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Emergent Conformal Symmetry and Geometric Transport properties of Quantum Hall States on Singular Surfaces

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We study quantum Hall states on surfaces with conical singularities. We show that the electronic fluid at the cone tip possesses an intrinsic angular momentum, which is due solely to the gravitational anomaly. We also show that quantum Hall states behave as conformal primaries near singular points, with a conformal dimension equal to the angular momentum. Finally, we argue that the gravitational anomaly and conformal dimension determine the fine structure of the electronic density at the conical point. The singularities emerge as quasi-particles with spin and exchange statistics arising from adiabatically braiding conical singularities. Thus, the gravitational anomaly, which appears as a finite size correction on smooth surfaces, dominates geometric transport on singular surfaces.

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Introduction In order to fully characterize the quantum Hall (QH) effect one must understand not only electromagnetic response, but also geometric response, that is the response of a QH state to varying spatial geometry [1–13]. Traditionally, the former has been more accessible and has thus received greater attention, indeed even giving the effect its name. However, with recent experimental advances in synthetic condensed matter systems and the observation of the QHE in lattice systems such as graphene, there is a real possibility of probing geometric response in the near future. The characteristics probed by the geometric response include the anomalous or odd viscosity, first proposed in [1], and the *gravitational anomaly* [6, 7, 9], a property only recently appreciated as being central to understanding QH physics.

Since the works of Laughlin [14, 15], it has been understood that magnetic field singularities (i.e. flux tubes) probe essential features of quantum Hall states. The response of QH states to flux tubes has two facets: (i) the quantized Hall conductance is observed as the charge transfer resulting from an adiabatic change of magnetic flux threading a punctured disk [14]; (ii) magnetic singularities can be seen as point-like coherent states, or quasi-particles, which transport a fractional charge [15] and posses a fractional spin, and statistics [16].

In this paper we show that in a similar manner further subtle properties of QH states are revealed by *geometric* singularities, overlooked in earlier studies. Geometric singularities are point-like concentrations of curvature, also known as conical singularities. The curvature could be extrinsic when electrons are placed on a curved surface, or intrinsic, created by various means such as mechanical stress of the material, an optical and acoustic environment, etc.

We will show that adiabatically threading a curvature flux through a punctured disk leads to charge and momentum transport. This is the *geometric transport*, which is characterized by transport coefficients that are quantized on QH plateaus in addition to the Hall conductance. Whereas Hall conductance arises from integer cohomology, the geometric transport arises from an adiabatic connection with rational cohomology [1–3, 13]. They and the Hall conductance are independent characteristics of QH states.

Similar to the relation of magnetic singularities to quasi-holes, the geometric singularities can be interpreted as coherent states, making them the geometric analogue to Laughlin quasi-holes. We compute their charge, spin and exchange statistics.

Our arguments stem from the observation that a QH state near the geometric singularity possesses an emergent conformal symmetry. In general, QH-states do not show conformal symmetry. They feature a scale - the magnetic length - and do not transform conformally, as two-point functions of the density exhibit exponential decay. However, the states appear to be conformal in the vicinity of a singularity. The emergent conformal symmetry and the universal physical properties it governs are new features of QHE which we address in this paper.

Geometric transport [1–5, 7–9, 11, 13], driven by an adiabatic change of the spatial geometry, is a fundamental probe of quantum liquids with topological characterization, complementary to the more familiar electromagnetic response. The singular geometry highlights the geometric properties of the state and serves as an ideal setting to probe these properties.

The gravitational anomaly is central to understanding the geometry of topological states [6–8, 10–13, 17], encoding the geometric characterization of states in the *central charge* c_H. We say more about this fundamental kinetic coefficient below.

On a smooth surface the gravitational anomaly is typically overshadowed by the dominant electromagnetic properties. We demonstrate that conical singularities bring geometric transport to the fore. There, the gravitational anomaly is a dominant effect with measurable consequences. We show that a small fluid parcel near the singularity spins with an *intensive* angular momentum, independent of the parcel volume and proportional to central charge. Moreover, near the singularity, the state is a conformal primary of the conformal field theory with the central charge $-c_H$. Its conformal dimension is equal to the angular momentum in units of $\hbar$.

