Smooth Horizonless Geometries Deep Inside the Black-Hole Regime

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We construct the first family of horizonless supergravity solutions that have the same mass, charges and angular momenta as general supersymmetric rotating D1-D5-P black holes in five dimensions. This family includes solutions with arbitrarily small angular momenta, deep within the regime of quantum numbers and couplings for which a large classical black hole exists. These geometries are well-approximated by the black-hole solution, and in particular exhibit the same near-horizon throat. Deep in this throat, the black-hole singularity is resolved into a smooth cap. We also identify the holographically-dual states in the $\mathcal{N} = (4,4)$ D1-D5 orbifold CFT. Our solutions are among the states counted by the CFT elliptic genus, and provide examples of smooth microstate geometries within the ensemble of supersymmetric black-hole microstates.

1. INTRODUCTION

The black-hole information paradox reveals a profound conflict between Quantum Mechanics and General Relativity [1]. Quantum mechanically, a black hole has an entropy given by the horizon area in Planck units, while in General Relativity the black hole is unique for a given mass, charge and angular momentum. Unitarity is violated because the enormous black-hole entropy is not visible at the black-hole horizon and so the information about the black-hole state cannot be encoded in the Hawking radiation. Thus, unitarity can only be preserved if there is new physics at the scale of the horizon [2]. However, constructing structure at the scale of the horizon is no easy task: The horizon is a null surface, and any classical matter or wave that can carry information will either fall in or dilute very fast.

One of the great successes of string theory has been a precise accounting of the entropy of certain black holes [3], and the identification of the microstates that give rise to this entropy, albeit in a regime of coupling where the classical black-hole solution is not valid. However, this is not enough to solve the information paradox. In order to create the required structure at the horizon, all the typical microstates of the black hole must become horizon-sized bound states that have the same mass and conserved charges as the black hole, and that exist in the same regime of parameters in which the classical black-hole solution is valid. Furthermore, microstates that are describable in supergravity should be horizonless.

For supersymmetric black holes it has been possible to construct large classes of supergravity solutions corresponding to such horizonless bound states, and these are known as "fuzzball" or microstate geometries [4, 5]. These solutions correspond to *some* of the microstates of the black hole, but have limitations, as we now discuss.

The microstate geometries constructed in [6–9], although carrying the same charges and angular momenta as a large black hole, have the following issues: (i) In all examples, these solutions carry an angular momentum that is a large fraction of the maximally allowed value for the black hole; (ii) Their CFT dual is not known and so their role in the ensemble of black-hole microstates remains unclear; (iii) It is not clear whether these configurations are generic and represent typical microstates of a black hole [10], nor whether the states of the black hole will continue to be described by such geometries when the black hole becomes non-extremal.

Another class of microstate geometries relevant for large supersymmetric black holes in five dimensions is discussed in [11–14]. While these solutions have known CFT duals, they also carry macroscopic five-dimensional angular momenta $j, \tilde{j}$.

The purpose of this Letter is to simultaneously resolve the first two issues described above by (i) constructing the first microstate geometries of rotating, supersymmetric D1-D5-P (BMPV) black holes in string theory [15] in which the angular momenta take arbitrary finite values, in particular including arbitrarily small values; and by (ii) identifying the dual CFT states. In doing so we also demonstrate, via an explicit example, that adding momentum to a two-charge solution describing a microstate of a string-size black hole can result in a large-scale, low-curvature supergravity solution.

2. BLACK-HOLE MICROSTATE GEOMETRIES

We work in type IIB string theory on $\mathbb{R}^{4,1}\times S^1 \times \mathcal{M}$, where $\mathcal{M}$ is T^4 or $K3$. We take the size of $\mathcal{M}$ to be microscopic, and that of S^1 to be macroscopic. The S^1 is parameterized by the coordinate y. We wrap n_1 D1-branes on the S^1 and n_5 D5-branes on $S^1\times\mathcal{M}$, and consider momentum charge, P, along the y direction. We work in the low-energy, six-dimensional supergravity theory obtained by reduction on $\mathcal{M}$.

