

This is the accepted manuscript made available via CHORUS. The article has been published as:

Skyrmions and Hall Transport

Bom Soo Kim and Alfred D. Shapere

Phys. Rev. Lett. **117**, 116805 — Published 9 September 2016

DOI: [10.1103/PhysRevLett.117.116805](https://doi.org/10.1103/PhysRevLett.117.116805)

Skyrmions and Hall Transport

Bom Soo Kim and Alfred D. Shapere

Department of Physics and Astronomy, University of Kentucky, Lexington, KY 40506, USA

(Dated: August 17, 2016)

We derive a generalized set of Ward identities that captures the effects of topological charge on Hall transport. The Ward identities follow from the 2+1 dimensional momentum algebra, which includes a central extension proportional to the topological charge density. In the presence of topological objects like Skyrmions, we observe that the central term leads to a direct relation between the thermal Hall conductivity and the topological charge density. We extend this relation to incorporate the effects of a magnetic field and an electric current. The topological charge density produces a distinct signature in the electric Hall conductivity, which is identified in existing experimental data, and yields further novel predictions. For insulating materials with translation invariance, the Hall viscosity can be directly determined from the Skyrmion density and the thermal Hall conductivity to be measured as a function of momentum.

PACS numbers: 73.43.Cd, 03.75.Lm, 11.30.-j, 75.70.-i

Introduction: Ward identities in quantum field theories [1] are relations among correlation functions that follow solely from conservation equations, and are thus independent of the properties of the Hamiltonian, other than its symmetries. Among their many applications in quantum field theory and many body physics (see for example [2–4]), Ward identities can be used to derive nontrivial relations among various measurable quantities such as conductivities and viscosities. Recently, Ward identities have been applied to 2+1 dimensional systems with broken parity to show that the Hall viscosity is equal to one-half of the angular momentum in the presence of a gap, along with other relations involving transport coefficients [5–11]. In this letter, we obtain a more general and powerful set of Ward identities for such systems by incorporating topological charges [12] that are not captured by conservation equations. These identities lead to useful relations between transport properties and the density of topological objects such as Skyrmions, with corresponding experimental signatures.

Skyrmions [13–15] in magnetic materials are stable, particle-like spin textures that are protected by topological quantum numbers. They have been studied theoretically [16–22] and have been realized experimentally in magnetic materials [23–26]. Their transport properties have been measured, including their electric Hall conductivity [27–32], thermal Hall conductivity, and angular momentum [33, 34]. The identification of these quantities is often subtle, due to the Skyrmions’ extended nature and their interactions with conduction electrons and other backgrounds, and has been based on phenomenological models rather than first principles. A better theoretical understanding of the transport properties of Skyrmions and their various relationships could be of great help in interpreting these experiments.

We will derive from the advertised Ward identities a set of simple and universal relations among the observables mentioned above, in the presence of baby Skyrmions (Skyrmions in 2+1 dimensions), which we hereafter re-

fer to simply as Skyrmions. In particular, we will argue that in insulators with translation and rotation invariance, the thermal Hall conductivity is proportional to the topological charge density, as in eq. (8). In the absence of translation invariance, these two quantities appear in a relation (11) which also involves the Hall viscosity and angular momentum. According to our Ward identity, the Hall viscosity, which has been previously overlooked in both theoretical and experimental studies of Skyrmion systems, can be directly determined from the Skyrmion density and the thermal Hall conductivity measured as a function of momentum, as explained in eq. (12).

For metallic materials with Skyrmions, the tight binding between Skyrmion spins and the spins of conduction electrons implies that the Skyrmion density makes a contribution to the electric Hall conductivity [27–32]. We identify this contribution in our context in eq. (18), along with new experimental implications that should be possible to verify without difficulty. An expression for the Hall viscosity in terms of the measured electric Hall conductivity as a function of momentum is given in (21). Skyrmion dynamics turns out to be robust against the presence of impurities [35]; thus our conclusions should apply to realistic materials.

Ward identities and central extension: Topologically non-trivial objects can lead to significant modifications in certain physical quantities. Baby Skyrmions carry topological charge that modifies the commutators of momentum operators [12, 36, 37]

$$[P^i(x^0), P^j(x^0)] = i\hbar C^{ij}, \quad (1)$$

where x^0 is time, $x^\mu = (x^0, \vec{x})$, and $i, j = 1, 2$. $P^i(x^0) \equiv \int d^2x T^{0i}(x^\mu)$ is the momentum 2-vector, defined as the space integral of the momentum density components of the stress energy tensor $T^{\mu\nu}$. The central term on the right side of (1) is proportional to the net topological charge C^{ij} , which may be written as the spatial integral

of the topological charge density $C^{ij} = \int d^2x c^{ij}(x^\mu)$. In terms of the spin configuration $\vec{n}(x^\mu)$,

$$c^{ij} = \vec{n} \cdot [\partial_i \vec{n} \times \partial_j \vec{n}]. \quad (2)$$

