Measuring the Second Chern Number from Nonadiabatic Effects

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Measuring second Chern number from non-adiabatic effects

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The geometry and topology of quantum systems have deep connections to quantum dynamics. In this paper, I show how to measure the non-Abelian Berry curvature and its related topological invariant, the second Chern number, using dynamical techniques. The second Chern number is the defining topological characteristic of the four-dimensional generalization of the quantum Hall effect and has relevance in systems from three-dimensional topological insulators to Yang-Mills field theory. I illustrate its measurement using the simple example of a spin-3/2 particle in an electric quadrupole field. I show how one can dynamically measure diagonal components of the Berry curvature in an over-complete basis of the degenerate ground state space and use this to extract the full non-Abelian Berry curvature. I also show that one can accomplish the same ideas by stochastically averaging over random initial states in the degenerate ground state manifold. Finally I show how this system can be manufactured and the topological invariant measured in a variety of realistic systems, from superconducting qubits to trapped ions and cold atoms.

Topological invariants such as the first Chern number have become relevant in condensed matter physics, describing novel states of matter [1–5]. While naturally defined in the Brillouin zone, these geometric concepts and the Berry phase on which they are based occur in a wide variety of systems. In particular, these ideas have been recently applied to engineer and measure topological properties of designed systems, such as many-body cold atomic systems [6–8] and few-body systems of qubits or random walkers [9–11].

It was noted early on [12, 13] that higher topological invariants could be defined, and particularly that a non-trivial second Chern number [14] characterizes systems with time-reversal symmetry. More recently, this has been connected the four-dimensional generalization of the quantum Hall effect [15] and three-dimensional topological insulators [16]. It is also intricately related to the axion electrodynamics used to define 3D topological insulators and non-perturbative instanton effects in Yang-Mills field theory.

This higher topological invariant has never been measured experimentally. Here I propose how the second Chern number may be measured using non-adiabatic effects similar to those used in Refs. [10] and [11] to measure the first Chern number. The proposal relies on time-reversal invariant Hamiltonians to enforce a doubly-degenerate ground state and thus the previous proposal must be extended to account for these degeneracies. This involves measuring a fundamentally non-Abelian topological object. I show two ways to account for this – one by deterministically sampling over degenerate ground states and another by stochastic sampling – which may be relevant for different experimental systems. I close by discussing how to access this physics in current experiments.

Dynamics with degeneracies - Consider a Hamiltonian $H(\boldsymbol{\lambda})$ that depends on parameters $\boldsymbol{\lambda}$. If one starts in the non-degenerate ground state $|\psi_0(\boldsymbol{\lambda}_i)\rangle$ at $\boldsymbol{\lambda}_i$ and ramps $\boldsymbol{\lambda}$ slowly with time, at zeroth order the system simply remains in it ground state and picks up both a dynamical and Berry phase [17]. If the ground state remains non-degenerate during the course of the ramp, the leading non-adiabatic correction giving population in the excited states can be calculated using adiabatic perturbation theory [18–24]. One starts by going to a moving frame $|\psi\rangle \to U(\boldsymbol{\lambda})^\dagger|\psi\rangle$, where $U(\boldsymbol{\lambda})$ diagonalizes the Hamiltonian ($H_d = U^\dagger H U$). In the moving frame, the Hamiltonian becomes

$$H_m = H_d - i\dot{\lambda}_\mu U^\dagger \partial_\mu U \equiv H_d - \dot{\lambda}_\mu \mathcal{A}^m_\mu, \tag{1}$$

where $\partial_\mu \equiv \partial/\partial\lambda_\mu$ and repeated indices are summed over. We refer to the second term $\mathcal{A}_\mu = U\mathcal{A}^m_\mu U^\dagger$ as the Berry connection operator as its matrix elements in the energy eigenbasis $|\psi_n\rangle$ are $\langle\psi_m|\mathcal{A}_\mu|\psi_n\rangle = i\langle\psi_m(\boldsymbol{\lambda})|\partial_\mu\psi_n(\boldsymbol{\lambda})\rangle$. Diagonal terms in H_m give rise to