The near-horizon geometry of a six-dimensional rotating, supersymmetric black string with the foregoing charges is S^3 fibered over the extremal BTZ black hole [16], whose metric is:

$$ds^2_{\text{BTZ}} = \ell^2_{\text{AdS}}\Big[\rho^2(-dt^2+dy^2)+\frac{d\rho^2}{\rho^2}+\rho_*^2(dt+dy)^2\Big]. \quad (1)$$

This metric is locally AdS_3 and it asymptotes to the standard AdS_3 form for $\rho\gg\rho_*$. It can be written as a circle of radius ρ_* fibered over AdS_2 in the near-horizon region $\rho\ll\rho_*$ (see, for example, [17]). Dimensional reduction on this circle yields the AdS_2 of the near-horizon BMPV solution. Following the usual abuse of terminology, we will refer to this region as the AdS_2 throat.

The BTZ parameters are related to the supergravity D1, D5, and P charges $Q_{1,5,P}$ and the radial coordinate r (to be used later) via $\rho = r/\sqrt{Q_1Q_5}$ and $\ell^2_{\text{AdS}}=\sqrt{Q_1Q_5}$. The horizon radius, ρ_*, of the BTZ solution (1) determines the onset of the AdS_2 throat (and thus the radius of the fibered S^1) and is given by $\rho_*^2 = Q_{\text{P}}/(Q_1Q_5)$. This value is determined by a competition between the momentum charge that exerts pressure on the geometry, and the D1 and D5 charges that exert tension.

Typical black-hole microstates should be very well-approximated by the black-hole solution until very close to the horizon. This requires a long, large, BTZ-like AdS_2 throat. To obtain such a throat, prior work has used bubbling solutions with multiple Gibbons-Hawking (GH) centers [6, 7]; the moduli space of these solutions includes "scaling" regions [8, 9, 18] in which the GH centers approach each other arbitrarily closely, whereupon the solution develops an arbitrarily long AdS_2 throat. It has been argued that quantum effects set an upper bound on the depth of such throats [9, 19], and a corresponding lower bound on the energy gap, which matches the lowest energy excitations of the (typical sector of the) dual CFT. This suggests that microstate geometries are capable of sampling typical sectors of the dual CFT.

Unfortunately, all the previously-known scaling microstate geometries involve at least three GH centers, whose dual CFT states are currently unknown. The holographic dictionary between supergravity solutions and CFT states has been constructed only for two-centered solutions [20]; we therefore construct new black-hole microstate solutions by adding momentum excitations to a certain two-charge seed solution. We do this using "superstratum" technology [13, 14, 21] to introduce deformations, with specific angular dependence, so as to modify the momentum and the angular momenta of the solution.

A particular sub-class of our deformations has the effect of reducing the angular momenta of the two-charge seed solution, while introducing no additional angular momentum. These deformations therefore allow us to obtain solutions that have arbitrarily small angular momenta and describe microstates of the non-rotating D1-D5-P (Strominger-Vafa) black hole. The solutions have an AdS_2 throat, which becomes longer and longer as the angular momenta $j,\tilde{j}\to 0$, thus classically approximating the non-rotating black hole to arbitrary precision.

3. THE NEW CLASS OF SOLUTIONS

The metric, axion and dilaton of our $\frac{1}{4}$-BPS solutions are determined by four functions, $Z_1, Z_2, Z_4, \mathcal{F}$ and two vector fields β,ω [22]:

$$ds_6^2 = -\frac{2}{\sqrt{\mathcal{P}}}\,(dv+\beta)\big(du+\omega+\tfrac{1}{2}\,\mathcal{F}\,(dv+\beta)\big) \;+\; \sqrt{\mathcal{P}}\,ds_4^2\,, \quad (2)$$

where ds_4^2 is the flat metric on $\mathbb{R}^4$ written in spherical bipolar coordinates,

$$ds_4^2 = \frac{\Sigma\,dr^2}{r^2+a^2}+\Sigma\,d\theta^2+(r^2+a^2)\sin^2\theta\,d\phi^2+r^2\cos^2\theta\,d\psi^2, \quad (3)$$

with $0\le\theta\le\pi/2$ and $0\le\phi,\psi<2\pi$. The coordinates u and v are light-cone variables related to the asymptotic time t and the S^1 coordinate y via:

$$u\equiv(t-y)/\sqrt{2}\,,\quad v\equiv\,(t+y)/\sqrt{2}\,,\quad y\cong y+2\pi R_y\,. \quad (4)$$

The functions Σ and $\mathcal{P}$ are defined by:

$$\Sigma\equiv r^2+a^2\cos^2\theta\,,\quad \mathcal{P}\equiv Z_1\,Z_2-Z_4^2\,, \quad (5)$$

and the dilaton and axion are given by:

$$e^{2\Phi}=Z_1^2\,\mathcal{P}^{-1}\,,\quad C_0 = Z_4Z_1^{-1}\,. \quad (6)$$

The tensor gauge fields are also related to these functions but we will not discuss their explicit form here.