We assume that the spin \vec{n} varies smoothly over space and that the continuum description is valid on length scales much larger than the lattice spacing [26]. Since c^{ij} is antisymmetric, we can write $c^{ij} \equiv c \epsilon^{ij}$. The modified commutator (1) implies that one cannot fully specify the momentum of the object due to uncertainty relations among different components. It is strongly reminiscent of the momentum algebra in the presence of a background magnetic field. As we will explain below, c^{ij} can be interpreted as an effective magnetic field produced by the Skyrmions.

A local form of the momentum commutator (1) is more convenient for our purposes. As proposed in [12],

$$\begin{aligned} [T^{0i}(x^0, \vec{x}), T^{0j}(x^0, \vec{x}')] \\ = i \left(-\partial_i T^{0j}(x^\mu) + \partial_j T^{0i}(x^\mu) + c^{ij} \right) \delta^2(\vec{x} - \vec{x}'), \end{aligned} \quad (3)$$

where the momentum density operators $T^{0i}(x^\mu)$ produce the derivative terms on the right side. Here and henceforth we set $\hbar = 1$.

We now proceed to generalize the Ward identities for systems with broken parity [5–11] by including the central extension in the equal time commutator. For simplicity, time and space translation symmetries are assumed along with rotational symmetry in the spatial plane, which is compatible with the central extension in eq. (3) in 2 + 1 dimensions [12]. We begin by considering the retarded correlator of momentum densities

$$G^{0i,0j}(x^\mu, x'^\mu) = i\theta(x^0 - x'^0) \langle [T^{0i}(x^\mu), T^{0j}(x'^\mu)] \rangle. \quad (4)$$

Applying two time derivatives $\partial_0 \partial'_0$ to this expression produces four terms. When both derivatives act on the momentum density, we use local momentum conservation $\partial_\mu T^{\mu i} = 0$ [49] to obtain a term with two spatial derivatives of the retarded correlator, $\partial_n \partial'_m G^{ni,mj}$. All other terms contain a delta function $\delta(x^0 - x'^0)$, the derivative of the step function in (4). These latter terms are precisely the contact terms that have been reported to be missing in some evaluations of Kubo formulas [8]. We stress that our Ward identities automatically produce all possible contact terms. For translation invariant systems, all the contact terms vanish [11] and the Ward identity becomes

$$\omega^2 \tilde{G}^{0i,0j} = -i\omega \epsilon^{ij} c + q_m q_n \tilde{G}^{mi,mj}, \quad (5)$$

where i, j, m, n are spatial indices, and a Fourier transform has been applied in the form $\tilde{G}^{0i,0j}(q_\mu) \equiv \int dx^0 d^2x e^{iq_\mu x^\mu} G^{0i,0j}(x^\mu)$, with $q_0 \equiv \omega$. We may treat the Skyrmion density c as a constant when discussing transport measurements on distance scales much larger than that of an individual Skyrmion.

Rotation invariance constrains the components of the retarded Green's function. In general, $\tilde{G}^{ni,mj}$ can have three independent contributions: a shear viscosity term $-i\omega\eta(\delta^{nm}\delta^{ij} + \delta^{nj}\delta^{im} - \delta^{ni}\delta^{mj})$, a bulk viscosity term $-i\omega\zeta\delta^{ni}\delta^{mj}$, and a Hall viscosity term $-\frac{i}{2}\omega\eta_H(\epsilon^{nm}\delta^{ij} + \epsilon^{nj}\delta^{im} + \epsilon^{im}\delta^{nj} + \epsilon^{ij}\delta^{nm})$ [38]. The coefficients η , ζ , and η_H can be complex functions of the frequency, whose real parts are the usual transport coefficients. The components with two spatial indices $\tilde{G}^{0i,0j}$ take the form

$$-i\omega [\delta^{ij} \kappa_\delta + \epsilon^{ij} \kappa_\epsilon + q^i q^j \kappa_q + (q^i \epsilon^{jn} + q^j \epsilon^{in}) q_n \kappa_{q\epsilon}], \quad (6)$$

where the form factors κ are the (complex) thermal conductivities, including the Hall component κ_ϵ . These form factors are analogous to the electric conductivities coming from a retarded current-current correlator that will be introduced later. By using these expressions to decompose equation (5) into independent tensor structures, one obtains

$$\begin{aligned} \omega^2 [\delta^{jl} \kappa_\delta + \epsilon^{jl} \kappa_\epsilon + q^j q^l \kappa_q + (\epsilon^{jo} q^l + \epsilon^{lo} q^j) q_o \kappa_{q\epsilon}] \\ = \epsilon^{jl} [c + q^2 \eta_H] + \delta^{jl} q^2 \eta + q^j q^l \zeta, \end{aligned} \quad (7)$$

which contains four distinct Ward identities corresponding to four independent tensor structures [11].

