Exact Short-Time Height Distribution in the One-Dimensional Kardar-Parisi-Zhang Equation and Edge Fermions at High Temperature

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Exact short-time height distribution in 1D KPZ equation and edge fermions at high temperature

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We consider the early time regime of the Kardar-Parisi-Zhang (KPZ) equation in $1+1$ dimensions in curved (or droplet) geometry. We show that for short time t, the probability distribution $P(H,t)$ of the height H at a given point x takes the scaling form $P(H,t) \sim \exp\left(-\Phi_{\rm drop}(H)/\sqrt{t}\right)$ where the rate function $\Phi_{\rm drop}(H)$ is computed exactly for all H. While it is Gaussian in the center, *i.e.*, for small H, the PDF has highly asymmetric non-Gaussian tails which we characterize in detail. This function $\Phi_{\rm drop}(H)$ is surprisingly reminiscent of the large deviation function describing the stationary fluctuations of finite size models belonging to the KPZ universality class. Thanks to a recently discovered connection between KPZ and free fermions, our results have interesting implications for the fluctuations of the rightmost fermion in a harmonic trap at high temperature and the full couting statistics at the edge.

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It is by now well known that many stochastic growth models in one dimension belong to the celebrated Kardar-Parisi-Zhang (KPZ) universality class [1–3]. These models are usually described by a field $h(x,t)$ that denotes the height of a growing interface at point x at time t. At the center of this class resides the continuum KPZ equation [1] where the height evolves as

$$\partial_t h = \nu\, \partial_x^2 h + \frac{\lambda_0}{2}\,(\partial_x h)^2 + \sqrt{D}\,\xi(x,t)\ , \tag{1}$$

where $\xi(x,t)$ is a Gaussian white noise with zero mean and $\langle \xi(x,t)\xi(x',t')\rangle = \delta(x-x')\delta(t-t')$. We use everywhere the natural units of space $x^* = (2\nu)^3/(D\lambda_0^2)$, time $t^* = 2(2\nu)^5/(D^2\lambda_0^4)$ and height $h^* = \frac{2\nu}{\lambda_0}$. At late times in all these growth models, including the KPZ equation itself, while the average height increases linearly with t, the typical fluctuations around the mean height grow as $\sim t^{1/3}$ [2]. Moreover even the probability distribution function (PDF) of the centered and scaled height is universal and is described by the Tracy-Widom (TW) [4] and Baik-Rains [6, 7] distributions, with a parameter that depends on the class of initial conditions (flat, droplet, stationary) [3, 5–11]. Some of these predictions have also been verified in experiments [12–14]. Recently, these late times studies have been extended to compute exactly the large deviations of the height distribution going beyond the TW regime [15], uncovering a third order phase transition [16, 17] in growth models belonging to the KPZ universality class, including the KPZ equation itself.

While the distribution of the height fluctuations is thus well understood at late times, in this Letter we study exactly the full distribution of the height fluctuations at small times $t \ll 1$ in the KPZ equation growing in the curved geometry. Remarkably, our short time results are directly relevant to an apriori different *quantum* problem of N non-interacting fermions in a one-dimensional harmonic trap. A recent mapping was indeed established between these two problems [18], where the time t in the KPZ equation corresponds to the temperature T in the fermion problem via the relation $t \sim N/T^3$ for large N [18]. Thus at $T=0$ ($t\to\infty$ limit in the KPZ problem), the rightmost fermion position is described by the TW distribution. Our results for short times t in the KPZ equation thus provide exact predictions for the distribution of the rightmost fermion position at high temperature, providing a generalization of the $T=0$ TW distribution.

Before presenting our exact results at short times, it is useful first to recall the late time results in the KPZ equation in the curved geometry. The distribution $P(H,t)$ of the height H at a given space point (suitably centered) takes the form at large times [15]

$$P(H,t) \sim \begin{cases} e^{-t^2\Phi_-(H/t)}\ , \ \Phi_-(z) = \dfrac{|z|^3}{12}\ ,\ z<0 & (2)\\[2mm] e^{-t\Phi_+(H/t)}\ ,\ \ \Phi_+(z) = \dfrac{4}{3}z^{3/2}\ ,\ z>0 & (3)\end{cases}$$

while the central region $H \sim t^{1/3}$ is governed by the TW distribution associated with the Gaussian Unitary Ensemble (GUE). The result in the right tail in (3), also holds for the flat initial condition.

