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General conditions for proximity-induced odd-frequency superconductivity in
two-dimensional electronic systems

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We obtain the general conditions for the emergence of odd-frequency superconducting pairing in a

two-dimensional (2D) electronic system proximity-coupled to a superconductor, making minimal

assumptions about both the 2D system and the superconductor. Using our general results we show

that a simple heterostructure formed by a monolayer of a group VI transition metal dichalcogenide,

such as molybdenum disulfide, and an s-wave superconductor with Rashba spin-orbit coupling will

exhibit odd-frequency superconducting pairing. Our results allow the identification of a new class

of systems among van der Waals heterostructures in which odd-frequency superconductivity should

be present.

Low-dimensional heterostructures hold the promise for

new technologies as well as granting us access to many

unconventional quantum states including: novel forms of

superfluidity6, manipulation of spin textures7,8, and un-

conventional superconductivity4,9–15. In addition, theo-

retical analyses have shown that Majorana bound states

can appear in heterostructures incorporating supercon-

cducting materials16–29. Given the variety of possible ex-

otic states in low dimensional heterostructures and that

the fabrication of layered heterostructures has rapidly ad-

vanced in recent years5 it is important to continue develop-

ing our understanding of their electronic properties.

One important facet of this understanding is the clas-

sification of the possible symmetries of the proximity-

induced superconductivity in these structures.

The symmetries of a superconductor can be charac-

terized by investigating the properties of the anomalous Green’s function

\[ F_{\alpha\beta}(r_1, t_1; r_2, t_2) = \langle Tc_{\alpha}(r_1, t_1)c_{\beta}^\dagger(r_2, t_2) \rangle \],

where \( c_{\alpha}(r, t) \) is the fermionic annihilation operator for an electron at position \( r \) at
time \( t \) with spin \( \sigma \), \( T \) is time ordering operator, and the angle brackets denote the expectation value.

Given the fermionic nature of the quasiparticles

\[ F_{\alpha\beta}(r_1, t_1; r_2, t_2) = -F_{\beta\alpha}(r_2, t_2; r_1, t_1). \]

Conventionally this is taken to imply that if the quasiparticle pair is in

a spin singlet state then the pairing amplitude is even in

parity while if it is a spin triplet the pairing amplitude is

odd in parity. However, if the pairing amplitude is odd in
time, or, equivalently, odd in frequency, spin triplet pairs can be even in parity and spin singlet pairs can be

odd in parity as was originally proposed for superfluid

\(^3\)He by Berezinskii20 and later for superconductivity by Balatsky and Abrahams30.

The study of odd-frequency superconductivity (SC)

has been hindered by the scarcity of experimental sys-
tems in which it can be realized. Soon after the origi-
nal suggestion that in general an odd-frequency pairing

term could be present it was realized that it would be

challenging to get such a term via electron-phonon in-
teractions and that a spin-dependent electron-electron

interaction would be necessary31. This fact greatly re-

stricts the number of systems in which odd-frequency SC

could be realized. However, in recent years it has be-

come apparent that odd-frequency SC can be obtained in

heterostructures9,12,14,15,32–39. Each of this works consid-
ered a different type of heterostructure. The recent im-
pressive explosion of the types of heterostructures that

can be realized has made this piecemeal approach unfea-
sible: a theoretical treatment able to provide the gen-

eral conditions in which odd-frequency SC should be

present in heterostructures has become necessary. In

this work we present such a general treatment. Our gen-

eral treatment also makes possible the identification of

novel, somehow unexpected, engineered systems in which

such pairing should be present, as exemplified by the het-

erostructure formed by one monolayer of MoS\(_2\) placed on

superconducting Pb, that we discuss in the second part

of the manuscript. In particular by showing what are the

necessary elements that a van der Waals heterostructure

must have to exhibit odd-frequency SC it adds this im-

portant class of systems to the odd-frequency playbook.

Our work also makes possible to select among such sys-
tems, the ones in which a direct observation – for ex-

ample via scanning tunneling microscopy (STMS) and an-

gle resolved photoemission spectroscopy (ARPES) – of

the signatures due to odd-frequency SC is more readily

achievable.

