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Stability of solitary waves and vortices in a 2D nonlinear Dirac model

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We explore a prototypical *two-dimensional massive* model of the nonlinear Dirac type and examine its solitary wave and vortex solutions. In addition to identifying the stationary states, we provide a systematic spectral stability analysis, illustrating the potential of spinor solutions to be neutrally stable in a wide parametric interval of frequencies. Solutions of higher vorticity are generically unstable and split into lower charge vortices in a way that preserves the total vorticity. These conclusions are found not to be restricted to the case of cubic two-dimensional nonlinearities but are found to be extended to the case of quintic nonlinearity, as well as to that of three spatial dimensions. Our results also reveal nontrivial differences with respect to the better understood non-relativistic analogue of the model, namely the nonlinear Schrödinger equation.

Introduction. In the context of dispersive nonlinear wave equations, admittedly the prototypical model that has attracted a wide range of attention in optics, atomic physics, fluid mechanics, condensed matter and mathematical physics is the nonlinear Schrödinger equation (NLS) [1–7]. By comparison, far less attention has been paid to its relativistic analogue, the nonlinear Dirac equation (NLD) [8], despite its presence for almost 80 years in the context of high-energy physics [9–13]. This trend is slowly starting to change, arguably, for three principal reasons. Firstly, significant steps have been taken in the nonlinear analysis of stability of such models [14–19], especially in the one-dimensional (1d) setting. Secondly, computational advances have enabled a better understanding of the associated solutions and their dynamics [20–24]. Thirdly, and perhaps most importantly, NLD starts emerging in physical systems which arise in a diverse set of contexts of considerable interest. These contexts include, in particular, bosonic evolution in honeycomb lattices [25, 26] and a growing class of atomically thin 2d Dirac materials [27] such as graphene, silicene, germanene and transition metal dichalcogenides [28]. Recently, the physical aspects of nonlinear optics, such as light propagation in honeycomb photorefractive lattices (the so-called photonic graphene) [29, 30] have prompted the consideration of intriguing dynamical features, e.g. conical diffraction in 2d honeycomb lattices [31]. Inclusion of nonlinearity is then quite natural in these models, although in a number of them (e.g., in atomic and optical physics) the nonlinearity does not couple the spinor components.

These physical aspects have also led to a discussion of potential 2d solutions of NLD in [25, 26]. However, a systematic and definitive characterization of stability and nonlinear dynamical evolution of the prototypical coherent structures in NLD models is still lacking, to the best of our knowledge. The present work is dedicated to offering analytical and numerical insights into these crucial mathematical and physical aspects of higher-dimensional nonlinear Dirac equations bearing in mind the physical relevance and potential observability of such waveforms. As our model of choice, in order to also be able to compare and contrast with the multitude of existing 1d results (e.g. [18, 23]), we select the well-established Soler model [32] (known in 1d as the Gross–Neveu model [33]), which is a Dirac equation with scalar self-interaction. Such self-interaction is based on including into the Lagrangian density a function of the quantity $\bar{\psi}\psi$ (which transforms as a scalar under the Lorentz transformations):

$$\mathcal{L}_{\mathrm{Soler}} = \bar{\psi}\left(i\gamma^{\mu}\partial_{\mu} - m\right)\psi + \frac{g}{2}\left(\bar{\psi}\psi\right)^{2}, \qquad (1)$$

where $m > 0$, $g > 0$, $\psi(x,t) \in \mathbb{C}^N$, $x \in \mathbb{R}^n$ and γ^{μ}, $0 \le \mu \le n$ are $N \times N$ Dirac γ-matrices satisfying the anticommutation relations $\{\gamma^{\mu}, \gamma^{\nu}\} = 2\eta^{\mu\nu}$, with $\eta^{\mu\nu}$ the Minkowski tensor [34], and $\bar{\psi} = \psi^{*}\gamma^{0}$. (The Clifford Algebra theory gives the relation $N \ge 2^{[(n+1)/2]}$ between the spatial dimension and spinor components [35, Chapter 1, §5.3].) The nonlinearity of the model is generalized in the spirit of [22], by using $g(\bar{\psi}\psi)^{k+1}/(k+1)$ with $k > 0$. The proof of existence of solitary waves in this model (in 3d) is in [36–38].