Conical singularities are not as exotic as they may seem, and occur naturally in several experimental settings. Disclination defects in a regular lattice can be described by metrics with conical singularities [18], and occur generically in graphene [19]. In a recent experiment, synthetic photonic Landau levels on a cone were created in an optical resonator [20].

A conical singularity of order $\alpha < 1$ is an isolated point ξ_0 on the surface with a concentration of curvature

$$R(\xi) = R_0(\xi) + 4\pi\alpha\delta(\xi - \xi_0), \tag{1}$$

where R_0 is the background curvature, a smooth function describing the curvature away from the singularity. Locally, if $\alpha > 0$ the singularity is equivalent to an embedded cone with the apex angle $2\arcsin\gamma$, where $2\pi\gamma = 2\pi(1-\alpha)$ is the cone angle (see Fig.2), and $2\pi\alpha$ is the deficit. There is no isometric embedding for $\alpha < 0$, which requires glueing multiple surfaces to obtain an *excess* rather than a deficit angle.

An especially interesting case occurs when γ or $1/\gamma$ is an integer. In this case the surface is also an orbifold, a surface quotiented by a discrete group of automorphisms. Then the conical singularities arise as fixed points of the group action [21].

Most of the formulas below are valid regardless of the sign of α in (1), although braiding of singularities on orbifolds is more involved (see [22, 23] for a similar issue in the context of CFT). We do not address it here.

To emphasize the difference between geometric and magnetic singularities we consider both simultaneously: a magnetic flux a threaded through the conical singularity α

$$eB(\xi) = eB_0(\xi) - 2\pi\hbar\, a\delta(\xi - \xi_0), \tag{2}$$

where $B_0 > 0$ is a smooth background magnetic field.

We express the results through fundamental geometric transport coefficients introduced in Ref.[13]. They can be defined through linear response in the planar geometry to a smooth inhomogeneous magnetic field and curvature. In such background the density and momentum of the ground state read

$$\bar\rho = \frac{\nu}{2\pi\hbar}eB + \frac{\mu_H}{4\pi}R, \quad P = \nabla\times\left(-\frac{\mu_H}{4\pi\hbar}eB + \frac{c_H}{96\pi}R\right), \tag{3}$$

with the curvature (1) and magnetic field (2) having $a = \alpha = 0$. For the j-spin Laughlin states with filling fraction ν the coefficients are [24]

$$c_H = 1 - 12(\mu_H^2/\nu), \quad \mu_H = \frac{1}{2}(1 - 2j\nu), \tag{4}$$

Lastly, before listing our main results, we comment on the inclusion of spin j. For the definition of spin see [8] and eq. (15) in the text. As discussed in [8, 13, 17] Laughlin states are characterized not only by the filling fraction but also by the spin. Spin does not enter electromagnetic transport. Nor does it enter local bulk correlation functions, such as the static structure factor. On a closed surface spin is coupled to the curvature of the surface and reflects an ambiguity of lifting the system from the plane to a curved surface. To the best of our knowledge, there is no experimental or numerical evidence that determines the spin in QH materials, nor are there any arguments that $j = 0$, as it silently assumed in earlier papers. For this reason, we keep spin as a parameter since it affects the physics of the QHE. For example, at the filling $\nu = 1/3$, the central charge vanishes at $j = 1$, and $j = 2$. The central charge equals 1 if $j = \frac{1}{2\nu}$, and equals -2 if $\nu = 1$ and $j = 0$ or 1.

Main results *a. Conformal dimensions.* In [7, 17] (see also [25]) it was shown that the magnetic singularity (2) is a conformal primary with the dimension

$$h_a = \frac{1}{2}a\left(2\mu_H - a\nu\right), \tag{5}$$

In this paper, we extend this result and show that the geometric singularity is also a conformal primary. In this case, its dimension is controlled solely the gravitational anomaly

$$\Delta_\alpha = \frac{c_H}{24}(\gamma^{-1} - \gamma), \quad \gamma = 1 - \alpha. \tag{6}$$

Formula (6) is familiar in conformal field theory: $-\Delta_\alpha$ (mind the opposite sign!) is the dimension of a vertex operator of a branch point in CFT [22, 23]. The same formula enters the finite size correction to the free energy of critical systems on a conical surface [26] and enters the formula for the spectral determinant of the Laplace operator (e.g., [27, 28]). These are not coincidences. In the neighborhood of a singularity, QH states and CFT share the same mathematics, but are by no means identical: the conformal dimension of QH states is opposite to that in CFT with the central charge given by (4).

b. Gyration and spin The conformal dimension determines transport near the singularity. A small piece of the fluid gyrates around the apex with an intensive angular momentum equal to the conformal dimension (6)

$$\mathrm{L}_\alpha = \hbar\Delta_\alpha. \tag{7}$$

A reason for this is that the angular momentum of the gyrating fluid gives the spin of the singularity. Since the state is holomorphic, its spin is identical to the dimension.