The Hamiltonian \( (H) \) describing the most general het-

erostructure formed by a 2D electron gas (2DEG) and a

superconductor can be written as

\[ H = H_{2D} + H_{SC} + H_t \]

where

\[ H_{2D} = \sum_{k, \sigma, \sigma'} c_{k, \sigma}^\dagger h_{0}(k) c_{k, \sigma} + h(k) \cdot \sigma \sigma_{\alpha} c_{k, \sigma}, \]

\[ H_{SC} = \sum_{k \sigma \sigma'} d_{k, \sigma}^\dagger h_{SC}(k) d_{k, \sigma}, + \sum_{k \sigma \sigma'} d_{k, \sigma}^\dagger \Delta_{k, \sigma} d_{-k, \sigma'}^\dagger + h.c. \]

\[ H_t = t \sum_{k, \sigma} d_{k, \sigma}^\dagger c_{k, \sigma} + h.c. \]
are the Hamiltonians describing the 2DEG, the superconductor, and the tunneling between the two systems, respectively. In Eqs. (1)-(3) $\sigma$ is the identity matrix in spin space, $\sigma = (\sigma_1, \sigma_2, \sigma_3)$ is the vector of Pauli matrices in spin space, $c_k^\dagger (d_k^\dagger)$ and $c_k (d_k)$ are the creation and annihilation operators, respectively, acting on the fermionic states in the 2DEG (SC) layer with momentum $k$ and spin $\sigma$, $h_0(k)$ is the spin-independent part of $H_{2D}$ and $h(k)$ is the field that describes its spin-dependent part due to an exchange field and/or spin-orbit coupling, $h^{SC}_{\sigma\sigma'}(k)$ describes the quasiparticle spectrum of the normal state of the superconductor, $\Delta_{k,\sigma,\sigma'}$ is the superconducting gap, and $t$ is the tunneling between the 2D system and the SC. We assume the tunneling to conserve both spin and momentum given that this is the most common situation and to be able to identify the most general condition to realize odd-frequency SC without having to resort to spin-active interfaces that are often difficult to realize experimentally. To keep the treatment general we make no assumptions on the form of $h(k)$, $h^{SC}_{\sigma\sigma'}(k)$, and $\Delta_{k,\sigma,\sigma'}$. The anomalous Green’s function associated with the superconductor described by Eq. (2) is given by $\hat{G}^{SC}_{k,i\omega_n} = \left(\hat{\Delta}^\dagger_{-k} - (i\omega_n + \hat{h}^{SC}(-k)^*) \hat{\Delta}^{-1}_{k} (i\omega_n - \hat{h}^{SC}(k))\right)^{-1}$. We can parameterize this matrix in terms of singlet and triplet parts:

$$\hat{G}^{SC}_{k,i\omega_n} = (S^{SC}_{k,i\omega_n} \sigma_0 + d_{k,i\omega_n} \cdot \sigma) i\sigma_2$$

where $\omega_n$ is the Matsubara frequency, and $S^{SC}_{k,i\omega_n}$ and the three-component complex vector $d_{k,i\omega_n}^{40}$ give the singlet and triplet superconducting amplitudes, respectively. The leading order contributions to the proximity-induced superconducting pairing in the 2DEG are given by:

$$\hat{F}^{2D}_{k,i\omega_n} = i^2 \hat{G}^{2D}_{k,i\omega_n} \hat{G}^{SC}_{k,i\omega_n} \left(\hat{G}^{2D}_{-k,i\omega_n}\right)^T$$

where

$$\hat{G}^{2D}_{k,i\omega_n} = \left(\frac{(i\omega_n - h_0(k))\sigma_0 + h(k) \cdot \sigma}{(i\omega_n - h_0(k))^2 - |h(k)|^2}\right)$$

is the Green’s function associated with the 2DEG. It is convenient to separate the anomalous Green’s function $\hat{G}^{2D}_{k,i\omega_n}$ into two parts

$$\hat{F}^{2D}_{k,i\omega_n} = A_{k,i\omega_n} \left(F^{odd}_{k,i\omega_n} + F^{even}_{k,i\omega_n}\right)$$