It is then natural to wonder: are these tails $P(H,t)\sim e^{-|H|^3/(12t)}$ (H large negative) and $\sim e^{-\frac{4}{3}H^{3/2}/t^{1/2}}$ (H large positive) visible only at late times, or do they appear even at early times? It is well known that the central typical regime at early times is described by a Gaussian – obtained from the Edwards-Wilkinson's (EW) equation [19] setting $\lambda_0 = 0$ in the KPZ equation. What about the tails? Recently, Meerson et al. [20] studied this question for the flat initial condition using the weak noise theory (WNT) (see also earlier results [21]), valid for short

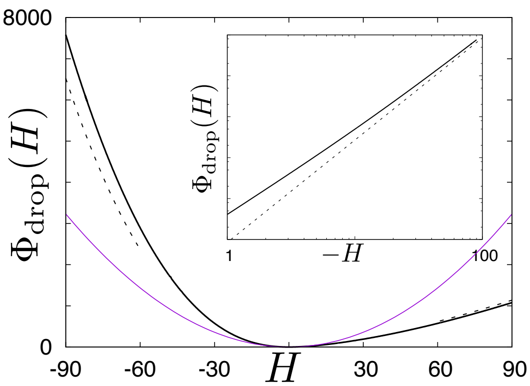

FIG. 1. The rate function $\Phi_{\rm drop}(H)$ (solid black line) which describes the distribution (4) of the KPZ height $H = H(t)$ at small time obtained in (20). The dashed black lines correspond respectively to the left and right tails given in (5), (7) and the solid (purple) line corresponds to the Edward-Wilkinson Gaussian regime (6). **Inset**: Log-log plot of the left tail compared with the asymptotics (5).

times. For the right tail they found $\sim e^{-\frac{4}{3}H^{3/2}/t^{1/2}}$, i.e. the same leading order result as late times. This shows that the asymptotic right tail is established even at early times. In contrast, for the left tail they found $P(H,t) \sim e^{-\frac{8}{15\pi}|H|^{5/2}/t^{1/2}}$ at early times (in our units). This $|H|^{5/2}$ tail behavior for large negative H for flat initial condition is manifestly different from the $|H|^3$ tail behavior at late times for the droplet initial condition. This raises the question whether this difference is due to the change in initial conditions, or whether early and late time left tails are different for any given initial condition – providing yet another motivation for studying the full PDF $P(H,t)$ at short times in the curved geometry.

In this Letter, we show that the early time PDF $P(H,t)$ for the droplet initial condition takes the form

$$P(H,t) \sim \exp\left(-\frac{\Phi_{\rm drop}(H)}{\sqrt{t}}\right), \tag{4}$$

which holds for any value of H and small time $t \ll 1$, and where $\Phi_{\rm drop}(H)$ is given explicitly by Eq. (20) below. The asymptotic behaviors of $\Phi_{\rm drop}(H)$ are obtained as

$$\Phi_{\rm drop}(H) \simeq \begin{cases} \frac{4}{15\pi}|H|^{5/2} \quad , \quad H \to -\infty & (5) \\ \frac{H^2}{\sqrt{2\pi}} \quad , \quad |H| \ll 1 & (6) \\ \frac{4}{3}H^{3/2} \quad , \quad H \to +\infty \, . & (7) \end{cases}$$

The first three cumulants of H obtained from Eqs. (4) and (20) are in agreement with the leading small time behavior obtained in [9], while here we obtain all cumulants. In the case of the flat initial condition one expects a similar form [20], $P(H,t) \sim e^{-\frac{\Phi_{\rm flat}(H)}{\sqrt{t}}}$. However, the function $\Phi_{\rm flat}(H)$ has not been obtained explicitly apart from the tails [22, 23] and the first two cumulants [20, 24]. Therefore the $|H|^{5/2}$ left tail seems to hold for a variety of initial conditions. Interestingly, as discussed below, our main results, Eqs. (4) and (20), turn out to be very reminiscent of the universal large deviation fluctuations obtained in the stationary regime of finite-size models in the KPZ universality class [25–31].

