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We show that there exists a special compactification of QCD on $\mathbb{R}^3 \times S^1$ in which the theory has a domain where continuous chiral symmetry breaking is analytically calculable. We give a microscopic derivation of the chiral lagrangian, the chiral condensate, and the Gell-Mann-Oakes-Renner relation $m_\pi^2 f_\pi^2 = -m_q \langle \bar{q}q \rangle$. Abelian duality, monopole operators, and flavor-twisted boundary conditions play the main roles. The flavor twisting leads to the new effect of fractional jumping of fermion zero modes among monopole-instantons. Chiral symmetry breaking is induced by monopole-instanton operators, and the Nambu-Goldstone pions arise by color-flavor transmutation from gapless "dual photons". We also give a microscopic picture of the "constituent quark" masses. Our results are consistent with expectations from chiral perturbation theory at large S^1, and yield strong support for adiabatic continuity between the small-S^1 and large-S^1 regimes. We also find concrete microscopic connections between $\mathcal{N} = 1$ and $\mathcal{N} = 2$ supersymmetric gauge theory dynamics and non-supersymmetric QCD dynamics.

Introduction. The importance of spontaneous chiral symmetry breaking (χSB) in QCD for our understanding of nature is hard to overstate. For example, χSB gives the main contribution to the rest masses of the light mesons and baryons, and strongly constrains their interactions. However, the microscopic mechanism of χSB is still mysterious. The basic problem is that in the situations known to date in QCD, continuous χSB at zero matter density only happens at strong coupling [1]. So, while there are many phenomenological models of χSB, such as Nambu-Jona-Lasinio models [2], truncated Schwinger-Dyson models [3], and instanton liquid models [4, 5], χSB happens outside of the regime where they are under systematic theoretical control.

Here we present a new mechanism of χSB, in a class of 4D non-Abelian gauge theories which is smoothly connected to standard 4D QCD with $N_c = 3$ and $N_f = 3$. Our mechanism operates in a weakly-coupled regime which is under systematic theoretical control thanks to the technique of adiabatic continuity [6–20]. We are thus able to give a controlled microscopic derivation of the chiral Lagrangian of the Nambu-Goldstone bosons (NGBs). Another historical mystery is the reason for the phenomenological successes of the naive quark model, built from "constituent quarks" with masses of the same scale as the gauge-sector mass gap. We show that constituent quark masses arise naturally from monopole-instantons.

The setting. We consider $SU(N_c > 2)$ gauge theory with a strong scale Λ coupled to $N_f \leq N_c$ flavors of fundamental Dirac fermions $\psi_i, i = 1, \cdots, N_f$ with a common mass $m_q \ll \Lambda$, as well as one heavy adjoint Dirac fermion λ with mass $m_\lambda \gg \Lambda$. On $\mathbb{R}^{1,3}$ the heavy adjoint fermion is a spectator field. On $\mathbb{R}^{1,2} \times S^1$, it plays a role in center stabilization, but otherwise it is still decoupled from the dynamics of light states. So for light states we deal with standard QCD if we set $N_c = 3$ and $N_f = 2, 3$. The idea of adiabatic continuity is to find a way to put an asymptotically-free gauge theory on

$\mathbb{R}^{1,2} \times S^1$ in such a way that its dependence on the spatial circle size L is smooth. If this condition is met, then one can get insight about the behavior of the theory for large L, where it is strongly coupled, by studying it for small L, where it is weakly coupled. Satisfying the adiabaticity condition has been especially challenging[9, 21] in theories with continuous chiral symmetries, but in this paper we present a method to ensure it, allowing us to address chiral symmetry breaking.