Our results show that the NLD in 2d admits different solutions involving a structure of vorticity $S \in \mathbb{Z}$ in the first spinor component, with the other spinor component bearing a vorticity $S + 1$. We identify such solutions for $S = 0, 1, \ldots$. While prior stability results have often been inconclusive (particularly in higher dimensions, see, e.g., [39]), our numerical computation of the spectrum of the corresponding linearization operator reveals that *only* the $S = 0$ solutions can be spectrally stable (the spectrum of the linearization contains no eigenvalues with positive real part), and that this stability takes place in a rather wide interval of the frequency of the solitary waves. On the contrary, we find that the states of higher vorticity are generically linearly unstable. Complementing the stability analysis results, our direct dynamical evolution studies show that the unstable higher vorticity solutions break up into lower vorticity waveforms, yet conserving the total vorticity. Importantly, the fundamental $S = 0$ solutions are found to be potentially stable in models both with a higher order (quintic) two-dimensional nonlinearity, as well as in higher dimensions (3d) under cubic nonlinearity. These features again reflect differences from the NLS model and as such suggest the particular interest towards a broader and deeper study of NLD models.

An important extension of our stability findings for higher dimensional $S = 0$ solutions, is that they remain valid for other types of nonlinearities. These include non-Lorentz-invariant ones such as most notably those arising in atomic [25, 26], and optical [29, 30] problems. The fundamental difference of those models is that they correspond to *massless* equations, contrary to the Soler model. For this reason, we have confirmed our stability conclusions by comparison with those emerging from the model for square binary waveguides [41] which leads to a *massive* nonlinear Dirac equation with the same nonlinearity as in [25, 26].

Theoretical Setup. We start from the prototypical 2d nonlinear Dirac equation system, derived from the Lagrangian density (1) with $k = 1$ and $m = g = 1$:

$$\begin{aligned} i\partial_t\psi_1 &= -(i\partial_x + \partial_y)\psi_2 + f(\bar\psi\psi)\psi_1, \\ i\partial_t\psi_2 &= -(i\partial_x - \partial_y)\psi_1 - f(\bar\psi\psi)\psi_2, \end{aligned} \tag{2}$$

where ψ_1, ψ_2 are the components of the spinor $\psi \in \mathbb{C}^2$ and the nonlinearity is $f(\bar\psi\psi) = 1-(\bar\psi\psi)^k = 1-(|\psi_1|^2-|\psi_2|^2)^k$. We note that (2) is a $\mathbf{U}(1)$-invariant, translation-invariant Hamiltonian system.

We simplify our analysis by using the polar coordinates, where Eq. (2) takes the form

$$\begin{aligned} i\partial_t\psi_1 &= -\mathrm{e}^{-i\theta}\left(i\partial_r + \frac{\partial_\theta}{r}\right)\psi_2 + f(\psi_1,\psi_2)\psi_1, \\ i\partial_t\psi_2 &= -\mathrm{e}^{i\theta}\left(i\partial_r - \frac{\partial_\theta}{r}\right)\psi_1 - f(\psi_1,\psi_2)\psi_2. \end{aligned} \tag{3}$$

The form of this equation suggests that we look for solutions as $\psi(\vec r,t) = \exp(-i\omega t)\phi(\vec r)$ with

$$\phi(\vec r) = \begin{bmatrix} v(r)\mathrm{e}^{iS\theta} \\ i\,u(r)\mathrm{e}^{i(S+1)\theta} \end{bmatrix}, \tag{4}$$

with $v(r)$ and $u(r)$ real-valued. The value $S \in \mathbb{Z}$ can be cast as the vorticity of the first spinor component.

Once solitary waves have been identified, we explore their stability. This approach has been previously developed in related settings including the multi-component NLS (see e.g. [42]), as well as a massless variant of the Dirac equation of [43]. The presence of a mass in our case allows not only a direct comparison with NLS (when $\omega \to m \equiv 1$), but also generates fundamental differences between our results and those of [25, 26, 43], as discussed below as well.

To examine its spectral stability, we consider a solution ψ in the form of a perturbed solitary wave solution:

$$\psi(\vec r,t) = \begin{bmatrix} (v(r)+\rho_1(r,\theta,t))\mathrm{e}^{iS\theta} \\ i\big(u(r)+\rho_2(r,\theta,t)\big)\mathrm{e}^{i(S+1)\theta} \end{bmatrix}\mathrm{e}^{-i\omega t}, \tag{5}$$

with $\rho = (\rho_1,\rho_2)^T \in \mathbb{C}^2$ a small perturbation. We consider the linearized equation on ρ,

$$\partial_t R = A_\omega R, \tag{6}$$

with $R(r,\theta,t) = (\operatorname{Re}\rho, \operatorname{Im}\rho)^T \in \mathbb{R}^4$ and with a matrix-valued first order differential operator $A_\omega(r,\theta,\partial_r,\partial_\theta)$ [44]. If the spectrum of the linearization operator A_ω contains an eigenvalue $\lambda \in \sigma(A_\omega)$ with $\operatorname{Re}\lambda > 0$, we say that the solitary wave is linearly unstable; in such cases, we resort to dynamical simulations of Eqs. (2) to explore the outcome of the unstable evolution. If there are no such eigenvalues, the solitary wave is called spectrally stable.