By isolating the momentum independent terms in (7) proportional to δ^{jl} and ϵ^{jl} , we arrive at the simple relations

$$\omega^2 \kappa_\delta^{(0)} = 0, \quad \omega^2 \kappa_\epsilon^{(0)} = c, \quad (8)$$

where the superscript (0) denotes the momentum independent part. Intuitively, the reason $\kappa_\delta^{(0)}$ vanishes and $\kappa_\epsilon^{(0)}$ does not is that Skyrmions are associated with spontaneously broken translation symmetry along with broken parity, whose imprints can only enter through the parity odd part of the conductivity at zero momentum. More precisely, the second identity predicts that the formation of a single Skyrmion results in the creation of a unit of thermal Hall conductivity $\kappa_\epsilon^{(0)}$ in units of the quantized topological charge density. The frequency dependence is a consequence of the pole structure of the Goldstone boson that manifests itself in the retarded momentum correlator. In the presence of disorder, the behavior $\kappa_\epsilon^{(0)} = c/\omega^2$ could in principle be lifted. However, recent numerical simulations have confirmed that Skyrmion motions are unaffected by impurities, in contrast to the case of domain walls [35]. The thermal Hall conductivity κ_ϵ is dissipationless and exists even at zero temperature. While our Ward identity relations are valid at finite temperatures as well, measurements will be cleaner at very low temperatures, where additional dissipative contributions are suppressed. Another interpretation of eq. (8) is that the Skyrmions carrying the thermal current propagate in an effective magnetic field given by the Skyrmion charge density c_{ij} , leading to a thermal Hall effect [39].

For the momentum dependent terms in (7), we obtain

$$\omega^2 \bar{\kappa}_\delta = q^2 \eta, \quad \omega^2 \bar{\kappa}_\epsilon = q^2 \eta_H, \quad \omega^2 \kappa_q = \zeta, \quad (9)$$

where the bar $\bar{}$ indicates the nonconstant momentum dependent part; for example, $\bar{\kappa}_\epsilon = \kappa_\epsilon - \kappa_\epsilon^{(0)} = q^2 \kappa_\epsilon^{(2)} + q^4 \kappa_\epsilon^{(4)} + \dots$. Thus, thermal conductivities are directly connected to the viscosities of the system, as previously confirmed [11]. Furthermore, it follows from (7) that $\kappa_{qe} = 0$.

If the system of interest is not translationally invariant, there will be additional contributions to the Ward identity (9); however, the zero momentum identity (8) will be unmodified. A particularly interesting contribution of this type arises in parity-breaking systems exhibiting spontaneously generated angular momentum ℓ [36, 40], where the momentum generator can develop an expectation value

$$\langle T^{0i} \rangle = \frac{1}{2} \epsilon^{ik} \partial_k \ell. \quad (10)$$

In the absence of translation invariance, the two time derivatives $\partial_0 \partial'_0$ acting on $G^{0i,0j}(x^\mu, x'^\mu)$ pick up the contact term $\frac{i}{2}(\partial'_0 - \partial_0)[\delta(x^0 - x'^0)\langle [T^{0i}(x^\mu), T^{0j}(x'^\mu)] \rangle]$ in addition to the terms appearing in (5). The commutator yields a tensor similar to η_H coming from the last term in (5), as one can check using (3) and (10) [11]. As a result, η_H in (9) is modified to $\eta_H + \frac{\ell}{2}$. In such cases, a coordinate space description might be more convenient. Similarly, the inclusion of pressure p , another universal contribution, would replace ζ in the last relation of (9) by the combination $\zeta - \frac{i}{\omega} p$.

Ward identities for insulators: Recently, Skyrmions have been observed in the insulating material Cu_2OSeO_3 [41], and various experiments regarding the Hall thermal conductivity and angular momentum have been carried out [34, 42, 43]. For insulators, our Ward identity provides a simple relation among parity violating transport coefficients,

$$\omega^2 \kappa_\epsilon = c + \partial^2 \left(\eta_H + \frac{\ell}{2} \right), \quad (11)$$

which is derived from eqs. (7) and (10) in the absence of translation invariance. Recent experiments have successfully measured the Skyrmion density, thermal Hall

conductivity and angular momentum in Skyrmion materials [34]. Such measurements could in principle be used to infer the existence of Hall viscosity [50].