where $A_{k,i\omega_n}$ is generally a function even in $\omega_n^{41}$, and $F^{odd}_{k,i\omega_n}$ and $F^{even}_{k,i\omega_n}$ are the odd- and even-frequency $2 \times 2$ matrices describing the spin structure of the induced superconducting pairs respectively. Let $h_{\pm}(k) \equiv h(k) \pm i h(-k)$. Then for $F^{even}_{k,i\omega_n}$ we find:

$$F^{even}_{k,i\omega_n} = (S^{even}_{k,i\omega_n} \sigma_0 + D^{even}_{k,i\omega_n} \cdot \sigma) i\sigma_2$$

where $S^{even}_{k,i\omega_n}$, $D^{even}_{k,i\omega_n}$ are the singlet and triplet components, respectively, given by:

$$S^{even}_{k,i\omega_n} = \left[\omega_n^2 + h_0^2(k) - \frac{i}{4} (|h_+(k)|^2 - |h_-(k)|^2)\right] S^{SC}_{k,i\omega_n}$$

$$- h_0(k) h_- + \frac{i}{2} h_+(k) \times h_-(k) \cdot d_{k,i\omega_n}$$

$$D^{even}_{k,i\omega_n} = \left[\omega_n^2 + h_0^2(k) + \frac{i}{4} (|h_+(k)|^2 - |h_-(k)|^2)\right] d_{k,i\omega_n}$$

$$- i h_0(k) h_+ \times d_{k,i\omega_n} - \frac{i}{2} h_+(k) (h_+(k) \cdot d_{k,i\omega_n})$$

$$+ \frac{i}{2} h_-(k) (h_-(k) \cdot d_{k,i\omega_n})$$

$$- h_0(k) h_- - \frac{i}{2} h_+(k) \times h_-(k) \cdot S^{SC}_{k,i\omega_n}.$$  (7)

The first line (three lines) of the expression for $S^{even}_{k,i\omega_n}$ ($D^{even}_{k,i\omega_n}$) show that, as expected a singlet (triplet) pairing is induced, via the proximity effect, in the 2DEG by a singlet (triplet) superconductor, regardless of the value of $h$. The last line for the expression of $S^{even}_{k,i\omega_n}$ ($D^{even}_{k,i\omega_n}$) shows that if $h_- \neq 0$, by proximity effect, we will have even-frequency superconductivity with both singlet and triplet pairing even if the substrate superconductor only has singlet or triplet pairing. It also shows that the strength of the pairing in the 2DEG with spin-structure different from the one of the substrate is proportional to $h_- (k)$ and is augmented when $h_+ \times h_+ \neq 0$. This result shows how the presence of spin-orbit coupling, that gives rise to $h_- \neq 0$, qualitatively affects the nature of the conventional (even-frequency) superconducting pairing induced by proximity. We then find that the interplay of the field $h$ in the 2DEG, and the superconducting pairing in the substrate gives rise to an odd-frequency pairing term:

$$F^{odd}_{k,i\omega_n} = i\omega_n \left(S^{odd}_{k,i\omega_n} \sigma_0 + D^{odd}_{k,i\omega_n} \cdot \sigma\right) i\sigma_2$$

with $S^{odd}_{k,i\omega_n}$, $D^{odd}_{k,i\omega_n}$ the odd-frequency singlet and triplet components, respectively, given by:

$$S^{odd}_{k,i\omega_n} = - h_+ (k) \cdot d_{k,i\omega_n}$$

$$D^{odd}_{k,i\omega_n} = - h_+ (k) S^{SC}_{k,i\omega_n} - i h_- (k) \times d_{k,i\omega_n}.$$  (8)