A convenient feature of NLS ground states is that the linearization operator at such states, albeit non-selfadjoint, has its point spectrum confined to the real and imaginary axes. This observation is at the base of the VK criterion [40]: a linear instability can thus develop when a positive eigenvalue bifurcates from $\lambda = 0$. More precisely, the loss of stability due to the appearance of a pair of positive and a pair of negative eigenvalues follows the jump in size of the Jordan block corresponding to the unitary invariance; this happens when the VK condition $\partial_\omega Q(\omega) = 0$ is satisfied, with $Q(\omega)$ being the charge of a solitary wave.

Crucially, in the NLD case, the spectrum of the linearization at a solitary wave is no longer confined to the real and imaginary axes; the linear stability analysis requires that one studies the *whole complex plane*. The key observation is that A_ω in (6) contains r, ∂_r, ∂_θ, but not θ; this allows to perform a detailed study of the spectrum of A_ω using the decomposition of spinors into Fourier harmonics corresponding to different $q \in \mathbb{Z}$ [44].

In the 3d case, we are not yet able to perform the general spectral analysis, but we studied the part of spectrum in the invariant subspace corresponding to perturbations of the same angular structure as the solitary waves [45], $[v(r)[1,0], iu(r)[\cos\theta, e^{i\phi}\sin\theta]]^T$; this invariant subspace seems most important since it is responsible for the linear instability in the non-relativistic limit $\omega \to 1$ which is a consequence of the instability of the 3d cubic NLS.

Numerical results. We have analyzed the existence and stability of solitary waves ($S = 0$, with its first component radially symmetric and the second component having vorticity 1) and vortex solutions ($S = 1$, with its components having vortices of order one and two, respectively). Both solitary waves and vortex solutions exist in the frequency interval $\omega \in (0, m = 1)$, a feature critically distinguishing our models from those of [25, 26]. An intriguing feature of the relevant waveforms is that both the radial profile of the solitary waves and that of the vortices possess a maximum that shifts from $r = 0$ to a larger r when ω approaches zero (see Fig. 1),

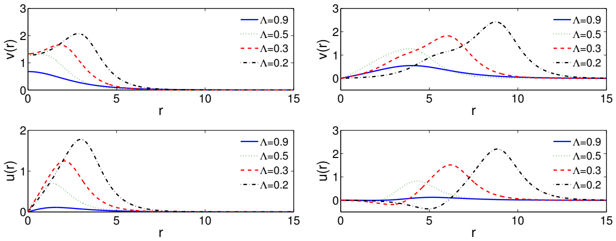

FIG. 1: Radial profiles of the spinor components for (left) $S = 0$ solitary waves and (right) $S = 1$ vortices for different values of ω.

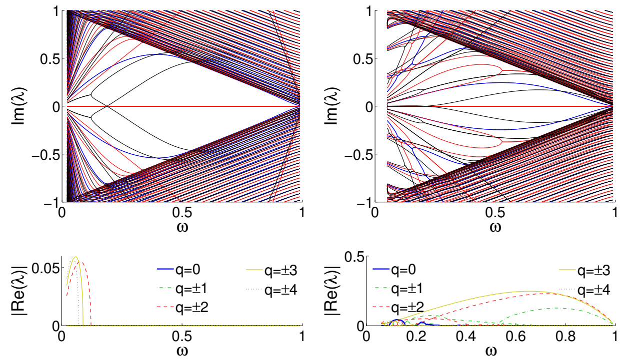

FIG. 2: 2d Soler model with cubic ($k = 1$) nonlinearity. Dependence of the (top) imaginary and (bottom) real part of the eigenvalues with respect to ω. Left (respectively, right) panels correspond to $S = 0$ solitary waves ($S = 1$ vortices). For the sake of clarity, we only included the values $|q| \leq 2$ for the imaginary part and $|q| \leq 4$ for the real part. In the former case, the imaginary part of the eigenvalues for $q = 0$, $q = \pm 1$ and $q = \pm 2$ are represented by, respectively, blue, red and black lines.