Large L expectations. If $m_q = 0$, the quantum theory has the global symmetry

$$G = SU(N_f)_L \times SU(N_f)_R \times U(1)_Q. \tag{1}$$

Here, we already factored out the anomalous $U(1)_A$ which reduces to $\mathbb{Z}_{2N_f}$ due to instanton effects. It is believed that G breaks spontaneously to $SU(N_f)_V \times U(1)_Q$, and the low-energy dynamics of the resulting NGBs are described by chiral perturbation theory

$$\mathcal{L}_{\chi PT} = \frac{f_\pi^2}{4} \text{tr} \, |\partial_\mu \Sigma|^2 - c \, \text{tr} \, (M_q^\dagger \Sigma + M_q \Sigma^\dagger) + \cdots, \tag{2}$$

where $\Sigma = e^{i\Pi/f_\pi}$ and $M_q = m_q 1_{N_f}$ is the quark mass spurion field, transforming as $\Sigma \rightarrow L \Sigma R^\dagger, M_q \rightarrow L M_q R^\dagger$ under chiral rotations. Equation (2) implies that the NGB masses obey the famous Gell-Mann-Oakes-Renner [22] (GMOR) relation $f_\pi^2 m_\pi^2 = -m_q \langle \bar{\psi}\psi \rangle$, where $\langle \bar{\psi}\psi \rangle = -2c$.

If generic imaginary chemical potentials μ are turned on for the Cartan subgroup $U(1)_L^{N_f - 1} \times U(1)_R^{N_f - 1}$ of G then $N_f - 1$ NGBs remain gapless, while the the rest pick up mass gaps $\sim |\mu|$ in the chiral limit $m_q = 0$. So, for instance, for $N_f = 2$, turning on an isospin chemical potential μ_I corresponds to shifting $\partial_\mu \Sigma \rightarrow \partial_\mu \Sigma + i[\mu_I \frac{\tau_3}{2} \delta_{\mu,0}, \Sigma]$, see e.g. [23]. With an imaginary μ_I the $\pi_\pm$ fields get positive mass gaps $\sim |\mu_I|$, while the π_0 remains gapless.

Vacuum structure at small L. To compactify the theory on S^1 we must choose boundary conditions for the

fields. As usual, we take the gluons $A_M, M = 1, \ldots, 4$ to be periodic. For the fermions, there are many more choices: one can demand periodicity up to global symmetry transformations, which is a topic which has been explored both in lattice QCD and continuum QFT discussions, see e.g. [24–39]. Twisted boundary conditions for fermions are equivalent to turning on expectation values for holonomies $\Omega_{\mathcal{F}} = \mathrm{P}e^{i\int_{S^1} \mathcal{A}_4}$ of background flavor gauge fields $\mathcal{A}_M$ valued in the Lie algebra of Eq. (1), and can also be interpreted as turning on imaginary flavor chemical potentials $\mu \sim 1/L$ [26].

In a gapped theory, one expects the effects of twisted boundary conditions to become negligible when $L\Lambda \gg 1$, but they can become important when $L\Lambda \ll 1$. In particular, it turns out that the circle-size dependence with some special choices of twists is smoother than the one associated to taking standard periodic or anti-periodic boundary conditions. The standard boundary conditions correspond to a trivial flavor background holonomy, and with this choice the theory has $N_f^2 - 1$ NGBs at large L, and no NGBs at small L, indicating the existence of a chiral phase transition at some intermediate value of L. But as we will see in this paper, one can choose a certain non-trivial background flavor holonomy that eliminates this phase transition. The way this works is that with the non-trivial holonomy the gaps of the lightest charged pseudoscalar states smoothly increase as the circle size decreases, but the lightest neutral pseudoscalar states remain gapless both at large L and small L. Indeed these neutral pseudoscalars are precisely the neutral NGBs of spontaneous χSB. The large L story was already given above, and we focus on the small L behavior in what follows.

Once $L\Lambda \ll 1$ the 4D gauge coupling at the scale $1/L$ becomes small, and the theory comes under semiclassical control, provided that the long-distance dynamics becomes Abelian. Whether this happens depends on the expectation value for the gauge holonomy in the compact direction $\Omega = \mathrm{P}e^{i\int_{S^1} A_4}$. At large L, we expect the "center-symmetric" result $\langle \mathrm{tr}\,\Omega^n \rangle \simeq 0, n = 1, \cdots, N_c - 1$ due to strong eigenvalue fluctuations (despite the lack of an exact center symmetry unless $N_f = 0$ or $N_f = N_c$, see [32, 38]). On the other hand, at small L the eigenvalue fluctuations become small, and $\langle \mathrm{tr}\,\Omega^n \rangle$ is determined by the minimum of the Wilsonian effective potential. The role of the massive λ field is to produce a $\mathbb{Z}_{N_c}$ center-symmetry-stabilizing contribution to the potential[8, 40–44], which can also be achieved by double-trace deformations [8]. The minimum of the potential is then attained with a $\mathbb{Z}_{N_c}$-center-symmetric configuration $\Omega \sim \mathrm{diag}(1, \omega, \cdots, \omega^{N_c-1})$, where $\omega = e^{2\pi i/N_c}$.

