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D. Belitz and T. R. Kirkpatrick

Phys. Rev. Lett. **117**, 236803 — Published 2 December 2016

DOI: [10.1103/PhysRevLett.117.236803](https://doi.org/10.1103/PhysRevLett.117.236803)

Scaling Theory of a Compressibility-Driven Metal-Insulator Transition in a Two-Dimensional Electron Fluid

D. Belitz¹ and T.R. Kirkpatrick²

¹*Department of Physics, Institute of Theoretical Science,
and Materials Science Institute, University of Oregon, Eugene, OR*

²*Institute for Physical Science and Technology, University of Maryland, College Park, MD 20742*
(Dated: October 14, 2016)

We present a scaling description of a metal-insulator transition in two-dimensional electron systems that is driven by a vanishing compressibility rather than a vanishing diffusion coefficient. A small set of basic assumptions leads to a consistent theoretical framework that is compatible with existing transport and compressibility measurements, and allows to make predictions for other observables. We also discuss connections between these ideas and other theories of transitions to an incompressible quantum fluid.

PACS numbers: 71.30.+h, 05.30.Rt, 71.27.+a

The metal-insulator transition (MIT) observed in two-dimensional (2D) electron (and hole) systems remains very incompletely understood [1–3]. Observationally, experiments on high-mobility Si MOSFETs and other 2D systems have shown, increasingly convincingly, that a quantum phase transition (QPT) from an insulator to a metal occurs with increasing carrier concentration [3]. Theoretically, various explanations have been proposed that include no true transition, percolation, and modifications of localization theory [1, 2]. Interesting recent developments include the realization that the transition in low-mobility/strong-disorder systems may be in a different universality class than the one in high-mobility/weak-disorder ones [3], and the observation that the thermopower in the metallic phase shows critical behavior at a critical value n_c of the electron density n that coincides, at least in high-mobility/weak-disorder systems, with the critical density for the MIT [4]. The latter observation was recently confirmed by a transport study of the insulating phase [5]. Another development are measurements of the thermodynamic density susceptibility $\partial n/\partial\mu$ [6–8], which is related to the electronic compressibility K_T via $\partial n/\partial\mu = n^2 K_T$. These experiments show that K_T vanishes in the insulating phase. Some also find $K_T < 0$ in the metallic phase, an observation that we will come back to below.

In this Letter we focus on weakly disordered systems and construct a scaling description of a 2D MIT that is driven by a vanishing compressibility. We will show that such a description is conceptually consistent and in agreement with existing experiments, and we will make predictions for the behavior of additional observables.

The electrical conductivity σ is given in terms of the diffusion coefficient D and $\partial n/\partial\mu \equiv \chi_n$ by the Einstein relation

$$\sigma = \chi_n D. \quad (1)$$

Our description of the 2D MIT is based on two basic assumptions, viz.,

Assumption 1: The observed MIT is indeed a continuous QPT characterized by a diverging correlation length ξ with critical exponent ν , $\xi \propto r^{-\nu}$, where $r = |n - n_c|/n_c$ is the dimensionless distance from the QPT.

Assumption 2: The diffusion coefficient D is uncritical, at least for weakly disordered systems. This implies that D has its naive scale dimension. Choosing the scale dimension of ξ as $[\xi] = -1$, we have $[D] = -2 + z_D$, with z_D an appropriate dynamical exponent. We note for later reference that there is more than one dynamical exponent. z_D is related to diffusive physics, hence $z_D = 2$ and $[D] = 0$. This assumption is motivated by the observation that the transition is not sensitive to weak disorder [3, 4, 9]. The experimental evidence for a MIT as interpreted by Assumption 1 then implies, via Eq. (1), that χ_n vanishes as the transition is approached from the metallic side and remains zero in the insulator, i.e., the insulator is an incompressible quantum fluid. Assigning a scale dimension $[\mu] = z$ to the chemical potential μ , we have $[\chi_n] = 2 - z$. This is a strong scaling assumption in the sense of the term defined in Ref. [10]: It assumes that $[\chi_n]$ simply follows from the fact that $1/\chi_n$ is an energy times a volume and is not affected by dangerous irrelevant variables. The dynamical exponent z is different from the diffusive z_D above. Its value is *a priori* unknown and only bounded by $z < 2$ (to ensure that χ_n vanishes). We will discuss plausible values of z below.