The angular momentum of a combined magnetic and geometric singularity is additive

$$\mathrm{L}_{\alpha,a} = \hbar\left(\frac{1}{\gamma}h_a + \Delta_\alpha\right). \tag{8}$$

c. Braiding singularities Just like Laughlin quasi-holes, conical singularities can be braided. Braiding two quasi-holes with charges a_1 and a_2 yields the phase $\Phi_{12} =$

$\pi(\nu a_1 a_2)$. This result has been known since the early days of the QHE [16] and is referred to as exchange statistics.

Braiding conical singularities is more involved. We show that the braiding phase of two cones of order α_1 and α_2 are determined exclusively by the central charge

$$\Phi_{12} = -\pi \frac{c_H}{24} \alpha_1 \alpha_2 \left(\frac{1}{\gamma_1} + \frac{1}{\gamma_2} \right) = \pi \left(\alpha_2 \Delta_{\alpha_1} + \alpha_1 \Delta_{\alpha_2} \right) + \pi \frac{c_H}{12} \alpha_1 \alpha_2. \quad (9)$$

The first two terms in (9) are the phase acquired by a particle with spin Δ_{α_1} (or Δ_{α_2}) going half way around a conical singularity with deficit angle $2\pi\alpha_2$ (or α_1). The last term $\frac{c_H}{12}\alpha_1\alpha_2$ is the exchange statistics. On an orbifold, where either γ or $1/\gamma$ is an integer, the phase for identical cones is $\Phi_{12} = \pi \frac{c_H}{12} \left(\sqrt{n} - 1/\sqrt{n}\right)^2$. It appears rational, even in the case of the integer QHE.

The formulae (6-9) are our main results: the braiding statistics of singularities and the angular momentum of the electronic fluid around a cone are solely due to the gravitational anomaly. Further results on transport and the fine structure of the density near a singularity are described below.

d. Moment of inertia The conformal dimension can be read off from the electronic density profile $\rho(r)$ at the singularity. Near the cone point, the density changes abruptly on the scale of the magnetic length and is singular in the limit of vanishing magnetic length. We plot $\rho(r)$ for the integer case in Fig. 1 using exact formulas of Supplemental Material [29]. It is thus properly characterized by the moments

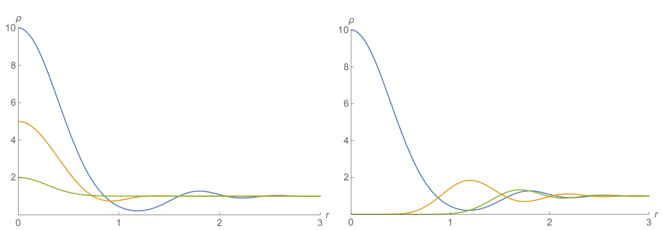

Figure 1. Density of the integer QH state on a cone. Left: $\gamma = 1/10$ (blue), 1/5 (yellow), 1/2 (green) with spin $j = 0$. Right: $\gamma = 1/10$, with spin $j = 0$ (blue), 1/2 (yellow), 1 (green).

$$m_{2n} = \int (r^2/2l^2)^n (\langle \rho(r) \rangle - \rho_\infty) dV. \quad (10)$$

Here, $\rho_\infty = \nu(e/h)B_0$ is the mean density away from the singularity, $l = \sqrt{\hbar/(eB_0)}$ is the magnetic length, r is the Euclidean distance to the singularity, and $dV = 2\pi\gamma r dr$ is the volume element.

The first moment, the 'charge' m_0, follows from the generalized Středa formula [30] (see also [8])- the number of particles in a patch of area dV is saturated by $\bar{\rho} dV$, where $\bar{\rho}$ is given by (3). Hence,

$$m_0 = \int \left(\bar{\rho}(r) - \rho_\infty \right) dV = -\nu a + \mu_H \alpha. \quad (11)$$

The cone accumulates electrons if $\mu_H \alpha > 0$.