Unlike the VEV of Ω, which is determined dynamically, the flavor holonomy is an external parameter and can be chosen at will. As emphasized above and in [45–50], some choices are better than others in the context of adiabatic continuity, and can lead to a smoother passage
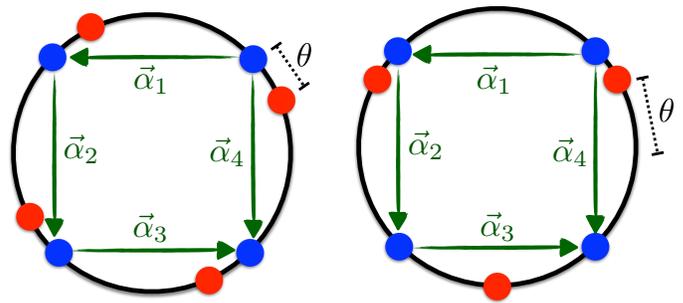

FIG. 1. Eigenvalue distributions for the gauge holonomy Ω (blue dots) $N_c = 4$ and flavor holonomy Ω_F (red dots) with $N_f = 4$ (on the left) and $N_f = 3$ (on the right). The $U(1)_Q$ holonomy $\Omega_Q = e^{i\theta}$ rotates Ω_F and Ω eigenvalues relative to each other. The green arrows represent the four monopole-instanton events, which are labeled by the affine roots $\vec{\alpha}_i$. Each time an arrow "passes" a flavor holonomy eigenvalue, the associated monopole-instanton operator picks up two fermion zero modes. So in the left $N_f = 4$ figure, all monopole instantons have two zero modes, while in the right $N_f = 3$ figure $\vec{\alpha}_1$ has no zero modes while $\vec{\alpha}_{2,3,4}$ each have two zero modes.

from large L to small L. At the two extremes, we can choose a trivial flavor-holonomy $\Omega_F = \mathbf{1}_{N_f}$, or a non-trivial flavor-holonomy $\Omega_F \sim \mathrm{diag}(1, \omega_F, \cdots, \omega_F^{N_f-1})$, where $\omega_F = e^{2\pi i/N_f}$, which preserves a $\mathbb{Z}_{N_f}$ flavor-center symmetry. Center-symmetric gauge and flavor holonomy configurations are illustrated in Fig. 1. We will see that choosing Ω_F to be $\mathbb{Z}_{N_f}$ symmetric has rather dramatic implications.

At small L, A_4 acts as an compact adjoint Higgs field. At long distances the center-symmetric VEV for Ω leads to an Abelianization of the dynamics:

$$SU(N_c) \to U(1)^{(N_c-1)}. \qquad (3)$$

This is similar to what happens in the 3D Polyakov model [51] and Seiberg-Witten analysis of $\mathcal{N} = 1$ and $\mathcal{N} = 2$ super-Yang-Mills (SYM) theories [52], but the fact that A_4 is a compact variable leads to important differences in the physics. The Abelianization happens at the scale of the lightest W-boson mass $m_W = \frac{2\pi}{LN_c}$.