In addition, in the context of finite temperature free fermions in a harmonic potential, our results provide exact predictions for the fluctuations of the position of the *rightmost fermion* $\mathsf{x}_{\rm max}$ near the edge $\mathsf{x}_{\rm edge}$ of the Fermi gas, via the following relation [18]

$$\frac{\mathsf{x}_{\rm max} - \mathsf{x}_{\rm edge}}{w_N} \equiv_{\rm in\,law} \frac{h(0,t) + \frac{t}{12} + \gamma}{t^{1/3}} \tag{8}$$

where $\equiv_{\rm in\,law}$ means identical PDF's. Here, on the l.h.s. N is large, $\mathsf{x}_{\rm edge} = \sqrt{2N}$ and $w_N = N^{-1/6}/\sqrt{2}$. On the r.h.s. γ is a Gumbel distributed random variable with PDF given by $P(\gamma) = e^{-\gamma - e^{-\gamma}}$, independent of the height $h(0,t)$. This equivalence in law is valid in the limit $N \to +\infty$, $T \to +\infty$ but with the ratio $t = N/T^3$ fixed. This leads to an exact prediction (27) for the high temperature behavior of the rightmost fermion.

For definiteness we focus on the narrow wedge initial condition, $h(x,0) = -|x|/\delta - \ln(2\delta)$, with $\delta \ll 1$. This initial condition gives rise to a curved (or *droplet*) mean profile as time evolves [3, 8–11]. We focus on the shifted height at a given space point, and define [32]

$$H(t) = h(x,t) + \frac{x^2}{4t} + \frac{t}{12} + \frac{1}{2}\ln(4\pi t) \, . \tag{9}$$

The starting point of our calculation is the exact formula, valid for all times t, for the following generating function, obtained in [8–11]

$$\left\langle \exp\left(-\frac{e^{H(t)-st^{1/3}}}{\sqrt{4\pi t}}\right)\right\rangle = Q_t(s) \tag{10}$$

$$Q_t(s) := {\rm Det}[I - P_0 K_{t,s} P_0] \tag{11}$$

where $\langle \dots \rangle$ denotes an average over the KPZ noise. Here $Q_t(s)$ is a Fredholm determinant associated to the kernel

$$K_{t,s}(r,r') := \int_{-\infty}^{+\infty} du\, Ai(r+u)Ai(r'+u)\sigma_{t,s}(u) \tag{12}$$

defined in terms of the Airy function $Ai(x)$ and the weight functions

$$\sigma_{t,s}(u) := \sigma(t^{1/3}(u-s)) \quad , \quad \sigma(v) := \frac{1}{1+e^{-v}} \, . \tag{13}$$

In (11), P_0 denotes the projector on the interval $r \in [0, +\infty[$. Even though the formal relation in Eq. (10) is valid at all times t, extracting from it the height distribution $P(H,t)$ explicitly is very hard. Previously, exact results have been extracted only for large t, both for typical [8–11] and atypical [15] height fluctuations. Below, we show that the full height distribution $P(H,t)$ for all H can be extracted from (10) at short time $t \ll 1$.

It is convenient to introduce the kernel

$$\bar{K}_{t,s}(u,u') = K_{\rm Ai}(u,u')\sigma_{t,s}(u') \tag{14}$$

defined in terms of the Airy kernel

$$K_{\rm Ai}(u,u') = \int_0^{+\infty} dr Ai(r+u)Ai(r+u') \,. \tag{15}$$

From (14) and (15), one checks that $\mathrm{Tr}\, \bar{K}^p_{t,s} = \mathrm{Tr}(P_0 K_{t,s} P_0)^p$ for any integer $p \geq 1$, which allows us to rewrite [33]

$$\ln \mathrm{Det}[I - P_0 K_{t,s} P_0] = \ln \mathrm{Det}[I - \bar{K}_{t,s}] = \sum_{p=1}^{+\infty} \frac{-1}{p} \mathrm{Tr}\, \bar{K}^p_{t,s} \tag{16}$$

a convenient form to study the small t limit. Indeed, as detailed in [34], we show that, in the limit $t \to 0$ and $s \to \infty$, keeping $\tilde{s} = s\, t^{1/3}$ fixed, the traces $\mathrm{Tr}\, \bar{K}^p_{t,s}$ in (16) take a rather simple form, for any $p \geq 1$, given by

$$\mathrm{Tr}\, \bar{K}^p_{t,s} \simeq I_p(\tilde{s})/\sqrt{t} \,, \tag{17}$$

where $I_p(\tilde{s}) := \frac{1}{\pi} \int_{-\infty}^0 dv \sqrt{|v|}\, (\sigma(v - \tilde{s}))^p$. The series (16) can then be summed up, leading to (using $\tilde{s} = s\, t^{1/3}$)

$$\ln Q_t(s) \simeq -\frac{1}{\sqrt{t}} \Psi(e^{-s\, t^{1/3}}) \,, \ \Psi(z) = -\frac{1}{\sqrt{4\pi}} Li_{\frac{5}{2}}(-z) \tag{18}$$

in terms of the poly-logarithm function $Li_\nu(x) = \sum_{p=1}^{+\infty} x^p/p^\nu$.