This result (for $j = 0$) is known (see, e.g., [9, 31, 32]) and there is a recent claim that the "charge" of the cone has been observed experimentally [20]. However, the gravitational anomaly does not enter here. It emerges in the *moment of inertia* m_2. We will see that

$$m_2 = (1 - j) m_0 + \gamma^{-1} h_a + \Delta_\alpha, \quad (12)$$

where h_a and Δ_α are the dimensions (5) and (6). We check this formula against the integer QH effect in the Supplemental Material.

The measurement of the second moment would constitute an observation of the gravitational anomaly. It is experimentally accessible within the framework of [20].

This relation between the moment of inertia (12) and the angular momentum (7) is expected. In QH states, the positions of particles determine their velocity. Consequently, the density determines the momentum of the flow [32–34]. In the next section we recall its origin. The relation reads

$$\nabla \times \mathrm{P} = -eB_0(\rho - \bar{\rho}) + \frac{\hbar}{2}(1 - j)\Delta\rho. \quad (13)$$

Here $\nabla \times = \epsilon_{ij}\nabla_j$, where ∇_j is a covariant derivative, Δ is the Laplace-Beltrami operator, and $\bar{\rho}$ is given by (3). The formula for the charge of the cone (11) is a consequence of (13). Away from the singularity the momentum rapidly vanishes. As a result the integral $\int (\nabla \times \mathrm{P})\, dV$ vanishes. Then (13) yields (11).

With the help of this formula we express the angular momentum $\mathrm{L} = \int (\xi \times \mathrm{P})\, dV$ in terms of the density

$$\mathrm{L} = (eB_0) \int \frac{r^2}{2} (\rho - \bar{\rho}) dV + \hbar (j - 1) \int \rho dV. \quad (14)$$

The first term in this formula is the diamagnetic effect of a fluid gyrating in a magnetic field, while the second term is the paramagnetic contribution. The angular momentum of the singularities is obtained from the difference between L at finite α, and L for $\alpha = 0$. Since ρ_∞ is the density for $\alpha = 0$, (10) implies that

$$L_{\alpha,a} = \hbar (m_2 + (j - 1) m_0).$$

Then (12) prompts (8). It remains to compute (5,6).

e. Transport at the singularity. Since the work of Laughlin [14] it was known that an adiabatic change of the magnetic flux $a(t)$ in (2) threading a disk causes a radial electric current flowing outward $\mathrm{I} = -\nu e \dot{a}$.

Adiabatic change of the cone angle $\alpha(t)$ also yields a current. It follows from (11) that the outward current is $\mathrm{I} = e\dot{m}_0$.

More interestingly, both evolving flux and the cone angle accelerate the gyration of the fluid, and produce a torque, the rate of change of the angular momentum $\dot{\mathrm{L}}$. From (7) it follows that the torque is proportional to the rate of change of the conformal dimension. We collect the formulae for electric and geometric transport

$$\text{e-transport: current} = -e\nu\dot{a}, \qquad \text{torque} = \hbar\dot{h}_a,$$
$$\text{g-transport: current} = e\mu_H\dot{\alpha}, \qquad \text{torque} = \hbar\dot{\Delta}_\alpha.$$

These formulas are the geometric transport in a nutshell.

QH states on a Riemann surface Before turning to singular surfaces, we recall some formulas about Laughlin states on a Riemann surface [7, 12].

The most compact form of the state appears in locally chosen complex coordinates $(z, \bar{z})$, where the metric is conformal $ds^2 = e^\phi |dz|^2$. In these coordinates the genus-0 unnormalized state for integer inverse filling ν^{-1} reads

$$\Psi = \prod_{1\leq i<k}^{N} (z_i - z_k)^\beta \exp \sum_{i=1}^{N} \frac{1}{2} \left[Q(z_i, \bar{z}_i) - j\phi(z_i, \bar{z}_i)\right], \quad (15)$$

where Q is the magnetic potential defined by $-\hbar\Delta Q = 2eB$ and $\beta = \nu^{-1}$.