Perturbative small L physics. In perturbation theory the Abelianized small L regime can be described by a 3D effective field theory. The lightest fields are the Cartan gluons A_μ^i, where $i = 1, \ldots, N_c - 1$ and from now on $\mu = 1, 2, 3$. In fact A_μ^i are massless to all orders in perturbation theory. This is easiest to see by using an Abelian duality transformation $F_{\mu\nu}^i = g^2/(2\pi L)\epsilon_{\mu\nu\alpha}\partial^\alpha \sigma^i$ relating the Cartan gluons to 3D scalars σ^i, which we call "dual photons". If we write $\sigma_i = \vec{\alpha}_i \cdot \vec{\sigma}$, where $\vec{\alpha}_i$ are the simple roots of the algebra $\mathfrak{su}(N_c)$ and $\vec{\cdot}$ denotes an N_c-vector, then the dual photons enjoy an emergent topological shift symmetry

$$[U(1)_J]^{N_c-1} : \vec{\sigma} \to \vec{\sigma} + \vec{\epsilon}, \qquad \vec{\mathcal{J}}_\mu = \partial_\mu \vec{\sigma} \qquad (4)$$

in perturbation theory. The Noether currents $\vec{\mathcal{J}}_\mu$ can be identified with the (Euclidean) magnetic field $\vec{B}_\mu = \epsilon_{\mu\nu\alpha}\vec{F}^{\nu\alpha}$ using the Abelian duality relation, and current conservation is just the statement of the absence of magnetic monopoles $\partial_\mu\vec{\mathcal{J}}_\mu = \partial_\mu\vec{B}_\mu = 0$. Therefore, to all orders in perturbation theory, the dual photons must remain gapless. Their action looks like

$$S_\sigma = \int d^3x \, \frac{g^2}{8\pi^2 L}(\partial_\mu\vec{\sigma})^2. \tag{5}$$

Quarks are charged under the Cartan subgroup of $SU(N_c)$, so one might be concerned that the resulting 3D QED theory, with gauge coupling $g_3^2 = g_{\mathrm{YM}}^2(m_W)/L$, might flow to strong coupling in the far infrared if $m_q = 0$. However, this does not happen for generic values of the $U(1)_Q$ holonomy $\Omega_Q = e^{i\theta}$. The long-distance perturbative action for the quarks is

$$S_\psi = L \int d^3x \left\{ \bar{\psi}_L \left[\gamma^\mu D_\mu + \gamma^4(A_4 + \mathcal{A}_4 + \theta/L) \right] \psi_L \right. \tag{6}$$
$$\left. + m_q\bar{\psi}_L\psi_R + L \leftrightarrow R \right\}.$$

So even if $m_q = 0$, the center-symmetric color and flavor holonomies give $\psi_{n,a}$ a chirally-invariant gap $\frac{2\pi}{L}(\frac{n}{N_c} + \frac{a}{N_f} + \theta)$. So long as $\theta \gtrsim g_3^2 L = g_{\mathrm{YM}}^2$, within perturbation theory all of the quarks decouple from the Cartan gluons at long distances, and there is no flow to strong coupling.

Monopole-instantons. Due to Eq. (3) our theory has N_c types of monopole-instantons [8, 53–55]. In the center-symmetric background, they all have identical Euclidean actions $S_0 = \frac{S_{\mathcal{I}}}{N_c} = \frac{8\pi^2}{g^2 N_c}$, where $S_{\mathcal{I}} = \frac{8\pi^2}{g^2}$ is the 4d instanton action. In this setting the 4d instanton is a composite configuration built from the N_c monopole-instantons. These solutions are associated with the affine root system of $\mathfrak{su}(N_c)$ Lie algebra: $N_c - 1$ correspond to the simple roots, while the remaining one is associated with the affine root, and is due to the compact nature of the adjoint Higgs field [53, 54].

To understand the contributions of these finite-action field configurations, we note that monopole-instantons carry two types of topological quantum numbers, magnetic and topological charge $(Q_m, Q_T) = \left(\frac{g}{4\pi}\int_{S_2} F \cdot dS, \frac{1}{16\pi^2}\int \mathrm{tr}\, F_{\mu\nu}\widetilde{F}^{\mu\nu} \right)$, given by $(Q_m, Q_T) = \pm(\alpha_i, 1/N_c)$ for monopoles and anti-monopoles.

