Geometric Heat Engines Featuring Power that Grows with Efficiency

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Phys. Rev. Lett. **116**, 160601 — Published 21 April 2016

DOI: 10.1103/PhysRevLett.116.160601

Geometric Heat Engines Featuring Power that Grows with Efficiency

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Thermodynamics places a limit on the efficiency of heat engines, but not on their output power or on how the power and efficiency change with the engine's cycle time. In this letter, we develop a geometrical description of the power and efficiency as a function of the cycle time, applicable to an important class of heat engine models. This geometrical description is used to design engine protocols that attain both the maximal power and maximal efficiency at the fast driving limit. Furthermore, using this method we also prove that no protocol can exactly attain the Carnot efficiency at non-zero power.

Introduction Heat engines - machines that exploit temperature differences to extract useful work, are modeled as operating in either a non-equilibrium steady-state, e.g. thermoelectric engine [1, 2], or as a cyclic engine, where external parameters and temperature are varied periodically in time, e.g. the Carnot, Otto, Stirling and the Diesel cycles [3]. Both types of engines are characterized by two main figures of merit: efficiency and power.

In recent years, many important properties of steady state heat engines were discovered. For example, it was shown that their power and efficiency cannot be maximized simultaneously, a property which we refer to as *power-efficiency trade-off* [4–10]. Less is known about the efficiency and power of cyclic heat engines, but a lot of research effort has been devoted to understanding them in the last decade [11–18]. The operation of a cyclic engine is characterized by a protocol that describes the time dependence of key variables along the cycle — e.g. piston position and temperature. The set of feasible protocols however, is strongly bounded by a set of engine specific and hence non-generic constraints. Maximizing power or efficiency is, therefore, a nontrivial constrained optimization problem. Nevertheless, there is a natural optimization problem which is both simpler and of practical importance: the efficiency and power of a heat engine with a fixed protocol as a function of the cycle time.

Do cyclic heat engines have a power-efficiency trade-off as a function of their cycle time, analogous to the trade-off in steady state heat engines? Analytical [15], numerical [19, 20] and experimental [21] results for certain driving protocols seem to suggest that this might be the case: increasing the cycle time increases the efficiency, with the maximal efficiency (which is possibly lower then the Carnot limit as in the Diesel, Miller and Sargent cycles) only attained in the quasi-static limit — namely at infinitely long cycle time, where the power vanishes. Driving the engine faster increases the power at the expense of efficiency, until eventually, at fast enough driving, the dissipation rate becomes significant and causes a decrease in power. Yet, as we demonstrate, this commonly adopted viewpoint [27] is not universally valid, and there is no inherent trade-off between power and efficiency as a function of the cycle time, although such a trade-off always exists in cycles that exactly attain the Carnot bound.

Here we analyze a class of cyclic heat engine models, referred to as *geometric heat engines*, which includes the paradigmatic examples of a Brownian particle in a parabolic potential [11, 12, 15, 21] and the two-state Markovian engines [23], but is not limited to these models. In this class, the work and heat can be interpreted as areas in state space (the space of all the possible states of the engine) defined through the periodic trajectory of the engine's state. Using geometrical insights, we construct a protocol whose power and efficiency are both maximized at the infinitely fast driving limit. This proves that cyclic heat engines do not have an inherent trade-off between power and efficiency as a function of their cycle time. Achieving the maximum efficiency at finite power, however, comes with a price: we prove that in this class of models the Carnot limit cannot be attained at positive power. Therefore, to avoid the trade-off, the efficiency must be lower then the Carnot limit. Similar relations between the Carnot limit and power were discussed in [11, 12] in the linear response regime, and very recently for systems with local detailed balance in [24].

Model description: We focus here on a specific model, and subsequently show that our results are valid for a larger class of models. This model consists of an overdamped Brownian particle confined to one spatial dimension, in a time dependent harmonic potential $V(x,t) = \frac{\Lambda(t)}{2}x^2$, coupled to a heat bath with a time dependent inverse temperature $\beta(t)$. This model was suggested in [15] and experimentally realized in [21]. Both $\Lambda(t)$ and $\beta(t)$ are periodic with a cycle time τ [28]. The engine's protocol, which is a time-parametrized closed curve in the *control space* - the space of external control parameters, is denoted by $\Gamma_t^{\beta,\Lambda} = (\beta(t), \Lambda(t))$ where the subscript t indicates that the protocol is parametrized by the time and the superscripts β, Λ indicate that this protocol is defined in the $[\beta, \Lambda]$ control space.

To describe the engine's properties when the same protocol is performed at different cycle times, it is useful to consider the protocol in terms of a dimensionless time parameter, $s = t/\tau \in [0, 1)$. Defining the protocol as $\Gamma_s^{\beta,\Lambda} = (\beta(s), \Lambda(s))$, allows us to treat the cycle time τ as a parameter, independent of other characteristics of the protocol.