We consider solutions that have a simple v-fibration:

$$\beta \;=\; 2^{-1/2}\,a^2R_y\,\Sigma^{-1}\,(\sin^2\theta\,d\phi-\cos^2\theta\,d\psi)\,. \quad (7)$$

We begin with the background of a maximally-rotating D1-D5 supertube [23, 24] and add deformations that depend upon the angles (v,ϕ,ψ) via the phase dependence:

$$\hat{v}_{k,m,n}\equiv\sqrt{2}\,R_y^{-1}\,(m+n)\,v+(k-m)\phi-m\psi\,, \quad (8)$$

where $k\in\mathbb{Z}_{>0}$ and $m,n\in\mathbb{Z}_{\ge0}$. These fluctuations modify the angular momenta $j,\tilde{j}$ and the momentum number $n_{\text{P}}=p_yR_y$ with p_y the momentum along the y circle. In order to obtain smooth solutions whose holographic duals we can identify, we add a fluctuating mode with strength $b_{k,m,n}$ using the "coiffuring" technique of [12–14, 25]:

$$Z_1 = \frac{Q_1}{\Sigma} + \frac{R_y^2}{2Q_5} b_{k,m,n}^2 \frac{\Delta_{2k,2m,2n}}{\Sigma} \cos \hat{v}_{2k,2m,2n} \,, \tag{9}$$

$$Z_2 = \frac{Q_5}{\Sigma} \,, \quad Z_4 = b_{k,m,n} R_y \frac{\Delta_{k,m,n}}{\Sigma} \cos \hat{v}_{k,m,n} \,, \tag{10}$$

where

$$\Delta_{k,m,n} \equiv a^k \, r^n (r^2 + a^2)^{-(k+n)/2} \cos^m \theta \, \sin^{k-m} \theta \,. \tag{11}$$

This coiffuring ensures that, while the tensor fields depend on $\hat{v}_{k,m,n}$, the metric does not. The remaining parts of the solution are given by

$$\mathcal{F} = b_{k,m,n}^2 \, \mathcal{F}_{k,m,n} \,, \quad \omega = \omega_0 + b_{k,m,n}^2 \, \omega_{k,m,n} \,, \tag{12}$$

where ω_0 is the value that ω takes in the undeformed supertube solution:

$$\omega_0 \; \equiv \; 2^{-1/2} \, a^2 \, R_y \, \Sigma^{-1} \left(\sin^2 \theta \, d\phi + \cos^2 \theta \, d\psi \right) . \tag{13}$$

The general expressions for $\mathcal{F}_{k,m,n}$ and $\omega_{k,m,n}$ are given in Appendix A and we leave the expressions of the tensor gauge fields to a subsequent publication.

Regularity and absence of closed timelike curves (CTCs) requires

$$Q_1 Q_5 / R_y^2 = a^2 + b^2/2 \,, \qquad b^2 = x_{k,m,n} \, b_{k,m,n}^2 \,, \tag{14}$$

with $x_{k,m,n}^{-1} \equiv \binom{k}{m}\binom{k+n-1}{n}$. The conserved charges of the solution are

$$j = \frac{\mathcal{N}}{2} \left(a^2 + \frac{m}{k} b^2 \right) , \qquad \tilde{j} = \frac{\mathcal{N}}{2} a^2, \qquad n_{\rm P} = \frac{\mathcal{N}}{2} \frac{m+n}{k} \, b^2 \tag{15}$$

where $\mathcal{N} \equiv n_1 n_5 R_y^2 / (Q_1 Q_5)$, with n_1, n_5 the numbers of D1 and D5 branes.