These field configurations couple to the dual photons, and, if $N_f = 0$ — that is, in pure Yang-Mills theory deformed by a heavy adjoint fermion — the associated amplitudes take the form

$$\mathcal{M}_i = e^{-S_0}e^{i\vec{\alpha}_i \cdot \vec{\sigma}}, \text{ no fermions}. \tag{7}$$

The usual 4d instanton amplitude is

$$\mathcal{I}_{4d} \sim \prod_{i=1}^{N_c} \mathcal{M}_i \sim e^{-\frac{8\pi^2}{g^2}}, \text{ no fermions}. \tag{8}$$

The 4d instanton has vanishing magnetic charge, and does not couple to dual photons. Consequently, it does not play any important role in the long-distance dynamics, which is determined by the monopole-instantons. The fact that the monopole-instantons carry magnetic charge means that, for $N_f = 0$, their proliferation in the vacuum leads to an explicit non-perturbative breaking of the topological $[U(1)_J]^{N_c-1}$ shift symmetry of the σ_i fields, $\partial_\mu\vec{\mathcal{J}}_\mu = \partial_\mu\vec{B}_\mu = \vec{\rho}_m$, where $\vec{\rho}_m$ is the (non-perturbative) magnetic charge density for the monopole-instantons. The monopoles and anti-monopoles generate a potential $V(\vec{\sigma}) \sim -m_W^3 e^{-S_0}\sum_i \cos(\vec{\alpha}_i \cdot \vec{\sigma})$, and fluctuations around its minimum $\vec{\sigma} = 0$ have mass $m_\sigma^2 \sim m_W^2 e^{-S_0}$, as discussed in Ref. [8], so that the deformed YM theory develops a non-perturbative mass gap. The long-distance $N_f = 0$ theory exhibits both confinement of electric charge with finite string tension and a non-perturbative mass gap for all gauge fluctuations.

Fermion zero modes. When $m_q = 0$, the chiral anomaly implies the presence of $2N_f$ fermion zero modes in the instanton background. This modifies the instanton amplitude from Eq. (8) to [56]

$$\mathcal{I}_{4d} \sim e^{-\frac{8\pi^2}{g^2}} \det_{a,b}\left[\bar{\psi}_{L,a}\psi_{R,b}\right] \tag{9}$$

which is invariant under $\mathbb{Z}_{2N_f}$ but not $U(1)_A$. However, the 't Hooft vertex in Eq. (9) is invariant under G from Eq. (1) because these symmetries are not anomalous, and must be preserved in all effective interaction vertices. The non-perturbative generation of explicit $U(1)_A$ breaking interactions in Eq. (9) and Eq. (10) below imply that no massless η' mode should be expected, see [57] and also [58] for a discussion of the $U(1)_A$ problem in a setting close to our current context.

At small S^1, the 4d instanton splits into monopole-instantons, and if $\Omega_F \sim 1_{N_f}$ and $\psi_j(x_4 + L) = e^{i\theta}\psi_j(x_4)$, the Nye-Singer index theorem implies that all the fermion zero modes localize on a single monopole [59, 60], say, $\mathcal{M}_1$. Then the monopole amplitudes are given by

$$\mathcal{M}_1 = e^{-S_0}e^{i\vec{\alpha}_1 \cdot \vec{\sigma}} \det_{a,b}\left[\bar{\psi}_{L,a}\psi_{R,b}\right]$$
$$\mathcal{M}_i = e^{-S_0}e^{i\vec{\alpha}_i \cdot \vec{\sigma}}, \qquad i = 2, \ldots N_c \tag{10}$$

All $\mathcal{M}_i$ are invariant under G from Eq. (1), and $\prod_i \mathcal{M}_i$ is equivalent to Eq. (9). With the choice $\Omega_F \sim 1_{N_f}$ there is no χSB at weak coupling. The reason is that all $2N_f$ fermi zero modes are localized at one monopole-instanton, and the semi-classical weight e^{-S_0} is not large enough to break chiral symmetry. In this case, there must be a chiral phase transition between small L and large L at the scale $L = \Lambda^{-1}/N_c$, as explained in [21].