In the presence of translation invariance, there is a simple way to measure the Hall viscosity. Combining eqs. (8) and (9), we get

$$\eta_H = c \frac{\bar{\kappa}_\epsilon}{\kappa_\epsilon^{(0)}} \rightarrow c \frac{\kappa_\epsilon^{(2)}}{\kappa_\epsilon^{(0)}}, \quad (12)$$

where we take the limit $q^2 \rightarrow 0$. Once the thermal Hall conductivity κ_ϵ is measured as a function of q^2 , the Hall viscosity is nothing but the Skyrmion density multiplied by the ratio between the slope and κ_ϵ -intercept $\kappa_\epsilon(q^2 = 0)$. Note that this is only applicable in the presence of nonzero Skyrmion density.

Ward identities for conductors: The neutral case discussed above provides a simple relation between the topological charge density and thermal Hall conductivity. However, since most realistic materials reveal Skyrmions in the presence of electric charge carriers, we need to generalize our discussion to include conducting materials. We will see that Skyrmions have a direct effect on charged dynamics as well, which can be accounted for by the inclusion of a conserved $U(1)$ current J^μ , $\partial_\mu J^\mu = 0$.

In the presence of a uniform external magnetic field B , the momentum density gets modified to

$$T_B^{0j} = T^{0j} - (B/2) \epsilon^j_k x^k J^0. \quad (13)$$

This modification is the expected minimal coupling in the presence of a constant magnetic field. Another important modification appears in the conservation equation

$$\partial_\mu T^{\mu i} = B \epsilon^i_j J^j. \quad (14)$$

These are the spatial components of the general relation $\partial_\mu T^{\mu\nu} = F^{\nu\rho} J_\rho$. The Ward identities are once again obtained by taking time derivatives of the correlator (4). The derivation is straightforward, and we present the details in the Supplemental Material. The resulting full Ward identity is

$$\begin{aligned} & \delta^{jl} \left[\omega^2 \kappa_\delta + i\omega B (\alpha_\epsilon + \alpha_\epsilon^* + q^2 [\alpha_{q\epsilon} - \alpha_{q\epsilon}^*]) + B^2 (\sigma_\delta + q^2 \sigma_q) \right] + \epsilon^{jl} \left[\omega^2 \kappa_\epsilon - i\omega B (\alpha_\delta + \alpha_\delta^* + q^2 [\alpha_q + \alpha_q^*]/2) + B^2 \sigma_\epsilon \right] \\ & + q^j q^l \left[\omega^2 \kappa_q - 2i\omega B (\alpha_{q\epsilon} - \alpha_{q\epsilon}^*) - B^2 \sigma_q \right] + (\epsilon^{jo} q^l + \epsilon^{lo} q^j) q_o \left[\omega^2 \kappa_{q\epsilon} + i\omega B (\alpha_q - \alpha_q^*)/2 - B^2 \sigma_{q\epsilon} \right] \\ & = \epsilon^{jl} [c - B\rho + q^2 \eta_H] + \delta^{jl} q^2 \eta + q^j q^l \zeta, \end{aligned} \quad (15)$$

where α, α^* are thermoelectric conductivity tensors re-

lated to the form factors of momentum-current corre-

lators $G^{0i,j} \sim \langle [T^{0i}, J^j] \rangle$ and $G^{i,0j} \sim \langle [J^i, T^{0j}] \rangle$, while the σ 's are electric conductivity tensors associated with current-current correlators $G^{i,j} \sim \langle [J^i, J^j] \rangle$. They arise due to the modifications eqs. (13) and (14) and the corresponding mix between the momentum T^{0i} and charge J^j densities. There are four independent tensor structures and four corresponding Ward identities in (15). These identities reduce to those of insulators when $B = 0$.

The momentum independent Ward identities give

$$\begin{aligned} \omega^2 \kappa_\delta^{(0)} + i\omega B(\alpha_\epsilon^{(0)} + \alpha_\epsilon^{*(0)}) + B^2 \sigma_\delta^{(0)} &= 0, \\ \omega^2 \kappa_\epsilon^{(0)} - i\omega B(\alpha_\delta^{(0)} + \alpha_\delta^{*(0)}) + B^2 \sigma_\epsilon^{(0)} &= c - B\rho, \end{aligned} \quad (16)$$

which reduce to eq. (8) when $B = 0$. At non-zero momentum, there are four independent relations connecting viscosities and conductivities as in the neutral case. In particular, the Hall viscosity [38] is

$$q^2 \eta_H = \omega^2 \bar{\kappa}_\epsilon + B^2 \bar{\sigma}_\epsilon - i\omega B \left[\bar{\alpha}_\delta + \bar{\alpha}_\delta^* + \frac{\alpha_q + \alpha_q^*}{2} \right]. \quad (17)$$

Ward identities for conductors at zero momentum: In [26, 45, 46] interactions between Skyrmions and conduction electrons are modeled by the ferromagnetic spin coupling. In the strong coupling limit, the spin wave function of the conduction electrons is identified with that of the localized spin $\vec{n}(x^\mu)$ of the Skyrmions. This limit is described by a tight binding model with Hund's rule coupling. More general interactions between conduction electrons and local magnetization may be considered [47].