Rotating D1-D5-P black holes with regular horizons exist when $n_1 n_5 n_{\rm P} - j^2 > 0$ and this cosmic censorship bound defines the "black-hole regime" for these parameters. Our solutions lie within this bound for

$$\frac{b^2}{a^2} > \frac{k}{n + \sqrt{(k-m+n)(m+n)}} \,. \tag{16}$$

Hence, in this regime of parameters, these solutions correspond to horizonless microstates of large-horizon-area BMPV black holes. They span the whole range of angular momenta that these black holes can have. This is a dramatic improvement over the earlier solutions [8, 9], which only have $j \gtrsim 0.88\sqrt{n_1 n_5 n_{\rm P}}$. The solutions with $m = 0$ are also remarkable because, as $a \to 0$, they give the first family of microstate geometries of the non-rotating D1-D5-P black hole. An explicit example (with $k = 1, m = 0$ and general n) is given by:

$$\begin{aligned} \mathcal{F}_{1,0,n} &= -a^{-2} \left(1 - r^{2n} (r^2 + a^2)^{-n} \right) \\ \omega_{1,0,n} &= 2^{-1/2} \, R_y \, \Sigma^{-1} \left(1 - r^{2n} (r^2 + a^2)^{-n} \right) \sin^2 \theta \, d\phi \,. \end{aligned} \tag{17}$$

One can easily show that the corresponding metrics are regular and have no CTCs. For our more general class of solutions, this proof becomes increasingly complicated, however our construction explicitly removes CTCs in the most dangerous regions (near $r = 0$ and $\theta = 0$ or $\pi/2$), and there is little reason to expect problems elsewhere.

4. THE DUAL CFT STATES

Our geometries are asymptotically $\mathrm{AdS}_3 \times \mathrm{S}^3$ and correspond holographically to 1/2-BPS states in a $(4,4)$ two-dimensional CFT with central charge $c = 6 n_1 n_5 \equiv 6N$. Since these states are supersymmetric, they should have a simple description at the locus in moduli space at which the CFT is realized as the symmetric orbifold $\mathcal{M}^N / S_N$.

The untwisted sector of this theory consists of N copies of the CFT with target space $\mathcal{M}$. The theory also contains twisted sectors, in which the elementary fields have non-trivial periodicities connecting different CFT copies: When k copies are cyclically permuted by the boundary conditions, we call the corresponding state a "strand of winding k". Following the conventions of [26], we denote by $|{+}{+}\rangle_1$ a strand of length 1 in the RR ground state that has $j = \tilde{j} = 1/2$. To describe our states we will also need the twisted-sector RR ground state, $|00\rangle_k$, which has winding k and is a scalar under all symmetries.

In our solutions, the momenta are carried by excitations of the $|00\rangle_k$ strands. These excitations can be described by $(4,4)$ superconformal algebras with central charge $6k$ living on each strand. Denoting the Virasoro generators as L_n and the R-symmetry $SU(2)$ generators as J_n^i, we excite the $|00\rangle_k$ strands with two mutually-commuting, momentum-carrying perturbations: $J_{-1}^+ = (J_{-1}^1 + i J_{-1}^2)$ and $(L_{-1} - J_{-1}^3)$.

The charges of our solutions (15) support the identification of the dual CFT states with coherent superpositions of states of the form:

$$\left(|{+}{+}\rangle_1\right)^{N_1} \left(\frac{(J_{-1}^+)^m}{m!} \frac{(L_{-1} - J_{-1}^3)^n}{n!} |00\rangle_k \right)^{N_{k,m,n}} \,, \tag{18}$$

for all values of N_1 such that $N_1 + k N_{k,m,n} = N$.

To find the exact coefficients of this superposition of states, one can straightforwardly generalize the derivation of [26] to states with $n > 0$. These coefficients are thereby determined in terms of the supergravity parameters a and $b_{k,m,n}$.

From this calculation one finds that the average numbers of $|{+}{+}\rangle_1$ and $|00\rangle_k$ strands are given by $\mathcal{N} a^2$ and $\mathcal{N} b^2/(2k)$ respectively, from which the strand quantum numbers immediately yield the supergravity momentum and angular momenta in (15). It is also possible to compute 3-point correlators between our heavy states and BPS states of low conformal dimension. These correlators depend not only on the average numbers of strands but also on the spread of the coherent superposition, providing an even more stringent check of the identification between the CFT states and the supergravity solutions.