While the wave function (15) explicitly depends on the choice of coordinates, the normalization factor

$$\mathcal{Z}[Q, \phi] = \int |\Psi|^2 \prod_i \exp\left[\phi(z_i, \bar{z}_i)\right] d^2 z_i, \qquad (16)$$

does not. It is an invariant functional depending on the geometry of the surface, and in particular on the positions and orders of the singularities. Eq.(16) encodes the correlations and the transport properties of the state and is therefore referred to as the generating functional. For example, a variation of $\mathcal{Z}$ over the magnetic potential Q at a fixed conformal factor ϕ is the particle density

$$\langle \rho \rangle \, dV = (\delta \log \mathcal{Z} / \delta Q) \, d^2 z.$$

In [13] it was shown that the variation over the metric at a fixed volume is the angular momentum

$$\mathrm{L} = -\hbar \int (\delta \log \mathcal{Z} / \delta \phi) \, d^2 z. \qquad (17)$$

Now we can obtain the relation (14). It follows from the observation that the magnetic potential and the conformal factor appear in (15,16) almost on equal footing, besides that under a variation over the conformal factor, the magnetic potential varies as $-\hbar\Delta\delta Q = 2\delta\phi(eB)$ [35].

QH state on a cone A surface has a conical singularity of order α if in the neighborhood of the conical point z_0 the conformal factor behaves as

$$\phi \sim -\alpha \log |z - z_0|^2. \qquad (18)$$

Locally a cone is thought of as a wedge of a plane with the deficit angle $2\pi\alpha$, whose sides are isometrically glued together (see Fig. 2).

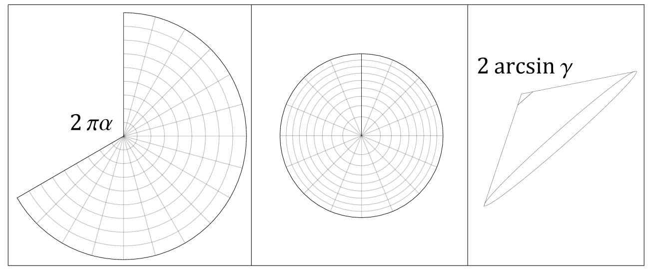

Figure 2. A cone with deficit angle $2\pi\alpha$ in ξ coordinates with restricted argument and Euclidian metric (Left), mapped to a plane z by (19) with a metric (20) (Middle), embedded in 3D (Right)

We denote the complex coordinate on the plane as ξ and the cone angle $2\pi\gamma = 2\pi(1-\alpha)$. The wedge is a domain $0 \leq \arg \xi < 2\pi\gamma$ with a Euclidean metric $ds^2 = |d\xi|^2$. A singular conformal map

$$z \to \xi(z) = \gamma^{-1}(z - z_0)^\gamma \qquad (19)$$

maps the wedge to the entire complex plane. The map introduces the complex coordinates $(z, \bar{z})$ where the metric is conformal

$$ds^2 = |z - z_0|^{-2\alpha} |dz|^2. \qquad (20)$$

Specifically, in the neighborhood of the conical singularity the conformal factor in (15) behaves as (18). A singularity in the wave function (15) can be interpreted as an insertion of the 'vertex operator' at the marked point of the surface. Then the generating functional $\mathcal{Z}_\alpha$ is the expectation value of this operator. We will show that this operator is a conformal primary. This means that under a dilatation, the functional transforms as $-\delta \log \mathcal{Z}_\alpha = \Delta_\alpha \delta\phi$, where Δ_α is the conformal dimension. Eq.(17) identifies the conformal dimension with the angular momentum (7).

Conformal Ward identity We obtain the dimensions (5,6) and the statistics (9) by employing the conformal Ward identity (CWI), a framework developed in [7, 36]. We sketch the major steps in the Supplemental Material. The CWI connects the momentum P to the conformal stress tensor

$$T = \frac{\nu}{2} (\partial_z \varphi)^2 - \mu_H \partial_z^2 \varphi, \qquad (21)$$

where $\varphi = -\beta \sum_i \log |z - z_i|^2 - Q$. The CWI reads

$$\frac{1}{\hbar} \int \frac{\mathrm{i}\langle P_{z'} \rangle - \frac{\mu_H}{2\pi} \partial_{z'}(eB_0)}{z - z'} dV_{z'} = \langle T \rangle. \qquad (22)$$

A Ward identity of this kind is commonly used in CFT.