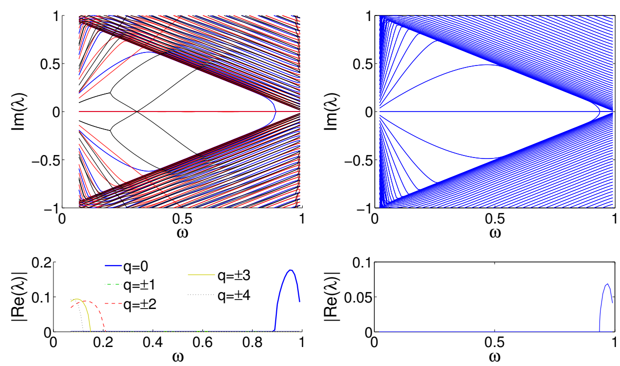

FIG. 3: Left: Solitary waves in the 2d Soler model with quintic ($k = 2$) nonlinearity. Dependence of the (top) imaginary and (bottom) real part of the eigenvalues with respect to ω in the same format as the previous figure. Right: Solitary waves in the 3d Soler model with cubic ($k = 1$) nonlinearity. Spectrum of the linearization in the one-dimensional invariant ($q = 0$) subspace which contains the eigenvalue that is responsible for the instability for $\omega \in (\omega_c, 1)$, with $\omega_c \approx 0.936$.

in a way reminiscent of the corresponding 1d solitary wave structures [22]. Here the relevant state will feature a stationary bright intensity ring. In order to obtain and analyze such coherent structures, we have made use of the numerical methods detailed in [44]. To confirm the results, we also computed the spectra using the Evans function approach of [23] adapted to the present problem.

We start by considering the stability of $S = 0$ solitary waves in the cubic ($k = 1$) case. Figure 2 shows the dependence of the real and imaginary parts of the eigenvalues with respect to the stationary solution frequency ω. From the spectral dependencies we can deduce several features of the 2d NLD equation: (1) It is known that the 2d NLS equation is charge-critical, and the zero eigenvalues are degenerate [6]: they have higher algebraic multiplicity. In the NLD case, however, this degeneracy is resolved: in the $S = 0$ case, as ω starts decreasing, two eigenvalues (corresponding to $q = 0$) start at the origin when $\omega = 1$ and move out of the origin for $\omega \lesssim 1$. The absence of the algebraic degeneracy of the zero eigenvalue *prevents solitary waves from NLS-like self-similar blow-up* which is possible in charge-critical NLS [46]. (2) The $\mathbf{U}(1)$ symmetry and the translation symmetry of the model result in zero eigenvalues with $q = 0$ and $|q| = 1$, respectively (in both $S = 0$ and $S = 1$ cases). (3) As in the 1d NLD equation, there are also the eigenvalues $\lambda = \pm 2\omega i$ which are associated with the $\mathbf{SU}(1, 1)$ symmetry of the model [47]. This eigenvalue pair corresponds to $q = -(2S + 1)$, i.e., to a highly excited linearization eigenstate. (4) Contrary to the 1d case, where the solitary waves corresponding to any $\omega < 1$ are spectrally stable, the $S = 0$ solitary wave is linearly unstable for $\omega < 0.121$ because of the emergence of nonzero real part eigenvalues via a *Hamiltonian Hopf bifurcation* in the $|q| = 2$ spectrum at $\omega = 0.121$. Another Hopf bifurcation occurs corresponding to $|q| = 3$ (at $\omega = 0.0885$), then yet another one corresponding to $|q| = 4$.

It is especially interesting that a wide parametric (over frequencies) interval of stability of solitary waves can *also* be observed in the quintic ($k = 2$) NLD case (see Fig. 3); while the quintic NLS solitary waves blow up (even in one dimension), the quintic NLD solitary waves are stable even in two dimensions, except for the interval $\omega < 0.312$ where the coherent structures experience the same Hopf bifurcation as in the cubic case, and for $\omega > 0.890$ where an exponential instability created by radial $q = 0$ perturbations emerges. Perhaps even more remarkably, the right panel of the Fig. 3 illustrates that this stability of NLD solitons against radial perturbations can be found in suitable frequency intervals *even in 3d* (see [48] for a discussion of the equations for existence and stability of radial perturbations in 3d). Both of the above cases (quintic 2d and cubic 3d NLD) are charge-supercritical i.e., the charge goes to infinity in the nonrelativistic limit $\omega \to m$. Contrary to the pure-power supercritical NLS whose solitary waves remain linearly unstable for all frequencies, solitary waves in the Soler model become spectrally stable when ω drops below some dimension-dependent critical value $\omega_c = \omega_c(n, k)$ [44]. The relevant unstable eigenvalue (associated with $q = 0$ and radially-symmetric collapse) is only present as real for $\omega \in (\omega_c, 1)$, where $\omega_c \approx 0.936$. This was identified in

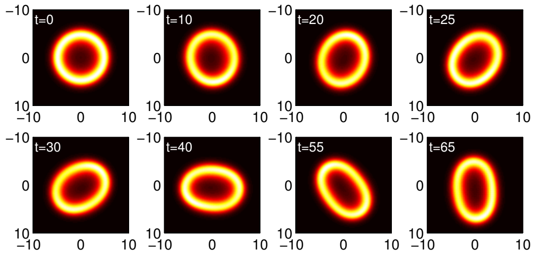

FIG. 4: Snapshots showing the evolution of the density of an unstable $S = 0$ solitary wave with $\omega = 0.12$. The soliton which initially had a circular shape becomes elliptical and rotates around the center of the original solitary wave.