We note that in standard theories of electron localization, with or without interactions, $[\mu] = d$ in d spatial dimensions, χ_n is uncritical, and the MIT is driven by a vanishing diffusion coefficient [11, 12]. In this sense our Assumption 2 is exactly complementary to localization physics.

The above assumptions immediately lead to homogeneity laws for both χ_n and σ ,

$$\chi_n(r, T) = b^{-(2-z)} F_\chi(r b^{1/\nu}, T b^z), \quad (2)$$

$$\sigma(r, T) = b^{-(2-z)} F_\sigma(r b^{1/\nu}, T b^z). \quad (3)$$

F_χ and F_σ are scaling functions, T is the temperature,

$b > 0$ is an arbitrary scale factor, and we have assumed $[T] = [\mu] = z$. This is the simplest possible dynamical-scaling assumption; we will discuss generalizations below.

We first focus on σ . Equation (3) implies

$$\begin{aligned}\sigma(r, T) &= r^{\nu(2-z)} F_\sigma(1, T/r^{\nu z}) \\ &= T^{(2-z)/z} F_\sigma(r/T^{1/\nu z}, 1) .\end{aligned}\quad (4)$$

Numerous sets of data have been fitted to a scaling law of this form, initially assuming $z = d = 2$, which was motivated by localization physics. Later it was realized that low- T data do not support this assumption. Determining the exponents precisely from scaling plots has proven difficult; current best estimates from transport measurements for high-mobility systems are $\nu(2-z) \approx 1 - 1.5$, $(2-z)/z \approx 1.5$, and $\nu z \approx 1 - 1.7$ [3, 13]. Combining these results we obtain $z \approx 0.8$ and $\nu \approx 0.83 - 2.1$. An independent value for $\nu z \approx 1$ is obtained from the thermopower data [4], which were interpreted by means of a scaling theory in Ref. [14]. Taken together, these data support scaling of the conductivity given by Eq. (4) with

$$z \approx \nu \approx 1 \quad (5)$$

within substantial error bars.

The critical behavior of $\partial n/\partial \mu$ affects the screening of the Coulomb interaction. The wave-number and frequency dependent dielectric function $\epsilon(\mathbf{q}, \omega)$ is [15]

$$\epsilon(\mathbf{q}, \omega) = \epsilon_0 [1 + v(\mathbf{q}) \chi_{\text{sc}}(\mathbf{q}, \omega)] . \quad (6)$$

$v(\mathbf{q}) = 2\pi e^2/\epsilon_0 |\mathbf{q}|$ is the bare 2D Coulomb potential, and ϵ_0 is the static background dielectric constant. The screened potential is

$$v_{\text{sc}}(\mathbf{q}, \omega) = \frac{v(\mathbf{q})}{\epsilon(\mathbf{q}, \omega)/\epsilon_0} = \frac{v(\mathbf{q})}{1 + v(\mathbf{q}) \chi_{\text{sc}}(\mathbf{q}, \omega)} . \quad (7)$$

χ_{sc} is the screened density susceptibility, which at small \mathbf{q} and ω is diffusive:

$$\chi_{\text{sc}}(\mathbf{q}, \omega) = \chi_n iD\mathbf{q}^2/(\omega + iD\mathbf{q}^2) . \quad (8a)$$

In particular,

$$\chi_{\text{sc}}(\mathbf{q} \rightarrow 0, \omega = 0) = \chi_n , \quad (8b)$$

which expresses the compressibility sum rule. The static screened potential thus is

$$v_{\text{sc}}(\mathbf{q}, \omega = 0) = \frac{2\pi e^2/\epsilon_0}{q + \kappa} , \quad (9)$$

with $\kappa = (2\pi e^2/\epsilon_0)\partial n/\partial \mu$ the 2D screening length, which has the same scale dimension as χ_n , $[\kappa] = 2 - z$. If $z = 1$, then κ scales as an inverse length and the static dielectric function, $\epsilon(\mathbf{q}, \omega = 0)/\epsilon_0 = 1 + \kappa/|\mathbf{q}|$, scales as a constant.