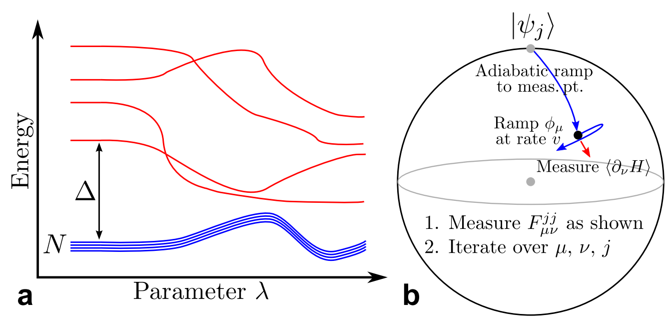

Figure 1. Measuring second Chern number dynamically. (a) General setup where measurement is possible. N degenerate ground states are separated by a non-zero gap from the excited states. The ground states must remain degenerate and gapped from the excited states, but with no restrictions on the excited states. (b) Illustration of the ramping protocol to find one component $F^{jj}_{\mu\nu}$ of the non-Abelian Berry curvature at measurement point $\boldsymbol{\phi}$.

the dynamical and Berry phases, since $A_\mu = \langle\psi_0|\mathcal{A}_\mu|\psi_0\rangle$ is the ground state Berry connection. Off-diagonal terms give rise to non-adiabatic occupation in the excited states, which can be derived by applying static perturbation theory to H_m: $c_{n\neq 0} \approx \dot{\lambda}_\mu \langle\psi_n|\mathcal{A}_\mu|\psi_0\rangle/(E_n - E_0)$, where $|\psi\rangle = \sum_n c_n|\psi_n\rangle$. Calculating the "generalized force" $M_\nu \equiv -\langle\partial_\nu H\rangle$ in this state gives the correction $M_\nu \approx M_\nu^0 - \dot{\lambda}_\mu F_{\mu\nu}$, where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the ground state Berry curvature [25–30]. This term is analogous to the "anomalous velocity" that appears semi-classicaly for Bloch electrons [31, 32] and can be thought of as a Lorentz force in parameter space. It has been used to experimentally measure the Berry curvature within non-degenerate ground state manifolds, from which the topologically-invariant first Chern number can be extracted [10, 11].

This formalism has been generalized to situations where the ground state is degenerate [33–36]. For the simplest case where the ground state remains N-fold degenerate throughout the process, we can reformulate these results through the above formalism. First note that, unlike the non-degenerate case, the connection and curvature are now non-Abelian, giving rise to unitary rotations within the ground state subspace. In our language of adiabatic perturbation theory, these non-Abelian effects can be seen as first-order degenerate perturbation theory; at a given point $\boldsymbol{\lambda}$ during the ramp, one must diagonalize $\mathcal{A}_\mu^m$ within the degenerate subspace, the eigenstates of which then just pick up separate Berry phases. The non-Abelian aspect comes as the diagonal basis of $\mathcal{A}_\mu^m$ changes with $\boldsymbol{\lambda}$. From integrating Eq. 1, we see that the anholonomy is given up to a dynamical phase by the path-ordered integral $\mathcal{P}\exp\left[i\int_{\boldsymbol{\lambda}_i}^{\boldsymbol{\lambda}} d\lambda'_\mu A_\mu(\boldsymbol{\lambda}')\right]$ [33]. Note that $A_\mu^{ij} \equiv i\langle\psi_{0i}|\partial_\mu\psi_{0j}\rangle$ is now an $N\times N$ matrix giving the non-Abelian Berry connection within the ground state sector.[37]

Fortunately, the off-diagonal terms responsible for excitations do not notice this degeneracy. Consider a path $\boldsymbol{\lambda}(s)$ such that an adiabatic traversal would yield a particular ground state $|\psi_{0A}(\boldsymbol{\lambda})\rangle$. Tracing the same path at a finite rate, the ground state component of the wave function is unchanged at order $\dot{\lambda}$. Excitations are given by the natural extension of the earlier formula:

$$|\psi(\boldsymbol{\lambda}(t))\rangle \approx |\psi_{0A}(\boldsymbol{\lambda})\rangle + i\dot{\lambda}_\mu \sum_{n\neq 0} |\psi_n(\boldsymbol{\lambda})\rangle \frac{\langle\psi_n|\partial_\mu\psi_{0A}\rangle}{E_n - E_0}.$$

One may readily confirm that the generalized force in this state sees the diagonal component of the non-Abelian Berry curvature matrix $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu]$, i.e., $M_\nu \approx M_{\nu A}^0 - \dot{\lambda}_\mu F_{\mu\nu}^{AA}$. Note that the adiabatic value $M_{\nu A}^0$ depends on the path $\boldsymbol{\lambda}(s)$. Thus our results for this observable are similar to the non-degenerate case, but with the important caveat that two different paths with the same value of $\boldsymbol{\lambda}$ and $\dot{\boldsymbol{\lambda}}$ at time t will not necessarily give the same Berry curvature correction to the generalized force.