As the $U(1)_Q$ holonomy is dialed, the $2N_f$ zero modes *jump collectively* from one monopole instanton to the next at discrete values of θ, e.g, at $\theta = \frac{2\pi k}{N_c}, k \in \mathbb{Z}$ for $N_f = N_c$ [61–63]. In other words, the index is only a piece-wise constant function, and exhibits jumps with respect to θ. This is interpreted as wall-crossing in [64].
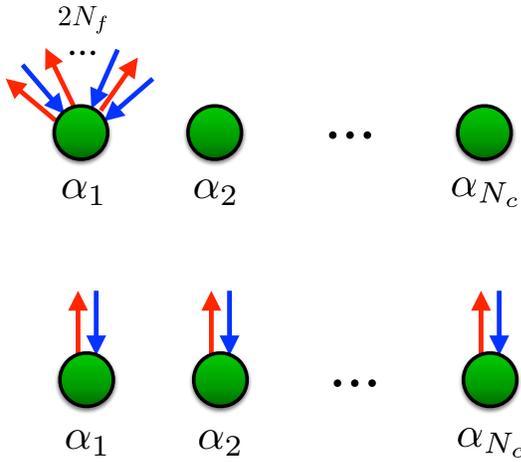
FIG. 2. The distribution of fundamental fermion zero modes for the monopole-instantons with a trivial flavor holonomy (top row) and a $\mathbb{Z}_{N_f}$-symmetric flavor holonomy (bottom row) for $N_f = N_c$. The latter distribution is identical to $\mathcal{N} = 1$ SYM on $\mathbb{R}^3 \times S^1$ in which monopole-instantons saturate the chiral condensate.

Fractional jumping. The main new idea of this work is a refinement of collective jumping. If we dial the Ω_F holonomy from the trivial configuration $\Omega_F = 1_{N_f}$ toward a configuration that respects the flavor center symmetry $\mathbb{Z}_{N_f}$, this shifts the frequency quantization of different flavor components, and leads to *fractional jumping* (in units of two) of the $2N_f$ fermionic zero modes onto the N_c monopole-instantons as evenly as possible. A similar phenomenon occurs in the 2D Schwinger model[65]. This fractional jumping phenomenon, as well as the collective jumping, is illustrated in Fig. 1. At long distances the twisted boundary conditions result in an explicit breaking of the global symmetry G down to its maximal Abelian subgroup

$$G_{\text{max-ab}} = [U(1)_V \times U(1)_A]^{N_f - 1} \times U(1)_Q. \quad (11)$$

With the twist, N_f of the monopole operators carry two fermi zero modes each, while the other $N_c - N_f$ monopole-instantons have no fermion zero modes. The monopole amplitudes are, schematically,

$$\mathcal{M}_i = e^{-S_0} e^{i\vec{\alpha}_i \cdot \vec{\sigma}} (\bar{\psi}_{L,i} \psi_{R,i}), \qquad i = 1, \ldots N_f, \quad (12)$$

$$\mathcal{M}_k = e^{-S_0} e^{i\vec{\alpha}_k \cdot \vec{\sigma}}, \qquad k = N_f + 1, \ldots N_c. \quad (13)$$

and are illustrated in Fig. 2. The individual zero mode vertices $(\bar{\psi}_{L,i} \psi_{R,i})$ transform non-trivially under $[U(1)_A]^{N_f - 1}$, and naively this could lead one to the puzzling conclusion that there is an anomaly for the corresponding $U(1)_A$ factors. But this is impossible, because $G_{\text{max-ab}}$ is a subgroup of a non-abelian chiral symmetry, which is non-anomalous.

The resolution of this puzzle comes from an axial-topological symmetry intertwining effect, also seen in

Ref. [12]. The $[U(1)_A]^{N_f - 1}$ symmetry intertwines with $[U(1)_J]^{N_f - 1}$ subgroup of Eq. (4), such that the symmetry phase of the fermion bilinear is cancelled exactly by a shift of the dual photon field, $\vec{\sigma}$:

$$\begin{aligned} (\bar{\psi}_{L,k} \psi_{R,k}) &\rightarrow e^{i\epsilon_k} (\bar{\psi}_{L,k} \psi_{R,k}), \\ e^{i\vec{\alpha}_k \cdot \vec{\sigma}} &\rightarrow e^{-i\epsilon_k} e^{i\vec{\alpha}_k \cdot \vec{\sigma}}. \end{aligned} \quad (14)$$

In this way, the monopole-instanton amplitude is invariant under the expected symmetries.

