Gheorghiu, L. Yu, and S. M. Cohen, "Local cloning of entangled states," Phys. Rev. A **82**, 022313 (2010).

[29] Mark Hillery, Vladimir Buzek, and Andre Berthiaume, "Quantum secret sharing," Phys. Rev. A **59**, 1829 (1999).

[30] Vlad Gheorghiu and Barry C. Sanders, "Accessing quantum secrets via local operations and classical communication," Phys. Rev. A **88**, 022340 (2013).

[31] D. P. DiVincenzo, T. Mor, P. W. Shor, J. A. Smolin, and B. M. Terhal, "Unextendible product bases, uncompletable product bases, and bound entanglement," Commun. Math. Phys. **238**, 379 (2003).

[32] J. Niset and N. J. Cerf, "Multipartite nonlocality without entanglement in many dimensions," Phys. Rev. A **74**, 052103 (2006).

[33] S. M. Cohen, "Local distinguishability with preservation of entanglement," Phys. Rev. A **75**, 052313 (2007).

[34] S. M. Cohen, "When a quantum measurement can be implemented locally, and when it cannot," Phys. Rev. A **84**, 052322 (2011).

[35] S. M. Cohen, "Local quantum protocols for separable measurements with many parties," Phys. Rev. A **87**, 052135 (2013).

[36] Matthias Kleinmann, Hermann Kampermann, and Dagmar Bruß, "Asymptotically perfect discrimination in the local-operation-and-classical-communication paradigm," Phys. Rev. A **84**, 042326 (2011).

[37] Andrew M. Childs, Debbie Leung, Laura Mančinska, and Maris Ozols, "A framework for bounding nonlocality of state discrimination," Communications in Mathematical Physics **323**, 1121–1153 (2013).

[38] Eric Chitambar and Min-Hsiu Hsieh, "Revisiting the optimal detection of quantum information," Phys. Rev. A **88**, 020302(R) (2013).

[39] Eric Chitambar, Runyao Duan, and Min-Hsiu Hsieh, "When do local operations and classical communication suffice for two-qubit state discrimination?" IEEE Trans. Inf. Theory **60**, 1549 (2014).

[40] Honghao Fu, Debbie Leung, and Laura Mančinska, "When the asymptotic limit offers no advantage in the local-operations-and-classical-communication paradigm," Phys. Rev. A **89**, 052310 (2014).

[41] Eric Chitambar and Min-Hsiu Hsieh, "Asymptotic state discrimination and a strict hierarchy in distinguishability norms," J. Math. Phys. **55**, 112204 (2014).

[42] B. Fortescue and H.-K. Lo, "Random bipartite entanglement from W and W-like states," Phys. Rev. Lett. **98**, 260501 (2007).

[43] Eric Chitambar, "Local quantum transformations requiring infinite rounds of classical communication," Phys. Rev. Lett. **107**, 190502 (2011).

[44] Eric Chitambar, Wei Cui, and Hoi-Kwong Lo, "Increasing entanglement monotones by separable operations," Phys. Rev. Lett. **108**, 240504 (2012).

[45] Eric Chitambar, Wei Cui, and Hoi-Kwong Lo, "Entanglement monotones for W-type states," Phys. Rev. A **85**,

315 062316 (2012).

[46] Eric Chitambar, Debbie Leung, Laura Mančinska, Maris Ozols, and Andreas Winter, "Everything you always wanted to know about LOCC (but were afraid to ask)," Commun. Math. Phys. **328**, 303 (2014).

[47] S. M. Cohen, "Devising local protocols for multipartite quantum measurements," Phys. Rev. A **93**, 042334 (2016).

[48] S. M. Cohen, "Structure of local quantum operations and classical communication: Finite versus infinite rounds," Phys. Rev. A **91**, 042106 (2015).

[49] William H. Press, Saul A. Teukolsky, William T. Vetterling, and Brian P. Flannery, *Numerical Recipes in Fortran 77*, Vol. 1 (Cambridge University Press, Cambridge, 1996).

[50] See the Supplementary Materials.

[51] See p. 25 in [17].

[52] C. H. Bennett, D. P. DiVincenzo, C. A. Fuchs, T. Mor, E. Rains, P. W. Shor, J. A. Smolin, and W. K. Wootters, "Quantum nonlocality without entanglement," Phys. Rev. A **59**, 1070 (1999).

[53] S. M. Cohen, "Extended necessary condition for local operations and classical communication: Tight bound for all measurements," Phys. Rev. A **91**, 062125 (2015).

[54] Lieven Vandenberghe and Stephen Boyd, "Semidefinite programming," Siam Review **38**, 49 (1996).