Second Chern number - The question then becomes what to make of the non-Abelian Berry curvature measurement if one can not easily predict the adiabatically-connected state. We want a quantity that is independent of the choice of basis; we find it in the topologically invariant Chern number. The simplest example is the first Chern number, defined for a closed two-dimensional manifold $\mathcal{M}_2$ as $C_1 = (2\pi)^{-1}\int_{\mathcal{M}_2} d\lambda_\mu \wedge d\lambda_\nu \mathrm{Tr}(F_{\mu\nu})$, where $\wedge$ denotes the wedge product . A novel topological invariant that appears for the four-dimensional manifold $\mathcal{M}_4$ is the second Chern number

$$C_2 = \int_{\mathcal{M}_4} \omega_2^{\mu\nu\rho\sigma} d\lambda_\mu \wedge d\lambda_\nu \wedge d\lambda_\rho \wedge d\lambda_\sigma \qquad (2)$$
$$\omega_2^{\mu\nu\rho\sigma} = \frac{\mathrm{Tr}(F_{\mu\nu}F_{\rho\sigma}) - \mathrm{Tr}(F_{\mu\nu})\mathrm{Tr}(F_{\rho\sigma})}{32\pi^2},$$

where ω_2 is the second Chern form. The trace is taken over the ground state (upper) indices, $\mathrm{Tr}(F_{\mu\nu}F_{\rho\sigma}) \equiv F_{\mu\nu}^{ij}F_{\rho\sigma}^{ji}$, rendering C_2 basis invariant. But one clearly requires knowledge of off-diagonal elements of F to take this trace, while our non-adiabatic scheme only yields diagonal elements. I will now discuss two schemes to fill in this gap.

First, let us see how we can deterministically reconstruct F by measuring its diagonal elements in an over-complete basis. For concreteness, assume there are $N = 2$ degenerate ground states, $|\psi_{0A}\rangle$ and $|\psi_{0B}\rangle$. $F_{\mu\nu}^{ij}$ is anti-symmetric w.r.t. exchange of the lower (parameter) indices and Hermitian w.r.t. the upper ones. Thus each matrix $F_{\mu\nu}$ is determined by N^2 real numbers. If we measure the diagonal components in the four states $|\psi_1\rangle = |\psi_{0A}\rangle$, $|\psi_2\rangle = |\psi_{0B}\rangle$, $|\psi_3\rangle = (|\psi_{0A}\rangle + |\psi_{0B}\rangle)/\sqrt{2}$, and $|\psi_4\rangle = (|\psi_{0A}\rangle + i|\psi_{0B}\rangle)/\sqrt{2}$, then

$$F_{\mu\nu} = \begin{pmatrix} F_{\mu\nu}^{AA} & F_{\mu\nu}^{AB} \\ F_{\mu\nu}^{BA} & F_{\mu\nu}^{BB} \end{pmatrix}$$
$$= \begin{pmatrix} F_{\mu\nu}^{11} & \frac{2iF_{\mu\nu}^{33}+2F_{\mu\nu}^{44}-(1+i)(F_{\mu\nu}^{11}+F_{\mu\nu}^{22})}{2i} \\ (F_{\mu\nu}^{AB})^* & F_{\mu\nu}^{22} \end{pmatrix}, \qquad (3)$$

from which evaluating the second Chern number integral is just math. This method is well-suited to controllable quantum systems where one has the ability to prepare arbitrary initial states. It trivially generalizes to arbitrary N.

Absent such a degree of control, a similar result may be achieved by stochastically sampling over ground states $|\psi\rangle$. This measurement technique is natural if one only has access to random snapshots of the system but may make multiple non-destructive measurements of the same state, as may be natural in the solid state. This method is discussed in the Supplementary Information.

Spin-3/2 model - I now demonstrate the applicability of these measurement techniques for the quintessential example of a system with non-trivial second Chern number: the quantum spin-3/2 in an electric quadrupole field.