We will discuss two different ways to model the effects of the interaction between the thermal and charge responses. First, we can modify the parameters of the Ward identities. The Skyrmion charge density produces an emergent magnetic field $b = c/2$ [26], which can change the dynamics of conduction electrons, similarly to B . For simplicity, we assume that the emergent magnetic field is homogeneous and constant, which is the case for all practical measurements. Due to the tight binding, the motion of the conduction electrons will also influence the thermal response of the Skyrmions. At vanishing momentum, by taking these effects into account, we get

$$\omega^2 \kappa_\epsilon^{(0)} - i\omega B_b(\alpha_\delta^{(0)} + \alpha_\delta^{*(0)}) + B_b^2 \sigma_\epsilon^{(0)} = c_b - B_b \rho. \quad (18)$$

This identity is of the same form as (16), with the modification $B \rightarrow B_b \equiv B + b$ contributing to the charge response, and $c \rightarrow c_b \equiv c + c_{el}$ incorporating an additional contribution to the thermal response from the conduction electrons c_{el} , without changing the topological charge density. The quantities c and b are constant and independent of B , while c_{el} (also measurable) is expected to be proportional to B and depends on the strength of the binding. b, c, c_{el} are expected to be readily identifiable experimentally. In particular, b can be identified from a step-function-like signature in the Hall conductivity σ_ϵ [27–30], as one passes into and out of a phase

in which Skyrmions develop a finite density c . Such behavior will also confirm the presence of a nonzero density c , which will likewise produce a similar step-function-like contribution in the thermal Hall conductivity κ_ϵ with an additional B -dependent c_{el} , by sweeping the magnetic field B or the temperature T independently.

In the absence of ferromagnetic binding between the Skyrmion and conduction electron spins, the electric Hall conductivity would only pick up contributions from the conduction electrons, and B_b would reduce to B . On the other hand, the thermal Hall conductivity would include both contributions, c and c_{el} , with the latter being independent of B .

A second, alternative way to incorporate the interaction between thermal and charged responses is to impose the following operator relation

$$T^{0i} = \mu J^i, \quad (19)$$

where μ parameterizes the strength of the coupling between the spins of the Skyrmion and the conduction electron. Then the momentum transport is tied to the charge transport as $\kappa = \mu \alpha = \mu \alpha^* = \mu^2 \sigma$. [51] The relation (19) implies a distinct experimental signature. At zero momentum, the analogue of (18) becomes

$$\sigma_\delta^{(0)} = -\frac{i\omega_c}{\omega} \frac{c - B\rho}{\mu^2(\omega^2 - \omega_c^2)}, \quad \sigma_\epsilon^{(0)} = \frac{c - B\rho}{\mu^2(\omega^2 - \omega_c^2)}, \quad (20)$$

where $\omega_c = \frac{B}{\mu}$. For small magnetic field $\omega_c \ll \omega$, the Hall conductivity is directly related to the topological charge density $\sigma_\epsilon^{(0)} \approx \frac{c}{\mu^2 \omega^2}$ and $\sigma_\delta^{(0)} \approx 0$. In the opposite limit with large magnetic field $\omega_c \gg \omega$, $\sigma_\epsilon^{(0)} \approx 0$ and $\sigma_\delta^{(0)} \approx \frac{-i\rho}{\mu\omega}$. Such behavior can easily be measured. It would be interesting to find a material with Skyrmions that displays these properties.

The momentum independent Ward identities are the same as (16) whether or not the system has translation invariance. If the system of interest has translation symmetry, one can use (17) with the modification $B \rightarrow B_b$ for momentum dependent Hall transport measurements.

In the presence of translation symmetry, it is also simple to measure the Hall viscosity similarly to eq. (12). Dividing eq. (17) by the second equation of (16) with $B \rightarrow B_b, c \rightarrow c_b$, and taking the approximation $\omega/B_b \rightarrow 0$ and the limit $q^2 \rightarrow 0$, we obtain

$$\eta_H = (c_b - B_b \rho) \frac{\sigma_\epsilon^{(2)}}{\sigma_\epsilon^{(0)}}. \quad (21)$$

In the opposite limit $B_b/\omega \rightarrow 0$, η_H reduces to eq. (12) with the modification $c \rightarrow c_b - B_b \rho$. Note that this identification of η_H can also be applied to systems without Skyrmions, such as quantum Hall systems.