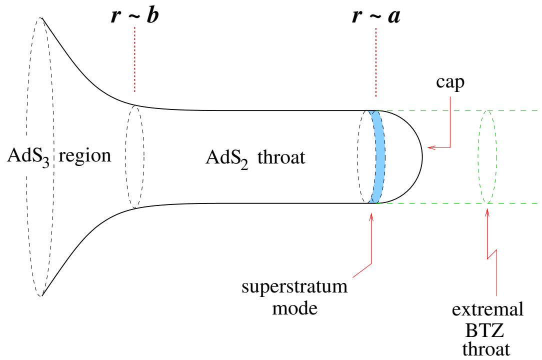

FIG. 1: *Sketch of the superstratum spatial geometry in the r-y plane, compared to the extremal BTZ geometry.*

5. THE STRUCTURE OF THE METRIC

In the AdS/CFT limit, one takes R_y to be the largest scale in the problem, implying that $Q_{\rm P} \ll \sqrt{Q_1 Q_5}$. We further focus on the regime $a^2 \ll Q_{\rm P} = (m+n)b^2/(2k)$, in which the structure of the cap lies deep inside the AdS$_2$ region discussed above.

In the $a \to 0$ limit, the AdS$_2$ throat tends to infinite depth and our solutions tend to the BMPV solution. Furthermore, when $m = 0$, the angular momentum vanishes and the solutions tend to that of the non-rotating D1-D5-P (Strominger-Vafa) black hole.

If, instead, we keep $a^2 \ll Q_{\rm P}$ small but finite, then the leading terms in the metric for $r \gg a$ are those of the corresponding black hole. The AdS$_2$ throat extends in the radial direction for a proper length of order $\ell_{\rm AdS} \log{(Q_{\rm P}/a^2)}$, and the geometry caps off smoothly in the region $r \ll a$, as shown in Fig. 1. In string units, the proper length of the y circle in the AdS$_2$ throat is of order $(g_s n_{\rm P})^{1/2}/N^{1/4}$ when the volume of the compact space $\mathcal{M}$ is of order one. Thus one can easily arrange that the proper length of the y circle in the AdS$_2$ region is large in string units, whereupon the supergravity approximation is valid.

The momentum charge is carried by a superstratum deformation (supergravity wave) concentrated deep inside the AdS$_2$ region. The wave profile is determined by the functions $\Delta_{k,m,n}$ [28]. Inside the support of the wave, the momentum density that stabilized the size of the y-circle quickly dilutes, and the circle starts to shrink until one gets to $r = 0$, where the coiffuring relations guarantee that the geometry caps off smoothly.

Our solutions therefore provide examples, with arbitrary finite angular momenta, of how the horizon of a D1-D5-P black hole can be replaced by a smooth cap. The solutions only differ significantly from the corresponding black hole metric near the cap; the difference is suppressed in the AdS$_2$ throat and further out into the asymptotic AdS$_3$ region. For example, in the $k=1$, $m=0$, general n solution, the leading corrections to the corresponding black hole metric have magnitude (in a local orthonormal frame) of order $\sqrt{n}\, a^2/r^2$ in the AdS$_2$ throat, and of order a^2/r^2 in the asymptotic AdS$_3$ region.

When a is exactly zero, from the dictionary (15) one can see that for any value of j and $n_{\rm P}$ there exists a one-parameter family of CFT states that should correspond to a bulk solution with $a = 0$. Since this solution is exactly the classical black-hole solution with an event horizon, one might naively conclude that certain pure CFT states have a bulk dual with an event horizon, which would contradict the intuition expressed in the Introduction. However, several hints indicate that the strong-coupling description of these particular states (and also of two-charge states of the form $(|00\rangle_k)^{\frac{N}{k}}$) requires ingredients beyond supergravity. For example, the supergravity approximation to the sequence of dualities used to derive the geometry [27] is not valid in these instances. Moreover, in the D1-D5 CFT, this class of states can be distinguished from the thermal ensemble only by the VEVs of non-chiral primary operators.