Hence the exact formula for the generating function (10) takes the following form at small time

$$\left\langle \exp\left(-\frac{z}{\sqrt{4\pi t}} e^{H(t)}\right) \right\rangle \sim e^{-\frac{1}{\sqrt{t}}\Psi(z)} \tag{19}$$

where we use $z = e^{-\tilde{s}}$. Note that the l.h.s. is finite only for $z > 0$ (for $z < 0$ it is infinite).

From this, assuming the form (4) and inserting it in (19) for any $z > 0$, we obtain $\Phi_{\rm drop}(H)$ by a saddle point analysis [34], as

$$\Phi_{\rm drop}(H) = \begin{cases} \frac{-1}{\sqrt{4\pi}} \min\limits_{z \in [-1,+\infty[} [z e^H + Li_{\frac{5}{2}}(-z)], \ H \leq H_c \\ \\ \frac{-1}{\sqrt{4\pi}} \min\limits_{z \in [-1,0[} [z e^H + Li_{\frac{5}{2}}(-z) \\ \qquad\qquad - \frac{8\sqrt{\pi}}{3}(-\ln(-z))^{\frac{3}{2}}], \ H \geq H_c \end{cases} \tag{20}$$

where $H_c = \ln \zeta(3/2) = 0.96026...$ Note that despite the two apparent branches, the function $\Phi_{\rm drop}(H)$ is analytic at $H = H_c$. From this expression one obtains the asymptotic behaviors given in Eqs. (5-7) [34]. One can also compute the cumulants of the height as, $\overline{H(t)^q}^c = t^{\frac{q-1}{2}} \phi^{(q)}(0)$, where $\phi^{(q)}$ is the q-th derivative of

$$\phi(p) = \max_H (pH - \Phi_{\rm drop}(H)) \,. \tag{21}$$

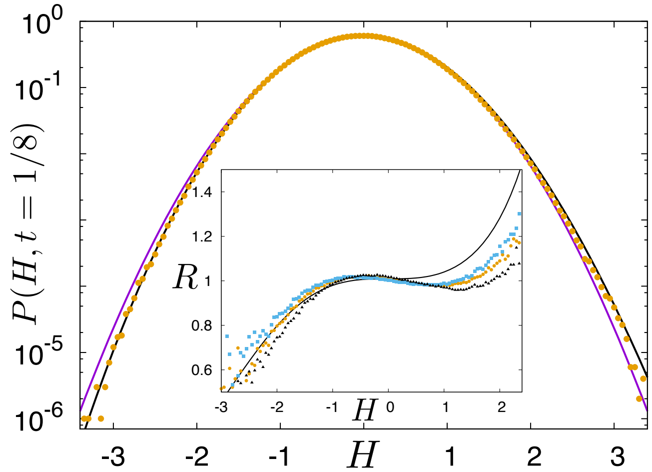

FIG. 2. Numerical determination of $P(H,t = 1/8)$. In the main figure, the solid (purple) line corresponds to the Edward-Wilkinson Gaussian regime (6). The symbols correspond to the numerical data for the discrete model with $\hat{t} = 256, \beta = 1/16$ (with 2.10^8 samples). Note that we have imposed $\langle H \rangle = 0$. The solid (black) line corresponds to $P(H,t) = c(t) e^{-\Phi_{\rm drop}(H)/\sqrt{t}}$ (20) where $c(t) = \int_0^\infty dH e^{-\Phi_{\rm drop}(H)/\sqrt{t}}$, with $c(1/8) = 1.23487\dots$ **Inset:** Plot of the ratio $R = P(H,t)/P_{\rm Gauss}(H,t)$ for $t = 1/8$, where $P_{\rm Gauss}(H,t)$ corresponds to the Gaussian regime (6). The triangles, circles and squares correspond respectively to $\hat{t} = 128$, $\hat{t} = 256$ and $\hat{t} = 512$. We see that when $\hat{t}$ increases the agreement with the short time continuum limit improves.