Since the N_f parameters ϵ_k must satisfy $\sum_{k=1}^{N_f} \epsilon_k = 0$ we can write $\epsilon_k = \boldsymbol{\alpha}_k \cdot \boldsymbol{\epsilon}$ where $\boldsymbol{\alpha}_k$ are the simple roots of $\mathfrak{su}(N_f)$ and bold symbols denote N_f-vectors. Writing $\vec{\sigma} = (\boldsymbol{\sigma}, \tilde{\sigma})$, where $\tilde{\sigma}$ is an $N_c - N_f$ vector, we find that $\boldsymbol{\sigma}$ enjoys an *unbroken* shift symmetry,

$$\boldsymbol{\sigma} \rightarrow \boldsymbol{\sigma} + \boldsymbol{\epsilon}, \quad (15)$$

while the shift symmetry of $\tilde{\sigma}$ is *explicitly* broken.

Chiral symmetry breaking. Now consider the fluctuations of $\boldsymbol{\sigma}$ around any point on its vacuum manifold $U(1)^{N_f - 1}$, along with the fluctuations of $\tilde{\sigma}$ around the bottom of its potential at $\tilde{\sigma} = 0$.

The choice of a point on the vacuum manifold *spontaneously* breaks the $[U(1)_A]^{N_f - 1}$. Then Eq. (15) implies that $N_f - 1$ dual photons enjoy a shift symmetry and remain *gapless* non-perturbatively. A more microscopic way to see is to note that all magnetically-charged topological molecules which couple to $\boldsymbol{\sigma}$ have uncompensated fermion zero modes, so here there is no analog of the magnetic bion mechanism of mass generation from adjoint QCD[7]. Thus the gapless dual photons become the "pions" of χSB, similarly to the phenomena seen in e.g. Ref. [66]. The remaining $N_c - N_f$ dual photons $\tilde{\sigma}$ pick up masses $m_\sigma \sim m_W e^{-S_0/2}$, as in pure YM theory.

It is also worth noting that in $\mathcal{N} = 2$ SYM theory compactified on $\mathbb{R}^3 \times S^1$ [67], the gaplessness of the dual photon is due to an identical mechanism of axial-topological symmetry intertwining. In Ref. [67], however, supersymmetry implies that an entire $\mathcal{N} = 2$ multiplet remains gapless, and there is a quantum moduli space associated to the elementary scalar fields as well. In our context, only the dual photons are protected, and there are no elementary scalars. Nevertheless, it is intriguing to see such close parallels between a theory with extended supersymmetry and QCD.

The basic physical phenomenon is that gapless dual photons are transmuted into the NGBs of the spontaneously broken Abelian chiral symmetry. This is the chiral symmetry group that remains exact with our choice of twisted boundary conditions. Indeed, if we define $\Pi' = \sum_{a=1}^{N_f} \pi_a T_a$, where T_a are the Cartan generators of $\mathfrak{su}(N_f)$, $\pi_a = g/(2\pi L)\sigma_a$, and $\Sigma' = e^{i\Pi'/f_\pi}$, then the action for $\boldsymbol{\sigma}$ can be written as

$$S_{\boldsymbol{\sigma}} = L \int d^3 x \left[\frac{f_\pi^2}{4} \text{tr} \, \partial_\mu \Sigma' \partial^\mu \Sigma'^\dagger - c \, \text{tr} \, (M_q^\dagger \Sigma' + \text{h.c.}) \right], \quad (16)$$

with the derivation of the c term given below. This precisely matches the dimensional reduction of Eq. (2) to 3D, with Σ' interpreted as the restriction of the $SU(N_f)$-valued chiral field Σ to its maximal torus, which parametrizes the exactly massless NGBs. In the $L\Lambda \ll 1$ regime the value of f_π can be calculated by gauging $G_{\text{max-ab}}$, giving

$$f_\pi^2 = \left(\frac{g}{\pi L \sqrt{6}}\right)^2 = \frac{N_c \, \lambda \, m_W^2}{24\pi^4}\,, \qquad (17)$$

where $\lambda = g^2 N_c$. This implies that starting from the weak-coupling, small-L, side of compactified QCD we have been able to derive the chiral Lagrangian expected from the strong coupling, large-L side. This supports the expectation that the compactification is indeed adiabatic, with no phase transitions as a function of L.