Ward identities for conductors without translation invariance: If spatial translation symmetry is broken (still

assuming time-translation and rotation invariance), more physical quantities can come into play. In particular, the Ward identity is given by eq. (17) with the replacement $\eta_H \rightarrow \eta_H + \frac{e}{2}$ as in the neutral case [52]. This Ward identity directly relates conductivities, angular momentum and Hall viscosity. Recent experiments on metallic MnSi have studied transport properties and angular momentum [33, 48]. While Hall viscosity has not previously been discussed in the context of Skyrmion physics, it might play an important role and has a chance to be observed for the first time in active experiments.

Acknowledgements: We thank Sumit Das, Lance De Long, Michael Eides, Ori Ganor, Petr Hořava, Carlos Hoyos, Seungjoon Hyun, Seok Kim, Elias Kiritsis, Kimyeong Lee, Keh-Fei Liu, Hitoshi Murayama, Yaron Oz, Sumiran Pujari, Shinsei Ryu, Sang-Jin Sin, David Tong, Oskar Vafeek, and Piljin Yi for helpful discussions. We are especially thankful to Jung Hoon Han for numerous illuminating discussions on Skyrmions and invaluable comments on the draft. Finally we would like to thank our referees for numerous helpful comments. BSK is grateful to the members of the Berkeley Center for Theoretical Physics, Berkeley, and KIAS, Seoul, for their warm hospitality during his visits. This work is partially supported by NSF Grant PHY-1214341.