As the transition is approached from the metallic phase, $\xi \propto r^{-\nu}$ and $1/\kappa \propto r^{-\nu(2-z)}$ both diverge, and

screening breaks down at the transition. For $z = 1$, $1/\kappa \propto \xi$, whereas the dielectric function is uncritical.

The above conclusions all follow from Assumptions 1 and 2. In order to make predictions about the behavior of other observables we need an additional assumption regarding the dynamics. The simplest possibility is

Assumption 3a: The strong-scaling assumption of Assumption 2 applies to other observables that have the dimensions of a density of states (DOS), including $\partial n/\partial \mu$, the single-particle DOS N , and the specific-heat coefficient γ (but not the spin susceptibility, which is uncritical in our picture of the 2D MIT), and they are all governed by the same dynamical exponent z . That is, apart from the diffusive exponent $z_D = 2$ there is only one dynamical exponent z [16, 17]. Both N and γ then obey scaling laws analogous to Eqs. (2, 3), which leads to

$$\chi_n(r, T) = r^{\nu(2-z)} G_\chi(T/r^{\nu z}) , \quad (10a)$$

$$\gamma(r, T) = r^{\nu(2-z)} G_\gamma(T/r^{\nu z}) , \quad (10b)$$

for χ_n and γ , and

$$N(r, \omega, T) = r^{\nu(2-z)} G_N(\omega/r^{\nu z}, T/r^{\nu z}) \quad (10c)$$

for N . Here $G_\chi(x) = F_\chi(1, x)$, and G_γ and G_N are analogous scaling functions. $N(r, \omega, T)$ is the single-particle DOS, which can be measured by tunneling, as a function of r , T , and the energy distance ω from the Fermi surface. According to Assumption 3a, the exponents ν and z are the same as those that govern the scaling of the conductivity, Eqs. (4, 5).

Also of interest is the thermal expansion coefficient $\alpha_V = -(1/n)(\partial n/\partial T)_p$, and the Grüneisen parameter $\Gamma = \alpha_V/\gamma T$. According to our assumptions, α_V scales the same as χ_n and γ , so at the QPT Γ is diverges as

$$\Gamma(r = 0, T \rightarrow 0) \propto 1/T . \quad (11)$$

The difference between this and the result of Ref. [18], which obtained $\Gamma \propto 1/T^{1/\nu z}$, can be understood as follows. We rewrite α_V as in Ref. [18], $\alpha_V = -(1/V)(\partial S/\partial p)_T$. Γ then scales as the inverse pressure. Since the compressibility is $K_T = (1/n)(\partial n/\partial p)_T$, $\chi_n \equiv \partial n/\partial \mu$ can be written $\partial n/\partial \mu = n(\partial n/\partial p)_T$. At the 2D MIT we have $[p] = [\mu] = [T] = z$, and we again obtain Eq. (11). At the magnetic QPTs considered in Ref. [18], on the other hand, the pressure scales as the control parameter, $[p] = [r] = 1/\nu$, and hence $\Gamma \sim 1/r \sim 1/T^{1/\nu z}$.

In addition to ν there are six other static exponents that describe the r -dependence of various thermodynamic quantities, see Ref. [10]. We use the definitions and notations put forward in that reference, some of which deviate by necessity from the standard ones used for classical phase transitions. The specific-heat exponent $\bar{\alpha}$ is obtained from Eq. (10b) as $\bar{\alpha} = -\nu(2-z)$, in agreement with one of the hyperscaling relations derived in [10]. The exponent α is given in terms of ν and z by

the hyperscaling relation $\alpha = 2 - \nu(2 + z)$. We define β by $\chi_n(r, T = 0) \propto r^\beta$, $\beta = \nu(2 - z)$ [19]. γ then follows from the Essam-Fisher relation $\gamma = 2 - \alpha - 2\beta$, δ is given by the Widom relation $\delta = 1 + \gamma/\beta$, and η by the Fisher relation $\eta = 2 - \gamma/\nu$. With $\nu = z = 1$ this yields $\bar{\alpha} = \alpha = -1$, $\beta = \gamma = \eta = 1$ and $\delta = 2$. α , γ , η , and δ describe the critical behavior of correlation functions that are non-standard and would be hard to measure.