Constituent quark masses and the chiral condensate. We now work out the effect of turning on light quark masses, as well as the origin of the "constituent quark" masses. First, consider turning on a small "current" quark mass m_q by adding $\mathcal{L}_m = m_q(\bar{\psi}_L \psi_R + \text{h.c.})$ to the 4D Lagrangian. The contribution of $\mathcal{L}$ to the 3D effective action can be obtained by saturating the integral over the fermion zero modes with a single mass insertion,

$$\mathcal{L}_{m,\text{eff}} = m_W^2 m_q e^{-S_0} e^{i\boldsymbol{\sigma}_k \cdot \boldsymbol{\sigma}} + \text{h.c.}\,, \qquad (18)$$

which leads to the c term in Eq. (16). The VEV of the monopole operator determines the vacuum energy density $\mathcal{E} = -c\,\text{tr}\,[M_q + \text{h.c.}]$ and the chiral condensate. We get $\langle \bar{\psi}\psi \rangle = -2m_W^3 e^{-S_0}$. The GMOR relation $f_\pi^2 m_\pi^2 = -m_q \langle \bar{\psi}\psi \rangle$ is of course also reproduced.

We note that if $N_f = N_c$ there are some remarkable relations to $\mathcal{N} = 1$ SYM. In both theories, all N_c monopole-instantons acquire 2 fermion zero modes, the leading order beta function is $b = 3N_c$ (on scales below m_λ), and the chiral condensate is saturated by the monopole-instantons. These observations may be the microscopic explanation of why the chiral condensate in $\mathcal{N} = 1$ SYM is so close to the one in real QCD [68].

Next, consider the notion of a constituent quark mass. For vanishing m_q the long-distance fermion effective Lagrangian includes the term $\sum_{k=1}^{N_f}(m_W e^{-S_0} e^{i\boldsymbol{\alpha}_k \cdot \boldsymbol{\sigma}} \bar{\psi}_{L,k} \psi_{R,k} + \text{h.c.})$. So once the magnetic flux part $e^{i\boldsymbol{\alpha}_k \cdot \boldsymbol{\sigma}}$ of the monopole-instanton operator acquires a VEV, the lightest quark modes pick up a non-perturbative chiral-symmetry breaking "constituent quark" mass $m_{\text{constituent}} \sim m_W e^{-S_0}$.

Outlook. We have described a new non-perturbative mechanism for χSB in 4D QCD which operates at weak coupling, and hence is under full theoretical control. The calculation is based on adiabatic compactification, and a key ingredient is the use of a $\mathbb{Z}_{N_f}$-symmetric flavor holonomy. Spontaneous chiral symmetry breaking arises from monopole-instanton vertices by color-flavor transmutation, and the NG pions originate from massless dual photons. This mechanism is somewhat reminiscent of chiral

symmetry breaking via color-flavor locking in high density QCD. We also emphasize that our mechanism operates within the regime where monopole-instanton gas is dilute, which is a major difference from phenomenological instanton liquid models. We provided a microscopic derivation of the chiral Lagrangian. The structure of the Lagrangian supports the idea that the small S^1 regime and the $\mathbb{R}^4$ limit are continuously connected.

Our results open a new playground for the exploration of 4D non-supersymmetric gauge theory dynamics in a fully calculable setting with confinement and χSB. Some of the issues which are ripe for exploration include generalizations to $N_f > N_c$, to other gauge groups, χSB in chiral gauge theories, explorations of the spectrum of non-Goldstone excitations, and many more.

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