We display here the first four cumulants

$$\overline{H^2}^c = \sqrt{\frac{\pi}{2}} t^{1/2} \quad , \quad \overline{H^3}^c = (\frac{8}{3\sqrt{3}} - \frac{3}{2})\pi t \tag{22}$$

$$\overline{H^4}^c = (18 + 15\sqrt{2} - 16\sqrt{6}) \frac{\pi^{3/2}}{3} t^{3/2} \,. \tag{23}$$

Remarkably, these cumulants are very similar to the ones obtained for the *stationary* fluctuations of the total integrated particle current for the TASEP on a finite ring [25, 26]. This similarity holds for all higher cumulants as well (see [34]). In fact, the generating function associated with these cumulants, called G in [25, 26] also appears in the stationary regime of the ASEP and of the KPZ equation on a finite ring [27–31], and is different, but structurally somewhat similar to our function Ψ. It remains a puzzle why this universal function G describing the late time stationary regime in a *finite* system should be similar to our short time large deviation function in an *infinite* system.

Our results also describe the high temperature limit of lattice directed polymer models (DP), which allows for a numerical test. We simulate a DP growing on a 2D square lattice with unit Gaussian site disorder. For inverse temperature $\beta \ll 1$, the number of steps $\hat{t}$ corresponds to the time of the continuum model as $t = 2\hat{t}\beta^4$ [9] (see [34] for details). The result for $P(H,t)$ is shown in Fig. 2: the data shows (slow) convergence to our prediction.

Fermions in an harmonic trap. Consider now the quantum problem of N non-interacting spinless fermions of mass m in an harmonic trap at finite temperature T, described by the Hamiltonian $H = \sum_{i=1}^{N} \frac{p_i^2}{2m} + \frac{1}{2} m\omega^2 x_i^2$. We use $\mathsf{x}^* = \sqrt{\hbar/m\omega}$ and $T^* = \hbar\omega$ as units of length and energy. At $T=0$, i.e., in the ground state, and for large N, the average fermion density is given by the Wigner semi-circle law, with a finite support $[-\mathsf{x}_{\rm edge}, \mathsf{x}_{\rm edge}]$ where $\mathsf{x}_{\rm edge} = \sqrt{2N}$. At finite temperature, the behavior of physical quantities in the bulk changes on a temperature scale $T \sim N$ (bulk scaling), while near the edge it varies on a scale $T = N^{1/3}/b$ (edge scaling), where b is a dimensionless parameter of order unity [18]. Here we are interested in the position $\mathsf{x}_{\max}(T)$ of the rightmost fermion (see [18] for a precise definition). Its cumulative distribution function (CDF) was shown [18] to be given by the *same* Fredholm determinant as in Eq. (11)

$$\text{Prob}\left(\frac{\mathsf{x}_{\max}(T) - \mathsf{x}_{\rm edge}}{w_N} < s\right) = Q_{t=b^3}(s) \qquad (24)$$

where $w_N = N^{-1/6}/\sqrt{2}$. Since we have already analysed the small time limit $t \ll 1$ of the Fredholm determinant $Q_t(s)$ (as in Eq. (18)), this provides us with an explicit formula for the fermion problem, valid in the high temperature region $b \ll 1$ of the edge scaling regime.

To use the result in (18), we first set $s = \tilde{s}/t^{1/3}$ in (24) where $t^{1/3} = N^{1/3}/T$. The regime $t \ll 1$ corresponds to $T \gg N^{1/3}$. This leads us to define a new random variable

$$\xi = \frac{\mathsf{x}_{\max}(T) - \mathsf{x}_{\rm edge}}{\ell_N(T)} \qquad (25)$$

with

$$\ell_N(T) := T N^{-1/3} w_N = T/\sqrt{2N} \qquad (26)$$

where $w_N = N^{-1/6}/\sqrt{2}$ is the scale of fluctuations of $\mathsf{x}_{\max}$ at $T=0$. Thus $\ell_N(T)$ in (26) sets the scale of fluctuations of $\mathsf{x}_{\max}$ for $T \gg N^{1/3}$. Using (18) in the limit $t = b^3 = N/T^3 \ll 1$, we find that the CDF of ξ takes the asymptotic form (replacing $\tilde{s}$ by s for convenience)

$$\text{Prob}(\xi < s) \sim \exp\left(\sqrt{\frac{T^3}{4\pi N}} Li_{5/2}(-e^{-s})\right) . \qquad (27)$$

Using $Li_{5/2}(y) \simeq y$ for small y, it is easy to see that the PDF of ξ is peaked around the typical value $\xi = \xi_{\rm typ} = \frac{1}{2}\ln(T^3/4\pi N)$, with typical fluctuations $\tilde{\xi} = \xi - \xi_{\rm typ}$ described by a Gumbel law, i.e., $P(\tilde{\xi}) = e^{-\tilde{\xi} - e^{-\tilde{\xi}}}$ (see [36] for a similar observation in a related model).