Another set of exponents describes the T -dependence of observables $r = 0$. They are given by $\omega_T = \omega/\nu z$, where ω can be $\bar{\alpha}$, α , β , γ , or ν . With $\nu z = 1$ from the thermopower experiment [4] we have $\omega_T = \omega$.

The exponents s and s_T that define the dependence of the electrical conductivity σ on r and T , respectively, are $s = \nu z s_T = \nu(2 - z)$, see Eq. (4). Note that σ does not obey the strong-scaling assumption that is valid in localization physics and leads to Wegner scaling, i.e., $s = \nu(d - 2)$, see Refs. [10–12]. Rather, D and χ_n in Eq. (1) are governed by separate strong-scaling assumptions.

So far we have assumed that there is only one dynamical exponent z . This is possible, and the experimental data to date are, within the error bars, consistent with this notion. However, in general QPTs are characterized by more than one dynamical critical exponent [10, 12, 20, 21]. For instance, the theory of a continuous 2D Mott transition in Ref. [22] contains two dynamical exponents, $z_1 = 2$ and $z_2 = 1$. The latter governs bosonic degrees of freedom that arise from decoupling the fermions, and the former is due to a gauge field associated with spinon degrees of freedom. Another example of a QPT with two dynamical exponents is the quantum ferromagnetic transition [10, 23]. This leads to an

Assumption 3b: The QPT is characterized by dynamical exponents z_n ($n = 1, 2, \dots$) such that $z_1 > z_2 > \dots$. The specific heat coefficient γ , which is the energy-energy susceptibility, will then be governed by the largest z . That is, γ obeys Eq. (10b) with $z = z_1$,

$$\gamma(r, T) = r^{\nu(2-z_1)} G_\gamma(T/r^{\nu z_1}). \quad (12)$$

For other observables no general statements can be made. Naive scaling suggests that z_1 governs the dynamics of all observables, but in general this is not true since some observables do not couple to the leading dynamical processes or, in an alternative interpretation, a subleading time scale acts as a dangerous irrelevant variable [12]. For instance, in quantum ferromagnets the order parameter (OP) and its derivatives are governed by a subleading time scale [10, 23], and so is the compressibility in Ref. 22. In both of these examples the specific heat is governed by the largest z , as expected.

We conclude with various discussion remarks, organized into four groups.

(1) Nature of the transition:

(i) Our most basic assumption is that the observed 2D MIT is a quantum phase transition from a metal to an

insulator that is an incompressible quantum fluid, which is one of the hallmarks of a transition of Mott type [24]. Another assumption is that the transition is continuous. For our scaling theory we assume, based on experimental evidence, that $\partial n/\partial \mu$ vanishes at the transition; explicit mechanisms for a vanishing $\partial n/\partial \mu$ have been proposed in Refs. [25] and [22].

(ii) We do not assume that there is an OP for this transition, nor do we specify what it is if there is one. All we assume is scaling, which is much more general. An example of a QPT with no OP, at least not a conventional one, is the Anderson transition of noninteracting electrons [26]. For the Mott-type transition discussed in Ref. [22] it is not obvious whether or not an OP exists, yet scaling works and the results agree with the general scaling description of QPTs developed in Ref. [10]. If an OP description of the 2D MIT is possible, then $\partial n/\partial \mu$ is an obvious candidate [19]. Another candidate is the single-particle DOS, which serves as the OP for the Anderson-Mott transition [27, 28]. Interestingly, in this OP theory for the Anderson-Mott transition $\partial n/\partial \mu$ is also critical and scales the same way as the DOS. Our premise is that the same is true for the 2D MIT.

(iii) An important consequence of our assumptions is that the density response remains diffusive as one approaches the critical point from the metallic side, only the prefactor of the diffusion pole vanishes.

(iv) Spin does not enter our discussion, and we therefore expect an in-plane magnetic field, even one strong enough to completely polarize the electrons, to have no qualitative effect on the transition. This is in agreement with the experimental observations [3, 5].

(v) An interesting question is the nature of the metallic phase. If it is a Fermi liquid (FL), then the transition is from a compressible FL to an incompressible quantum fluid, along the lines of Refs. [14, 29]. However, this requires an unknown mechanism by which strong correlations suppress localization in 2D. A candidate for such a mechanism is discussed in point (3) (ii) below.