-
- [1] J. C. Ward, “An Identity in Quantum Electrodynamics,” *Phys. Rev.* **78**, 182 (1950).
 - [2] Y. Takahashi, “On the generalized ward identity,” *Nuovo Cimento, Ser 10*, 6 (1957) 370.
 - [3] Y. Nambu, “Quasi-Particles and Gauge Invariance in the Theory of Superconductivity,” *Phys. Rev.* **117**, 648 (1960).
 - [4] H. K. Janssen, B. Schaub and B. Schmittmann, “New universal short-time scaling behaviour of critical relaxation processes,” *Z. Phys. B. Condensed Matter* **73**, 539 (1989).
 - [5] N. Read, “Non-Abelian adiabatic statistics and Hall viscosity in quantum Hall states and $p(x) + ip(y)$ paired superfluids,” *Phys. Rev. B* **79**, 045308 (2009) [arXiv:0805.2507].
 - [6] N. Read and E. H. Rezayi, “Hall viscosity, orbital spin, and geometry: paired superfluids and quantum Hall systems,” *Phys. Rev. B* **84**, 085316 (2011) [arXiv:1008.0210 [cond-mat.mes-hall]].
 - [7] C. Hoyos and D. T. Son, “Hall Viscosity and Electromagnetic Response,” *Phys. Rev. Lett.* **108**, 066805 (2012) [arXiv:1109.2651 [cond-mat.mes-hall]].
 - [8] B. Bradlyn, M. Goldstein and N. Read, “Kubo formulas for viscosity: Hall viscosity, Ward identities, and the relation with conductivity,” *Phys. Rev. B* **86**, 245309 (2012) [arXiv:1207.7021 [cond-mat.stat-mech]].
 - [9] M. Geracie, D. T. Son, C. Wu and S. F. Wu, “Spacetime Symmetries of the Quantum Hall Effect,” *Phys. Rev. D* **91**, 045030 (2015) [arXiv:1407.1252 [cond-mat.mes-hall]].
 - [10] C. Hoyos, B. S. Kim and Y. Oz, “Ward Identities for Hall Transport,” *JHEP* **1407**, 054 (2014) [arXiv:1407.2616 [hep-th]].
 - [11] C. Hoyos, B. S. Kim and Y. Oz, “Ward Identities for Transport in 2+1 Dimensions,” *JHEP* **1503**, 164 (2015) [arXiv:1501.05756 [hep-th]].
 - [12] H. Watanabe and H. Murayama, “Noncommuting Momenta of Topological Solitons,” *Phys. Rev. Lett.* **112**, 191804 (2014) [arXiv:1401.8139 [hep-th]].
 - [13] T. H. R. Skyrme, “A Non-Linear Theory of Strong Interactions,” *Proc. Roy. Soc. London, Ser. A* **247**, 260 (1958).
 - [14] A.A. Belavin and A.M. Polyakov, “Metastable states of two-dimensional isotropic ferromagnets,” *JETP Lett.* **22**, 245 (1975).
 - [15] F. Wilczek and A. Zee, “Linking Numbers, Spin, and Statistics of Solitons,” *Phys. Rev. Lett.* **51**, 2250 (1983).
 - [16] A. Bogdanov and D. Yablonskii, “Thermodynamically stable “vortices” in magnetically ordered crystals. The mixed state of magnets,” *Zh. Eksp. Teor. Fiz.* **95**, 178 (1989) [*Sov. Phys. JETP* **68**, 101 (1989)].
 - [17] A.N. Bogdanov, U.K. Rössler, M. Wolf, and K.-H. Müller, “Magnetic structures and reorientation transitions in noncentrosymmetric uniaxial antiferromagnets,” *Phys. Rev. B* **66**, 214410 (2002).
 - [18] U. Rössler, A. Bogdanov, and C. Pfleiderer, “Spontaneous skyrmion ground states in magnetic metals,” *Nature (London)* **442**, 797 (2006).
 - [19] B. Binz, A. Vishwanath, and V. Aji, “Theory of the Helical Spin Crystal: A Candidate for the Partially Ordered State of MnSi,” *Phys. Rev. Lett.* **96**, 207202 (2006).
 - [20] B. Binz and A. Vishwanath, “Theory of helical spin crystals: Phases, textures, and properties,” *Phys. Rev. B* **74**, 214408 (2006).
 - [21] S. Tewari, D. Belitz, and T.R. Kirkpatrick, “Blue Quantum Fog: Chiral Condensation in Quantum Helimagnets,” *Phys. Rev. Lett.* **96**, 047207 (2006).
 - [22] I. Fischer, N. Shah, and A. Rosch, “Crystalline phases in chiral ferromagnets: Destabilization of helical order,” *Phys. Rev. B* **77**, 024415 (2008).
 - [23] S. Mühlbauer, B. Binz, F. Jonietz, C. Pfleiderer, A. Rosch, A. Neubauer, R. Georgii, P. Böni, “Skyrmion lattice in a chiral magnet,” *Science* **323**, 915 (2009).
 - [24] X. Z. Yu, Y. Onose, N. Kanazawa, J. H. Park, J. H. Han, Y. Matsui, N. Nagaosa and Y. Tokura, “Real-space observation of a two-dimensional skyrmion crystal,” *Nature* **465**, 901 (2010).
 - [25] S. Heinze, K. von Bergmann, M. Menzel, J. Brede, A. Kubetzka, R. Wiesendanger, G. Bihlmayer, and S. Blügel, “Spontaneous atomic-scale magnetic skyrmion lattice in two dimensions,” *Nature Phys.* **7**, 713 (2011).
 - [26] N. Nagaosa and Y. Tokura, “Topological properties and dynamics of magnetic skyrmions,” *Nature Nanotechnology*, **8**, 899 (2013).
 - [27] M. Lee, W. Kang, Y. Onose, Y. Tokura, and N. P. Ong, “Unusual Hall anomaly in MnSi under pressure,” *Phys. Rev. Lett.* **102**, 186601 (2009).
 - [28] A. Neubauer, C. Pfleiderer, B. Binz, A. Rosch, R. Ritz, P. G. Niklowitz, and P. Böni, “Topological Hall effect in the A phase of MnSi,” *Phys. Rev. Lett.* **102**, 186602 (2009).
 - [29] N. Kanazawa, Y. Onose, T. Arima, D. Okuyama, K. Ohoyama, S. Wakimoto, K. Kakurai, S. Ishiwata and Y. Tokura, “Large topological Hall effect in a short-period helimagnet MnGe,” *Phys. Rev. Lett.* **106**, 156603 (2011).
 - [30] T. Schulz, R. Ritz, A. Bauer, M. Halder, M. Wagner, C. Franz, C. Pfleiderer, K. Everschor, M. Garst and A.