This result clearly shows that it is possible to get an odd-frequency singlet term provided the substrate is a triplet superconductor with a $d$ vector that is not perpendicular to the even component of $h$, $h_+$. Notice that because $h$ and $d$ belong to different layers they are not constrained to be in any specific relation. Eq. (8) also shows that an odd-frequency triplet term will be present if both $h_+$ and the singlet pairing in the substrate $s^{SC}$ are not zero, as shown previously$^{3,32,33}$. Eq. (8) therefore shows that when $h_+ \neq 0$, and $h_- = 0$, by proximity effect, we will have odd-frequency superconductivity in the 2DEG that has the “opposite” spin structure from the superconductivity in the substrate: triplet if the substrate is a singlet superconductor, singlet if the substrate is a triplet superconductor (with $d$ not orthogonal to $h_+$). A very interesting and novel result is that even when $h_+ = 0$, i.e. no
These monolayers have been shown to possess a direct band gap of 1.8eV, as high as 200 cm$^2$/V$-s$, and have exhibited electron mobilities as high as 200 cm$^{-1}$.

A single monolayer is composed of three covalently bonded layers trigonally coordinated with a layer of transition metal dichalcogenide (TMD). A heterostructure formed by a group-VI dichalcogenide monolayer and a superconductor’s surface can have odd-frequency superconductivity by adding a whole new class of heterostructures in which to look for odd-frequency pairing. In general this situation is realized

consider the heterostructure shown in Fig. 1 composed of a transition metal dichalcogenide (TMD) monolayer on top of a superconductor. The low-energy electronic states of an TMD monolayer are well described by the following valley-dependent Hamiltonian:

$$
\hat{H}_{k,\lambda}^{TMD} = \left[ a \gamma \left( \lambda \hbar k_x \tau_1 + k_y \tau_2 \right) + \frac{u}{2} \tau_3 - \mu \sigma_0 \right] \otimes \sigma_0 
- \frac{\lambda \alpha}{2} \left( \tau_3 - \tau_0 \right) \otimes \sigma_3
$$

where $\tau_i$ are Pauli matrices acting on the orbital space of the TMD monolayer, $a$ is the lattice constant, $\gamma$ is the effective hopping integral, $u$ is the energy gap between the valence and conduction bands, $\alpha$ is the strength of the spin-orbit coupling, $\lambda = \pm 1$ is the valley index ($\lambda = 1$ denotes the $K$ valley, $\lambda = -1$ denotes the $K'$ valley, see Fig. 1), $\kappa = (k_x, k_y, 0)$ is a vector describing small deviations from the $K$ or $K'$ point in $k$-space, and $\mu$ is the chemical potential. For MoS$_2$, $a = 3.193 \, \text{Å}$, $\gamma = 1.10 \, \text{eV}$, $u = 1.66 \, \text{eV}$, and $2\alpha = 0.15 \, \text{eV}$.  

The Hamiltonian in Eq. (9) possesses four eigenstates at the $K$ and $K'$ points; two spin-degenerate conduction states separated by an eV-scale gap from two spin-polarized valence states, as shown in Fig. 1. For our analysis the most interesting case is when MoS$_2$ is hole doped. For this reason in the following we will use an effective 2-band model in which we include only the valence bands. Considering the large gap between the valence and the conduction bands this does not introduce any inaccuracy. For small $k$ the valence band Hamiltonian can be written in spin space as:

$$
\hat{H}_{k,\lambda}^{TMD} = -\left( \frac{a^2 \gamma^2}{u} k^2 + \frac{u}{2} + \mu \right) \sigma_0 + \lambda \alpha \sigma_3.
$$

Notice that, taking into account the valley index $\lambda$, for the parity operator, $P$, acting on a function, $f(k, \lambda)$, we have $P f(k, \lambda) = f(-k, -\lambda)$. Using the notation used in Eqs (7) and (8) we then find that in this case $h_0(k) = -\left( \frac{a^2 \gamma^2}{u} k^2 + \frac{u}{2} + \mu \right)$, $h_+(k, \lambda) = 0$, and $h_-(k, \lambda) = 2\lambda \alpha \hat{\tau}$, where $\hat{\tau}$ is the unit vector normal to the TMD monolayer. Starting from the general Eqs (7) and (8) we then find:

$$
S_{k,\lambda;\omega n}^{\text{even}} = (\omega_n^2 + \xi_k^2 + \alpha^2) s_{k,\lambda;\omega n}^{SC} - 2\lambda \alpha \xi_k \hat{\tau} \cdot \mathbf{d}_{k,\lambda;\omega n}
$$

$$
D_{k,\lambda;\omega n}^{\text{even}} = (\omega_n^2 + \xi_k^2 - \alpha^2) d_{k,\lambda;\omega n} + 2\alpha^2 (\hat{\tau} \cdot \mathbf{d}_{k,\lambda;\omega n}) \hat{\tau}
- 2\lambda \alpha \xi_k s_{k,\lambda;\omega n}^{SC} \hat{\tau}
$$

$$
S_{k,\lambda;\omega n}^{\text{odd}} = 0
$$

$$
D_{k,\lambda;\omega n}^{\text{odd}} = -i2\lambda \alpha \hat{\tau} \times \mathbf{d}_{k,\lambda;\omega n}
$$

In accordance with Eq. (8) we find that, given that $h_+ = 0$, to get odd frequency superconductivity in the TMD we need a substrate with non-zero triplet superconducting pairing. This is a result that significantly enlarges the set of materials that is readily available, easily manufactured and incorporated into heterostructures, but also because of its strong spin-orbit coupling.
in non-centrosymmetric superconductors. Additionally, this condition can be realized at the surface of centrosymmetric singlet superconductors with spin-orbit coupling since the surface breaks inversion symmetry leading to the appearance of a Rashba spin-orbit term that in turn induces a superconducting triplet component. This is expected to be the case for the surface of superconducting Pb.

Considering the case in which the superconductor in Fig. 1 (b) has Rashba spin-orbit coupling, the Hamiltonian matrix describing the single particle spectrum of the superconductor is \( \hat{h}^{\text{SC}}(k) = \mathbf{\tilde{c}} \sigma_0 + \eta \hat{z} \cdot (\mathbf{\sigma} \times \mathbf{k}) \) where \( \eta \) is the Rashba spin-orbit coupling in the superconductor surface, \( \sigma_0 \) is the dispersion of the normal state quasiparticles in the absence of spin-orbit coupling, and \( \mathbf{k} \) is the momentum measured from the Brillouin zone center. Considering that the dominant pairing is intraband we obtain\(^{41,49}\)

\[
F^{\text{SC}}_{k;i\omega_n} = \frac{\Delta}{(s_{k;i\omega_n}^2 - |d_k|^2)^2} \left[ \frac{(s_{k;i\omega_n}^2 + d_k \cdot \sigma)i\sigma_2}{s_{k;i\omega_n}^2 - |d_k|^2} \right]
\]

where \( \Delta \) is the substrate’s superconducting gap, \( s_{k;i\omega_n}^2 = \Delta^2 + \omega_n^2 + \mathbf{\tilde{c}}^2 + \eta^2 \mathbf{k}^2 \) and \( d_k = 2\mathbf{\xi}_{k}(\mathbf{\tilde{c}}_y, \mathbf{\tilde{c}}_x, 0) \). The key point of Eq. (13) is that thanks to the Rashba spin-orbit coupling induced by the breaking of the inversion symmetry at the surface of the Pb substrate a triplet term appears in the \( F^{\text{SC}} \) and that in addition the d vector for such triplet component is perpendicular to the field \( h_\perp \) in the TMD monolayer. The interplay of such triplet component with the spin-orbit coupling of the TMD monolayer gives rise to odd-frequency SC in the TMD.

With the above definitions we can follow the same steps leading to Eqs 8 and 7 and obtain the leading order contribution to the proximity-induced anomalous Green’s function in the TMD layer as

\[
A^{\text{TMD}}_{\mathbf{k}, \lambda; i\omega_n} = F^{\text{odd}}_{\mathbf{k}, \lambda; i\omega_n} + F^{\text{even}}_{\mathbf{k}, \lambda; i\omega_n}
\]

\[
A^{\text{TMD}}_{\mathbf{k}, \lambda; i\omega_n} = \frac{\Delta f^2}{(i\omega_n - \xi_k)^2 - \alpha^2} \left[ \frac{(s_{k+\mathbf{K}_\lambda;i\omega_n}^2 - |d_{k+\mathbf{K}_\lambda}|^2)}{(s_{k;i\omega_n}^2 - |d_k|^2)^2} \right]^{\text{TMD}}
\]