This model was proposed by Avron et al. [12, 13] as containing a "quaternionic singularity" [38] giving C_2 much as the monopole singularity of the Berry curvature in a spin-1/2 yields non-trivial C_1. The Hamiltonian may be written as $H = -\boldsymbol{\lambda} \cdot \mathbf{H}$, where $\mathbf{H} = (H_0, H_1, \ldots, H_4)$ denotes an orthonormal basis of spin-3/2 quadrupole operators and $\boldsymbol{\lambda}$ denotes vector of coupling parameters. In particular, we choose the basis in Ref. 13: $H_0 = (-J_x^2 - J_y^2 + 2J_z^2)/3$, $H_1 = (J_xJ_z + J_zJ_x)/\sqrt{3}$, $H_2 = (J_yJ_z + J_zJ_y)/\sqrt{3}$, $H_3 = (J_x^2 - J_y^2)/\sqrt{3}$, and $H_4 = (J_xJ_y + J_yJ_x)/\sqrt{3}$. These Hamiltonians are invariant under time reversal, so the eigenvalues come in two degenerate pairs. By construction, the energy eigenvalues of each are ± 1; due to orthonormality, this is also true for arbitrary unit 5-vector $\boldsymbol{\lambda}$.

It is clear from the above discussion that the only way for all four eigenvalues to be degenerate is to have $\boldsymbol{\lambda} = \mathbf{0}$; this is the "quaternionic monopole" that gives a non-zero second Chern number. Avron et al. showed that for a 4-sphere surrounding this degeneracy, the second Chern number is equal to 1. Furthermore, due to time reversal symmetry this system has $C_1 = 0$, so C_2 is its defining topological invariant. I will now show how the above ideas can be used to measure C_2 directly.

Let us begin by fixing $|\boldsymbol{\lambda}| = 1$ and re-parameterizing in terms of the spherical angles $\boldsymbol{\phi} = (\phi_1, \phi_2, \phi_3, \phi_4)$, where $\phi_4 \in [0, 2\pi)$, $\phi_{1-3} \in [0, \pi)$, $\lambda_0 = \cos\phi_1$, $\lambda_1 = \sin\phi_1 \cos\phi_2$, $\ldots$, $\lambda_4 = \sin\phi_1 \sin\phi_2 \sin\phi_3 \sin\phi_4$. To obtain the Chern form at some point $\boldsymbol{\phi}$, we begin with one of the states $|\psi_{1-4}\rangle$ described earlier for the value $\boldsymbol{\phi} = \mathbf{0}$ (the North pole). Here the Hamiltonian has the simple form $H = 5/4 - J_z^2$, so that ground states are just the $m_z = \pm 3/2$ eigenstates. Starting from one of these states, say $|\psi_1\rangle$, we ramp slowly along some arbitrary path $\boldsymbol{\phi}_1(s)$ to the measurement point $\boldsymbol{\phi}$. Then to measure the component $F_{\mu\nu}^{11}$, we ramp $\phi_\mu = \phi_\mu^m + v(t - t_m)t^2/t_m^2$, where ϕ_μ^m is its value at the point to be measured. This ramp is chosen to start slowly ($\dot{\phi}_\mu(0) = 0$) at $\phi_\mu(0) = \phi_\mu^m$ and return to ϕ_μ^m at time t_m with velocity v. Repeating this ramp multiple times with velocity $v/2$ and measuring the expectation values $M_\nu(v)$ and $M_\nu(v/2)$, the Berry curvature is $F_{\mu\nu}^{11} \approx 2[M_\nu(v/2) - M_\nu(v)]/v$. This protocol, illustrated in Fig. 1, must be repeated for all pairs (μ, ν) and all initial states $|\psi_i\rangle$ to obtain the second Chern form at the point $\boldsymbol{\phi}$ via Eq. 3. Crucially, for a given point $\boldsymbol{\phi}$, the same path $\boldsymbol{\phi}_1$ must be taken for each component of the tensor to ensure that the appropriate phase relations between the $|\psi_i\rangle$'s remain once ramped to $\boldsymbol{\phi}$. From the second Chern form, the second Chern number may be obtained by the integral in Eq. 2. Carrying out the above procedure with Monte Carlo integration over $\boldsymbol{\phi}$ yields $C_2 = 0.9926 \pm 0.0073$, consistent with the exact value of 1.