- Rosch, “Emergent electrodynamics of skyrmions in a chiral magnet,” *Nature Physics*, **8**, 301 (2012).
- [31] Y. Li, N. Kanazawa, X. Z. Yu, A. Tsukazaki, M. Kawasaki, M. Ichikawa, X. F. Jin, F. Kagawa, and Y. Tokura, “Robust Formation of Skyrmions and Topological Hall Effect Anomaly in Epitaxial Thin Films of MnSi,” *Phys. Rev. Lett.* **110**, 117202 (2013).
- [32] S. X. Huang and C. L. Chien, “Extended skyrmion phase in epitaxial FeGe(111) thin films,” *Phys. Rev. Lett.* **108**, 267201 (2012).
- [33] F. Jonietz, S. Mühlbauer, C. Pfleiderer, A. Neubauer, W. Mnzer, A. Bauer, T. Adams, R. Georgii, P. Böni, R. A. Duine, K. Everschor, M. Garst, A. Rosch, “Spin Transfer Torques in MnSi at Ultralow Current Densities,” *Science* **330**, 1648 (2010).
- [34] M. Mochizuki, X. Z. Yu, S. Seki, N. Kanazawa, W. Koshibae, J. Zang, M. Mostovoy, Y. Tokura and N. Nagaosa, “Thermally driven ratchet motion of a skyrmion microcrystal and topological magnon Hall effect,” *Nature Materials*, **13**, 241 (2014).
- [35] J. Iwasaki, M. Mochizuki and N. Nagaosa, “Universal current-velocity relation of skyrmion motion in chiral magnets,” *Nature Communications*, **4**, 1463 (2013).
- [36] N. Papanicolaou and T. N. Tomaras, “Dynamics of Magnetic Vortices,” *Nucl. Phys. B* **360**, 425 (1991).
- [37] D. J. Thouless, P. Ao and Q. Niu, “Transverse Force on a Quantized Vortex in a Superfluid,” *Phys. Rev. Lett.* **76**, 3758 (1996).
- [38] J. E. Avron, R. Seiler and P. G. Zograf, “Viscosity of Quantum Hall Fluids,” *Phys. Rev. Lett.* **75**, 697 (1995). [arXiv:cond-mat/9502011]
- [39] M. Stone, “Magnus force on skyrmions in ferromagnets and quantum Hall systems,” *Phys. Rev. B* **53**, 16573 (1996). [arXiv:cond-mat/9512010]
- [40] H. Liu, H. Ooguri, B. Stoica and N. Yunes, “Spontaneous Generation of Angular Momentum in Holographic Theories,” *Phys. Rev. Lett.* **110**, 211601 (2013) [arXiv:1212.3666 [hep-th]].
- [41] S. Seki, X. Z. Yu, S. Ishiwata and Y. Tokura, “Observation of Skyrmions in a Multiferroic Material,” *Science* **336**, 198 (2012).
- [42] S. Seki, J.-H. Kim, D. S. Inosov, R. Georgii, B. Keimer, S. Ishiwata, and Y. Tokura, “Formation and rotation of skyrmion crystal in the chiral-lattice insulator Cu_2OSeO_3 ,” *Phys. Rev. B* **85**, 220406(R) (2012).
- [43] K. A. van Hoogdalem, Y. Tserkovnyak, and D. Loss, “Magnetic texture-induced thermal Hall effects,” *Phys. Rev. B* **87**, 024402 (2013).
- [44] Y.-T. Oh, H. Lee, J.-H. Park, J. H. Han, “Dynamics of magnon fluid in Dzyaloshinskii-Moriya magnet and its manifestation in magnon-Skyrmion scattering,” *Phys. Rev. B* **91**, 104435 (2015).
- [45] G. E. Volovik, “Linear momentum in ferromagnets,” *J. Phys. C* **20**, L83 (1987)
- [46] J. Zang, M. Mostovoy, J.H. Han, and N. Nagaosa, “Dynamics of Skyrmion Crystals in Metallic Thin Films,” *Phys. Rev. Lett.* **107**, 136804 (2011).
- [47] S. Zhang and Z. Li, “Role of Nonequilibrium Conduction Electrons on the Magnetization Dynamics of Ferromagnets,” *Phys. Rev. Lett.* **93**, 127204 (2004).
- [48] K. Everschor, M. Garst, B. Binz, F. Jonietz, S. Mühlbauer, C. Pfleiderer and A. Rosch, “Rotating skyrmion lattices by spin torques and field or temperature gradients,” *Phys. Rev. B* **86**, 054432 (2012).
- [49] Although translation symmetry is spontaneously broken due to the presence of Skyrmions, the conservation equations can be checked to hold with the model in [12]. Translation symmetry of the Skyrmion configuration can be restored in the continuum limit.
- [50] In real materials, κ_ϵ has a further contribution from the magnon [41], whose dynamics has been considered recently [44]. Modeling the interaction between the Skyrmion and the magnon in this context of Ward identity is an interesting problem, which is beyond the scope of this letter due to the non-conservation of the magnon spin and the dependence on details of the interaction.
- [51] Similar relations to (19) have been used to impose Galilean invariance in related systems [5–11].
- [52] Likewise, the magnetization M , from the expectation value of the current $\langle J^i \rangle = \epsilon^{ik} \partial_k M$, can contribute in the combination $\zeta - \frac{i}{\omega} (p - BM)$, replacing the bulk viscosity ζ [11].