The angular momentum follows from the Ward identity for dilatations, which is contained in (22). Multiplying (22) by $\frac{z dz}{2\pi i}$ and integrating over the boundary of a singular patch yields $\hbar^{-1} \int \mathrm{Im}(z \langle P \rangle) dV = \mathrm{res}(z \langle T \rangle)$. The

LHS of this equation is proportional to the angular momentum since $\int \mathrm{Im}(z\langle P\rangle)dV = -\gamma\int(\xi\times\langle P\rangle)\,dV = -\gamma L$, so that

$$\gamma L = \hbar\,\mathrm{res}(z\langle T\rangle). \tag{23}$$

Gravitational Anomaly In the Supplemental Material

we show that to leading order in N, the conformal stress tensor equals

$$\langle T\rangle = \frac{\nu}{2}\left(\langle\partial_z\varphi\rangle\right)^2 - \mu_H\partial_z^2\langle\varphi\rangle + \frac{1}{12}\mathcal{S}[\phi], \tag{24}$$

where $\mathcal{S}[\phi] = -\frac{1}{2}(\partial_z\phi)^2 + \partial_z^2\phi$ is the Schwarzian of the metric. The last term is the effect of the gravitational anomaly [7].

Geometric singularity We evaluate the stress tensor in the leading approximation $\rho\approx\bar\rho$. In this approximation the first two terms in (24) are proportional to the Schwarzian which effectively change the coefficient of the anomalous (last) term in (24)

$$\langle T\rangle = \frac{c_H}{12}\mathcal{S}[\phi]. \tag{25}$$

We compute the singular part of the stress tensor by evaluating the Schwarzian on the singular metric (18), which is equal to the Schwarzian derivative of the singular conformal map (19), $\mathcal{S}[\phi] = \frac{\xi'''}{\xi'} - \frac{3}{2}\left(\frac{\xi''}{\xi'}\right)^2 = \frac{\alpha(2-\alpha)}{2z^2}$. Thus,

$$\langle T\rangle = \frac{c_H}{24}\frac{\alpha(2-\alpha)}{z^2}. \tag{26}$$

Using (23), we arrive at our main result (7).

Magnetic singularity The stress tensor receives the contribution from the magnetic potential of the flux tube $Q_a = 2a\log|z|$

$$\langle T\rangle = -\frac{\nu}{2}(\partial_z Q_a)^2 - \mu_H\partial_z^2 Q_a = \frac{h_a}{z^2}, \tag{27}$$

where h_a is the conformal dimension (5).

When the flux tube sits on top of a conical singularity, the stress tensor is the sum of (27) and (26) $\langle T\rangle = (\gamma\Delta_\alpha + h_a)z^{-2}$. This implies the relation (8).

Exchange statistics Now consider adiabatically exchanging two singularities. The state will acquire a phase. Since the state is a holomorphic function of singularity position, its holonomy is encoded in the normalization factor [37]. The exchange statistics is then $\Phi_{12} = (\mathrm{i}/2)\int_C d\log\mathcal{Z}$, where contour C traces the adiabatic path of the singularity. The adiabatic connection $d\log\mathcal{Z}$ is a differential in the configuration space of singularities, and has a pole when two singularities coincide. Therefore, the exchange phase is the residue of the pole $\Phi_{12} = -\pi\,\mathrm{res}[d\log\mathcal{Z}]$.

For conical singularities, the residue arises entirely from the gravitational anomaly (see Supplemental Materials). Explicitly

$$d\log\mathcal{Z} = \frac{c_H}{24}\alpha_1\alpha_2\left(\frac{1}{\gamma_1}+\frac{1}{\gamma_2}\right)\frac{dz_{12}}{z_{12}}, \tag{28}$$

where z_{12} is the difference in position of two singularities in the z-plane. It prompts the formula (9) for the exchange statistics.

Spectral determinant We comment that the formula (28) describes the holonomy of the spectral determinant of the Laplacian (see, e.g., [28]).

$$d\log\mathcal{Z} = \frac{c_H}{2}d\log\left[\mathrm{Det}(-\Delta)\right]$$

This is consistent with the result of Refs.[8, 13, 17] where it was found that the generating functional $\mathcal{Z}$ is proportional to $[\mathrm{Det}(-\Delta)]^{c_H/2}$.

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