(2) Exponent values:

(i) The experiments are consistent with $z = 1$, see Eq. (5). If the metallic phase is a FL, then there will be at least one z that is equal to 1, since FLs are characterized by ballistic soft modes whose frequency scales as the wave number [30–32]. However, the metallic phase may well be a “strange metal” rather than a FL: Tuning the electron-electron interaction to zero will turn the metal into an insulator due to weak-localization effects, so it is unlikely to be a FL. The observations of a negative compressibility are also indicative of a non-standard metal.

(ii) If metallic soft modes related to those in a FL are indeed the source of $z = 1$, then disorder will change this exponent to $z = 2$, which is characteristic of the diffusive soft modes in a disordered metal. This suggests that the scale dimension of $\partial n/\partial \mu$ will be smaller ($2 - z = 0$ plus a

likely small anomalous dimension) in disordered samples than in clean ones. However, even in the disordered case there are likely still soft modes with $z = 1$, e.g., the bosonic excitations in Ref. [22]. It is thus possible that there are various degenerate dynamical exponents in the clean case whose degeneracy is lifted by disorder.

(iii) Irrespective of the origin of the observed $z = 1$, the thermopower experiment of Ref. [4] implies $\nu = 1$, provided there either is only one dynamical exponent, or the thermopower depends on a z that is equal to 1. The remaining static exponents then follow from various (hyper)scaling relations, as discussed in the main text. Some of these exponents values are unusual, e.g., $\delta = 2$ is unusually small. This reflects long-range correlations that are effectively built into our scaling scenario.

(iv) A generic mechanism for generating a $z > 1$, which will then determine the critical dynamics of the specific heat, see Eq. (12), is Landau damping, which occurs if bosonic degrees of freedom couple to fermionic ones. This mechanism is very general and thus expected to be widespread; explicit examples include the OP fluctuations in quantum ferromagnets, which have a $z = d$ due to Landau damping [10, 23], and the gauge fluctuations in Ref. [22], which have a $z = 2$ by the same mechanism.

(3) The role of disorder:

(i) Our scaling scenario is motivated by the observed 2D MIT in high-mobility samples, where experiments indicate that the transition is not qualitatively affected by disorder [3, 4]. We point out, however, that these samples are not weakly disordered in any conventional sense. While the mean-free path is long due to the high mobility, the Fermi wavelength $1/k_F$ is equally long due to the low Fermi temperature (typically, a few K [33]). As a result, $k_F \ell$ is not large compared to 1 even in high-mobility samples, which raises the question of what suppresses localization effects. This issue is not understood; a possible resolution is discussed in the following discussion point.

(ii) An important observation is the negative compressibility in the metallic phase [6, 8, 34], which is also found in quantum Hall systems [35], and in a quasi-3D metal [36], and the physical meaning of which is not understood. $K_T < 0$ makes the system mechanically unstable and most likely indicates a spontaneous rearrangement into a spatially inhomogeneous state [37]. This may be an important ingredient for stabilizing a metallic phase in 2D system, as it will introduce a new length scale that cuts off the weak-localization singularities. As a result, the effects of quenched disorder may be much weaker than in other 2D systems. We note that our scaling scenario does not hinge on the nature of the metallic phase, or on what stabilizes it, but merely on its existence.

(4) Suggestions for experiments:

(i) It would be very interesting to measure the tunneling DOS of the same samples that show critical behavior

of the electrical conductivity and the thermopower. It follows from Assumption 3a that the DOS will obey a homogeneity law, Eq. (10c), with the same exponents ν and z as $\partial n / \partial \mu$ and hence σ .

(ii) Of equal interest would be measurements of the specific heat of the same samples. In the presence of multiple dynamical exponents the specific heat depends on the largest one, Eq. (12), so a specific-heat coefficient that obeys Eq. (12) with $z_1 > 1$ would be a clear indication of multiple dynamical exponents.

We thank Myriam Sarachik for discussions. This work was supported by the NSF under Grants No. DMR-1401410 and No. DMR-1401449. Part of this work was performed at the Aspen Center for Physics and supported by the NSF under Grant No. PHY-1066293.

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