Our formula (27), however, holds beyond the typical fluctuation regime and also describes the large deviations away from $\xi_{\rm typ}$. While the right tail is exponential, as given by the Gumbel distribution, using $Li_{\frac{5}{2}}(-z) \simeq_{z\to+\infty} -\frac{8}{15\sqrt{\pi}}(\ln z)^{\frac{5}{2}}$ in (27), we find that the left tail exhibits a distinct, stretched exponential decay

$$\text{Prob}(\xi < s) \sim \exp\left(-\frac{4}{15\pi}\sqrt{\frac{T^3}{N}}|s|^{5/2}\right) . \qquad (28)$$

Note that in this edge regime quantum correlations are still important. At much higher temperatures $T \sim N$, the positions of the fermions become completely independent variables, and the fluctuation of $\mathsf{x}_{\max}(T)$ is also described by a Gumbel distribution, albeit different from the one obtained here [37].

The above method is easily extended to obtain the full counting statistics (FCS) of the fermions near the edge for temperatures $T \gg N^{1/3}$. This is a generalisation of the $T=0$ result for the FCS in the edge regime [38]. Denoting by $N(s)$ the number of fermions in the interval $[\mathsf{x}_{\rm edge} + s\,\ell_N(T), +\infty[$ we obtain the characteristic function (see Eq. (18) in [34]) and, from it, the cumulants

$$\langle \left(N(s)\right)^p \rangle^c \simeq -\sqrt{\frac{T^3}{4\pi N}} Li_{\frac{5}{2}-p}(-e^{-s}) \qquad (29)$$

for all positive integer $p \geq 1$. In the typical region defined above, $s - \xi_{\rm typ} = \mathcal{O}(1)$ the statistics is Poisson with mean $\langle N(s) \rangle \simeq e^{\xi_{\rm typ} - s}$ [34]. There are deviations from Poisson in the tails, in particular for $s - \xi_{\rm typ} \to -\infty$ where the distribution becomes peaked around $\langle N(s) \rangle \simeq \sqrt{\frac{T^3}{4\pi N}} \frac{4(-s)^{3/2}}{3\sqrt{\pi}}$ with $\langle N(s)^2 \rangle^c \simeq \sqrt{\frac{T^3}{4\pi N}} \frac{2\sqrt{-s}}{\sqrt{\pi}}$ and zero higher cumulants [34].

In conclusion we have studied the statistics of the height fluctuations for the continuum KPZ equation at short time with the droplet initial condition. We obtained the exact analytical rate function $\Phi_{\rm drop}(H)$ and compared with numerics. It confirms, and extends, through an exact solution, recent approaches using weak noise theory developed for the flat geometry and unveils puzzling similarities with other large deviation results for finite-size system. We demonstrate that, remarkably, the right tail coincides with the Tracy-Widom result already at short time. This result agrees with rigorous bounds [39] valid at any fixed time t, $P(H > s) \leq e^{-\frac{4}{3}s^{3/2}/t^{1/2}}$. By contrast the convergence towards the left TW tail $\sim (-H)^3$ appears to be much slower, with $\sim (-H)^{5/2}$ behavior at short time. Our short-time results for the KPZ equation also provide exact asymptotic predictions for the PDF of the rightmost fermion in a harmonic trap at high temperature $T \gg N^{1/3}$. We hope that the present results will stimulate further investigations of extreme value questions in the KPZ class [42] and also in cold atom systems.

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[33] We recall that, for a trace-class operator $K(x,y)$ such that ${\rm Tr} K = \int dx K(x,x)$ is well defined, $\det(I-K) = \exp[-\sum_{n=1}^{\infty} {\rm Tr}\, K^n/n]$, where ${\rm Tr}\, K^n = \int dx_1 \cdots \int dx_n K(x_1,x_2)K(x_2,x_3)\cdots K(x_n,x_1)$. The effect of the projector P_0 in (11) is simply to restrict the integrals over x_i's to the interval $[0,+\infty)$.
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