For the even-frequency singlet and triplet components of \( F^{\text{TMD}} \) we find:

\[
S^{\text{even}}_{\mathbf{k}; i\omega_n} = (\omega_n^2 + \varepsilon_{k}^2 + \alpha^2) s_{k+\mathbf{K}_\lambda;i\omega_n}^2
\]

\[
D^{\text{even}}_{\mathbf{k}; i\omega_n} = - (\omega_n^2 + \varepsilon_{k}^2 - \alpha^2) d_{k+\mathbf{K}_\lambda} - 2\lambda \alpha \xi_k s_{k+\mathbf{K}_\lambda;i\omega_n}^2 \xi_{k+\mathbf{K}_\lambda}^2
\]

Given that \( h_+ = 0 \), see Eq. (12), the odd-frequency singlet component vanishes whereas for the triplet component we find:

\[
D^{\text{odd}}_{\mathbf{k}; i\omega_n} = i \lambda \alpha \eta \varepsilon_{k+\mathbf{K}_\lambda}(k + \mathbf{K}_\lambda)
\]

where \( \mathbf{K}_\lambda \) is the momentum vector at the \( K \) (\( K' \)) point for \( \lambda = 1 \) (\( \lambda = -1 \)). Eq. (15) shows that in the TMD the odd-frequency triplet component has a d-vector pointing in the direction of the momentum. One can verify that this corresponds to an equal-spin spin triplet amplitude given by \( F^{\text{TMD}}_{\uparrow\downarrow} \sim i\omega_n \eta \alpha \mathbf{\tilde{c}} \cdot \mathbf{\tilde{c}}^\dagger \) which is proportional to the product of the spin-orbit couplings in the two materials. Consistent with the general case, we see that the emergence of this term requires the spin-orbit couplings in the two media to be non parallel.

Our results add a new class of systems, Van der Waals (VdW) heterostructures, to the odd-frequency playbook. Van der Waals systems have many advantages: i) the 2DEG in which odd-frequency pairing is present lives in a layer with an exposed surface, a fact allows for ideal STS and ARPES measurements; ii) as shown by the example of the MoS\(_2\)/Pb heterostructure, it is possible to realize VdW systems with no ferromagnetic layers, or spin-active interfaces that exhibit odd-frequency SC; iii) the 2DEG in which odd-frequency pairing is present can be just one atom thick, this fact removes many of the complications associated with the interpretation of STS and ARPES data done in heterostructures in which each layer is several nanometers thick; iv) because the top layer is just one atom thick the electrons are truly confined in 2D, this fact, combined with the fact that according to our results the top layer can be a semiconductor, rather than a ferromagnetic metal as in previous proposals, ensures that the DOS of the normal state is quite low and therefore allows for an easier observation of the features in the DOS due to the presence of odd-frequency pairing.

In conclusion, in this work we investigated the symmetries of proximity-induced superconducting pairing amplitudes in a 2DEG coupled to a superconductor. We arrived at a general expression relating the induced pairing amplitudes to the components of the anomalous Green’s function of the superconducting substrate and the elements of the 2DEG Hamiltonian matrix, \( \hat{h}(k) = h_0(k)\sigma_0 + h(k) \cdot \sigma \). We have shown that the interplay of the spin-orbit coupling in the 2DEG and the superconducting pairing of the substrate can give rise, via proximity effect, to unusual superconducting pairings in the 2DEG. We find that even when no ferromagnetism is present in the 2DEG, and there is no spin-active interface, odd-frequency superconductivity can be induced in the 2DEG provided the 2DEG has spin-orbit coupling and the substrate has some triplet superconductivity. We then showed that this condition can be realized in a MoS\(_2\)/Pb heterostructure. This result, combined with the general equations that we obtain, adds a new class of systems, Van der Waals (VdW) heterostructures, to the odd-frequency playbook.

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