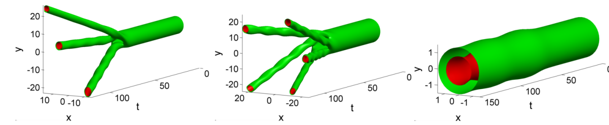

FIG. 5: Isosurfaces for the density of an $S = 1$ (left) and $S = 2$ (center) vortex with $k = 1$, $\omega = 0.6$ and (right) an $S = 0$ solitary wave with $k = 2$ and $\omega = 0.94$.

[32] as the value at which both the energy and charge of solitary waves have a minimum. Hence, we indeed find that the radially-symmetric collapse-related instability ceases to be present below this critical point. Finally, as regards two dimensions, $S = 1$ vortices are unstable for every ω, because of the presence in the spectrum of quadruplets of complex eigenvalues. These quadruplets emerge (and disappear) for different values of q via direct (inverse Hopf) bifurcations; see the right panel of Fig. 2. The spectrum for $S = 2$ vortex is quite similar to that of $S = 1$; for this reason, we do not analyze it further.

In order to analyze the result of instabilities in 2d settings, we have probed the dynamics of unstable solutions directly (see [44] for details). Prototypical examples of unstable $S = 0$ solitary waves and $S = 1$ vortices for $k = 1$ are shown in Figs. 4 and 5. As can be observed, the $S = 0$ solitary waves spontaneously amplify perturbations breaking the radial symmetry in their density and, as a result, become elliptical and rotate around the center of the circular density of the original solitary wave in line with the expected amplification of the $q = 2$ unstable eigenmode. On the other hand, the $S = 1$ vortices split into three smaller ones. Let us mention that in the latter case, the first spinor component splits into structures without angular dependence, whereas the second component splits into corresponding ones with angular dependence $\propto e^{i\theta}$, in accordance with the ansatz of Eq. (4). This preserves the total vorticity across the two components, as is also shown in Fig. 5. Along a similar vein, the instability of an $S = 2$ vortex eventually leads to the emergence of five $(0, 1)$ pairs, again preserving the total vorticity. Finally, we have analyzed the outcome of the instabilities caused by radially-symmetric perturbations in the $k = 2$ case for $\omega > \omega_c$ (see Fig. 5). We can observe the typical behavior of such solutions, i.e. the density width (and amplitude) oscillate leading to a "breathing" structure, but there is no collapse. This phenomenology is reminiscent of the 1d case [49].

Conclusions and Future Challenges. We have illustrated that solitary waves of vorticity $S = 0$ in one spinor component and $S = 1$ in the other are *spectrally stable within a large parametric interval*, suggesting their physical relevance. In that connection, we highlight that although our models of choice may bear a particular nonlinearity, our results suggest that under different nonlinearities including the more physically relevant ones of e.g., [25, 26] and massive models [41] *still* bear stable solitary waves for a suitable *wide parametric range* of frequencies. Thus, the conclusion of higher dimensional stability is more general than the specifics of our particular nonlinearity and hence of broad interest. We also showcased the significant difference of NLD from the focusing NLS equation, where solitary waves are linearly unstable in the charge-supercritical cases. When the NLD solutions were found to be unstable, their dynamical evolution suggested breathing oscillations in the $S = 0$ case and splitting into lower charge configurations for $S = 1$ and $S = 2$.

It is of interest to extend present considerations to numerous settings. From a mathematical physics perspective, it would be useful to explore further the 3d stability and associated dynamics. This is especially timely given that the 3d analogue of photonic graphene has been experimentally realized very recently [50]. Admittedly, the latter setting does not feature a mass in the model, thus the generalization of NLD models such as those appearing in the works of [25, 26, 43] would be particularly important there. It would also be of interest to compare more systematically the present findings with models associated with different nonlinearities, including the case of honeycomb lattices in atomic and optical media or, e.g., those stemming from wave resonances in low-contrast photonic crystals [51].

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