6. DISCUSSION

In this Letter, we have constructed a new family of black hole microstate geometries that solve the ten-year-old problem of lowering the angular momentum j arbitrarily below the cosmic censorship bound, and we have identified the dual CFT states. Our results demonstrate how adding momentum can transform a two-charge solution describing a microstate of a string-size black hole into a smooth low-curvature solution with a long AdS$_2$ throat. We are confident that all the solutions one can build by generalizing the present work to include more general fluctuations will continue to share these properties. The generic black-hole microstate differs from the states we have constructed in the distribution and type of momentum carriers – our solutions correspond in the CFT to using a very limited set of generators of the chiral algebra (see (18)) to carry the momentum. It is a very interesting question to ask how closely one can approach the generic state using our techniques.

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Appendix A: Details of the general solution

The form of $\mathcal{F}_{k,m,n}$ and $\omega_{k,m,n}$ for general k, m, n is

$$\mathcal{F}_{k,m,n} = 4\bigg[\frac{m^2(k+n)^2}{k^2}\,F_{2k,2m,2n} + \frac{n^2(k-m)^2}{k^2}\,F_{2k,2m+2,2n-2}\bigg], \;\; \omega_{k,m,n} = \mu_{k,m,n}\,(d\psi + d\phi) + \zeta_{k,m,n}\,(d\psi - d\phi)\,, \quad \text{(A1)}$$

$$\mu_{k,m,n} = \frac{R_y}{\sqrt{2}}\,\bigg[\frac{(k-m)^2(k+n)^2}{k^2} F_{2k,2m+2,2n} + \frac{m^2n^2}{k^2} F_{2k,2m,2n-2} - \frac{r^2+a^2\,\sin^2\theta}{4\,\Sigma}\mathcal{F}_{k,m,n} - \frac{\Delta_{2k,2m,2n}}{4\,\Sigma} + \frac{x_{k,m,n}}{4\,\Sigma}\bigg]\,, \quad \text{(A2)}$$

where

$$F_{2k,2m,2n} \;=\; -\sum_{j_1,j_2,j_3=0}^{j_1+j_2+j_3\le k+n-1} \binom{j_1+j_2+j_3}{j_1,j_2,j_3} \frac{\binom{k+n-j_1-j_2-j_3-1}{k-m-j_1,\,m-j_2-1,\,n-j_3}^2}{\binom{k+n-1}{k-m,\,m-1,\,n}^2}\,\frac{\Delta_{2(k-j_1-j_2-1),2(m-j_2-1),2(n-j_3)}}{4(k+n)^2(r^2+a^2)}\,, \qquad \text{(A3)}$$

and where

$$\binom{j_1+j_2+j_3}{j_1,j_2,j_3} \equiv \frac{(j_1+j_2+j_3)!}{j_1!j_2!j_3!}\,. \qquad \text{(A4)}$$

It should be understood that in $\mathcal{F}_{k,m,n}$ and $\mu_{k,m,n}$, when the coefficient of an F function is zero, the term is zero.

The expression for $\zeta_{k,m,n}$ can be obtained from $\mu_{k,m,n}$ by quadrature using the BPS equations for ω, which now reduce to an integrable system of differential equations, as was the case for the $n = 0$ solutions studied in [13].

For regularity, $\mu_{k,m,n}$ must vanish at $r = 0, \theta = 0$; this fixes $x_{k,m,n}$ to the value given below Eq. (14).

One might worry that the warp factor Z_1 could become negative and render the solution singular if the amplitude of the fluctuations becomes too large. However, the minimal value of Z_1 occurs when $\cos\hat{v}_{2k,2m,2n} = -1$. Then the regularity conditions in Eq. (14) and the identity

$$\frac{\Delta_{2k,2m,2n}}{x_{k,m,n}} \le \sum_{\substack{m,n=0\\k=1}} \delta_{k+n-1,p}\,\frac{\Delta_{2k,2m,2n}}{x_{k,m,n}} \;=\; \frac{a^2}{(r^2+a^2)} \le 1$$

ensure that $b^2_{k,m,n}\Delta_{2k,2m,2n} < b^2$ and hence $Z_1 > 0$.

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[28] To see this, note that the $\Delta_{k,m,n}$ are peaked around $r_{\rm max} \sim a\sqrt{n/k}$ (and around a band of latitude on S^3), so for instance for $n \sim k$ and n, k large, the wave profile is sharply peaked near $r \sim a$ (the blue band in Fig. 1).