To demonstrate the robustness of this topological invariant, we may induce a topological transition by adding a constant offset $\Lambda_0 H_0$ to the previous Hamiltonian. This shifts the unit sphere by an amount Λ_0, and for $|\Lambda_0| > 1$ the sphere fails to surround the degeneracy at the origin. Therefore, the second Chern number jumps to being trivial. This topological transition is seen in the simulations in Fig. 2d; the transition appears broadened for a finite velocity v due to higher-order non-adiabatic corrections near the gapless transition point.

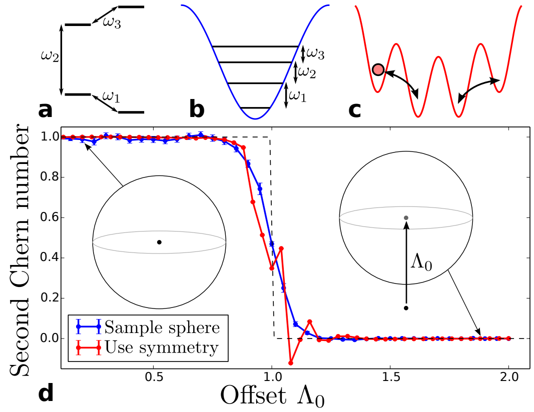

Figure 2. Experimental realizations of four-level systems where the second Chern number may be measured: (a) atomic hyperfine levels, (b) bound states of an artificial atom, (c) a particle hopping in a 4-site lattice. While all sites/levels must be coupled to realize arbitrary 4×4 Hamiltonians, only the indicated drives/hoppings are necessary in the presence of symmetry about the λ_0 axis. (d) Second Chern number measured dynamically for a spin-3/2 in an electric quadrupole field, as described in the text, either without utilizing symmetry (blue points) or with symmetry (red points). The deviation from quantization near the transition at $\Lambda_0 = 1$ is due to using finite time protocols; I ramp from the North pole to the measurement point in time $t = 100$ then ramp for measurement with $v = 0.01$ and $t_m = 100$. The inset shows how as $\Lambda_0 H_0$ is added, the 4-sphere shifts away from the origin until eventually the (4-fold) degeneracy at $\boldsymbol{\lambda} = \mathbf{0}$ is no longer enclosed.

Experiments - The above procedure naturally lends itself to controllable quantum systems such as superconducting qubits, ultracold atoms, ions, and solid state defects. For such systems, more detailed topological and geometric properties such as the Wilson loop may be measured via full tomography of adiabatic protocols; the dynamic second Chern number measurement supplements this natural toolkit. However, the dynamical measurement trivially generalizes to more complicated systems where full tomography is not possible, requiring neither strict adiabaticity nor tomographic measurements that scale exponentially with system size.

A useful experimental trick is to use the symmetry of the sphere to reduce the number of measurements that must be made. As the simplest example of this, consider the case $\Lambda_0 = 0$ where the problem has full spherical symmetry. Then, since all points on the sphere are identical, the Chern number can be obtained by measuring the Berry curvature at a single point. Starting in the ground state at the North pole, there are four orthonormal tangent vectors: $\hat{\lambda}_1$, $\hat{\lambda}_2$, $\hat{\lambda}_3$, and $\hat{\lambda}_4$. We can measure the response to these parameters as we did with $\boldsymbol{\phi}$. For instance, we obtain $F_{\lambda_\mu\lambda_\nu}$ by ramping λ_μ and measuring $M_{\lambda_\nu} = -\langle H_\nu\rangle$. By symmetry, $\mathrm{Tr}(F_{\lambda_1\lambda_2}F_{\lambda_3\lambda_4} - F_{\lambda_1\lambda_3}F_{\lambda_2\lambda_4} + F_{\lambda_1\lambda_4}F_{\lambda_2\lambda_3}) = 3\mathrm{Tr}(F_{\lambda_1\lambda_2}F_{\lambda_3\lambda_4}) \sim \omega_2$ will be constant at any point on the sphere. So we simply multiply by the surface area of the unit 4-sphere to get[39]

$$C_2 = \frac{3A_{S^4}\mathrm{Tr}(F_{\lambda_1\lambda_2}F_{\lambda_3\lambda_4})}{4\pi^2} = 2\mathrm{Tr}(F_{\lambda_1\lambda_2}F_{\lambda_3\lambda_4}). \quad (4)$$

We thus expect that $\mathrm{Tr}(F_{\lambda_1\lambda_2}F_{\lambda_3\lambda_4}) = 1/2$, which is readily confirmed numerically.

The argument must be slightly modified in the presence of an offset Λ_0, but symmetry still significantly reduces the number of measurements required. λ_0 is now distinct from the other axes, which translates into a ϕ_1-dependence of the second Chern form. The axes tangent to a point $\boldsymbol{\phi} = (\phi_1, 0, 0, 0)$ are now $\hat{\phi}_1$, $\hat{\lambda}_2$, $\hat{\lambda}_3$, and $\hat{\lambda}_4$, so the non-trivial terms $F_{\phi_1\lambda_2}$ and $F_{\lambda_3\lambda_4}$ are obtained by ramping ϕ_1 and λ_3 and measuring $\langle H_2\rangle$ and $\langle H_4\rangle$ respectively. For a given ϕ_1, the remaining parameters trace out a 3-sphere of radius $\sin\phi_1$ with surface area $2\pi^2\sin^3\phi_1$. By the same logic as Eq. 4, one finds

$$C_2 = \frac{3}{2}\int_0^\pi d\phi_1 \sin^3\phi_1 \mathrm{Tr}\left[F_{\phi_1\lambda_2}(\phi_1)F_{\lambda_3\lambda_4}(\phi_1)\right]. \quad (5)$$

The resulting Chern number is shown in Fig. 2d.

In addition to reducing the number of measurements, Eq. 5 reduces the number of control axes required; one need only ramp ϕ_1(i.e. λ_1 and λ_2) and λ_3, by symmetry any three λs will do. This is reduced further to only two λs in the fully symmetric case. While one may in principle realize arbitrary 4×4 Hamiltonians given four levels fully-connected by drives [40], a more natural situation is partially-connected levels like those illustrated in Fig. 2 [41–43]. For such couplings, not all of the terms can be easily realized, but fortunately a sufficient number can be realized to allow the measurement using symmetry. This can be seen from the representations of H_i in the J_z basis:

$$H_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \; H_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \text{ etc.}$$

If we think of the four states as positions of a particle on a 4-site chain, then H_0 represents on-site chemical potentials, while H_1 and H_2 represent nearest-neighbor hopping. This is well within the capacity of the driven system, and could even be realized with a 4-site superlattice via lattice-shaking schemes analogous to those used to generate artificial gauge fields [6, 7].

Instead of imprinting the Hamiltonian structure on Hilbert space by hand, we might instead be given a system with natural structure of its own, say two coupled spins-1/2 [44–46]. In this case, the Hamiltonians may be written as $H_0 = \sigma_1^z\sigma_2^z$, $H_1 = \sigma_1^x\sigma_2^z$, $H_2 = \sigma_1^y\sigma_2^z$, $H_3 = \sigma_2^x$, and $H_4 = \sigma_2^y$. Of these operators, H_0, H_3 and H_4 are naturally realized in both trapped ions [46] and transmon qubits [47]. Similarly, we can imagine directly obtaining $J = 3/2$ by fusing three spin-1/2's. In this language, H_0 is proportional to an Ising-like interaction $\sum_{<ij>}\sigma_i^z\sigma_j^z$, although the other H_i's seem less natural. This three-qubit realization may become possible in this fast-developing field, but the two-qubit version is already experimentally feasible.

Discussion - In conclusion, I have shown how to measure topological transitions of the non-Abelian second Chern number C_2 in experimentally-realizable systems. This topological invariant is necessarily quantized, which could prove valuable in metrological applications much as topologically-protected response of the quantum Hall effect allows one to obtain e^2/h with unprecedented precision [48]. An interesting open question is how to connect these results in the language of generic many-body quantum systems with parameters $\boldsymbol{\lambda}$ to solid state physics, where single-particle Bloch states with momentum $\mathbf{k}$ provide a natural parameter space [49, 50]. In particular, the four-dimensional Chern insulator analogous to our spin-3/2 example has a quantized non-linear electromagnetic response [15] and has recently become relevant in artificial 4D systems of cold atoms, photons, and quasicrystals [51–54]. Our method for measuring C_2 involves *linear* response, followed by classical post-processing. How these linear and non-linear responses arise from the same topological invariant remains an intriguing open question.

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