As described in [15], when this engine model reaches its periodic state, the probability distribution of the particle's position is a centered Gaussian whose width (variance) $w = \langle x^2 \rangle$ evolves according to [29]:

$$\frac{dw}{ds} = \tau\Big(\beta^{-1}(s) - \Lambda(s)w(s)\Big). \quad (1)$$

The infinitesimal system-bath heat exchange is given by $dQ = \frac{\Lambda}{2}\frac{dw}{ds}ds$, where $dQ > 0$ implies that heat flows into the system [15, 25]. By energy conservation, the total work extracted in a cycle and the corresponding power are:

$$W = \int_0^\tau dQ = \int_0^1 \frac{\Lambda}{2} dw; \;\; P = \tfrac{W}{\tau}. \quad (2)$$

The work W has a geometrical interpretation: it is half the oriented area bounded by $\Gamma_s^{w,\Lambda} = (w(s), \Lambda(s))$, which is the trajectory the system follows in its *state space*, $[w, \Lambda]$. An important consequence of the geometrical interpretation is that the work is parametrization independent: if some other driving protocol, $\hat{\Gamma}_s^{\beta,\Lambda}$, happens to trace the same contour in the $[w, \Lambda]$ space as $\Gamma_s^{w,\Lambda}$ but at a different s parametrization, then the extracted work is equal in the two protocols, even though $\Gamma_s^{w,\Lambda} \neq \hat{\Gamma}_s^{w,\Lambda}$.

To define efficiency, the cost of any protocol in terms of the extracted heat during a cycle must be quantified. This can be done by accounting for only the sections of the cycle during which heat flows from the bath into the system:

$$Q_{in} = \int_0^1 \frac{\Lambda}{2}\frac{dw}{ds}\Theta\left[\Lambda\frac{dw}{ds}\right]ds, \quad (3)$$

where $\Theta[\cdot]$ is the Heaviside step function. Q_{in} can be interpreted geometrically as half the area under the sections of $\Gamma_s^{w,\Lambda}$ in which w decreases [30]. With these definitions, which are consistent with the laws of thermodynamics [15], the efficiency is given by $\eta = \frac{W}{Q_{in}}$, and it can be interpreted as the ratio between the two corresponding areas [31].

The work and heat are interpreted as areas in state space $[w, \Lambda]$, whereas the engine's driving protocol is defined in the $[\beta, \Lambda]$ control space. This makes it difficult to directly relate the protocol to the areas associated with work and heat. However, the protocol can also be defined as $\Gamma_s^{\Omega,\Lambda} = (\Omega(s), \Lambda(s))$, where $\Omega = (\beta\Lambda)^{-1}$. Note that Ω is the width of the Boltzmann distribution for the potential $V = \Lambda\frac{x^2}{2}$. The main advantage of defining the protocol as $\Gamma_s^{\Omega,\Lambda}$ is that in the quasi-static limit, $\tau \to \infty$, Eq.(1) implies that $w(s) \to \Omega(s)$ and $\Gamma_s^{w,\Lambda} \to \Gamma_s^{\Omega,\Lambda}$, therefore we can unify the control space and state space. With decreasing τ, the contour that defines the protocol, $\Gamma_s^{\Omega,\Lambda}$, continuously deforms into the contour $\Gamma_s^{w,\Lambda}$, which has the geometrical interpretation for work and heat.

Finite cycle time: Consider next a simple driving protocol, in which $\Gamma_t^{\beta,\Lambda}$ traces a circle at a uniform rate in the control space $[\beta, \Lambda]$, as shown in the upper inset of Fig(1). The upper panel shows the efficiency and power as a function of the cycle time τ. This protocol has the typical behavior described above: the efficiency is a monotonically increasing function of τ, attaining its maximum as $\tau \to \infty$, where the power is zero. The power is maximal at $\tau \approx 25$. Below about $\tau = 11$, the work – and hence the power and efficiency – becomes negative. For this to happens, the area bounded by $\Gamma_s^{w,\Lambda}$ must changes its orientation. How can this comes about?

There are only two generic ways in which the area orientation of a closed curve in 2D can change through continuous deformations: a cusp singularity or a self-tangent singularity [26]. We next analyze the formation of cusps, whereas the self-tangent case is analyzed in the SI. A cusp in $\Gamma_s^{w,\Lambda}$ emerges when both $w(s)$ and $\Lambda(s)$ have an extreme point at the same value of s [26]: if varying τ causes the value of s^* at which $\frac{dw}{ds}(s^*) = 0$ to pass through s^{**} for which $\frac{d\Lambda}{ds}(s^{**}) = 0$, then a cusp is formed when $s^* = s^{**}$, and developed into a loop with an inverted orientation. This loop decreases the power and efficiency, enabling the work to vanish at some non-zero τ. In the example described in Fig.(1), a cusp is generated at $\tau \approx 17$. Below this τ, the cusp evolves into a negatively oriented loop, reducing the power and efficiency.

Realizing that the negative power can be related to negatively oriented areas immediately raises the question: can we design a protocol in which they never occur? To avoid singularities, the locations $s_i^*(\tau)$ of the extreme points of $w(s)$, where the index i label the different extreme points, must be considered. However, even if we know $s_i^*(\tau)$, manipulating the protocol to avoid the singularities is challenging: varying either $\Lambda(s)$ or $\Omega(s)$ varies $s_i^*(\tau)$ too. To simplify the analysis, we take advantage of the parametrization invariant property of areas. Instead of the actual parameterization, we consider the reparametrized quantities

$$\bar{w}(s) = w\Big(\lambda(s)\Big), \;\; \bar{\Lambda}(s) = \Lambda\Big(\lambda(s)\Big), \;\; \bar{\Omega}(s) = \Omega\Big(\lambda(s)\Big) \quad (4)$$

where

$$\lambda(s) = \frac{\int_0^s \bar{\Lambda}(x)^{-1}dx}{\int_0^1 \bar{\Lambda}(x)^{-1}dx}. \quad (5)$$

Note that $\lambda(s)$ is monotonically increasing with s, and moreover $\lambda(0) = 0$ and $\lambda(1) = 1$. Also note that $\lambda(s)$ is given in terms of $\bar{\Lambda}(s)$ rather then $\Lambda(s)$. Although

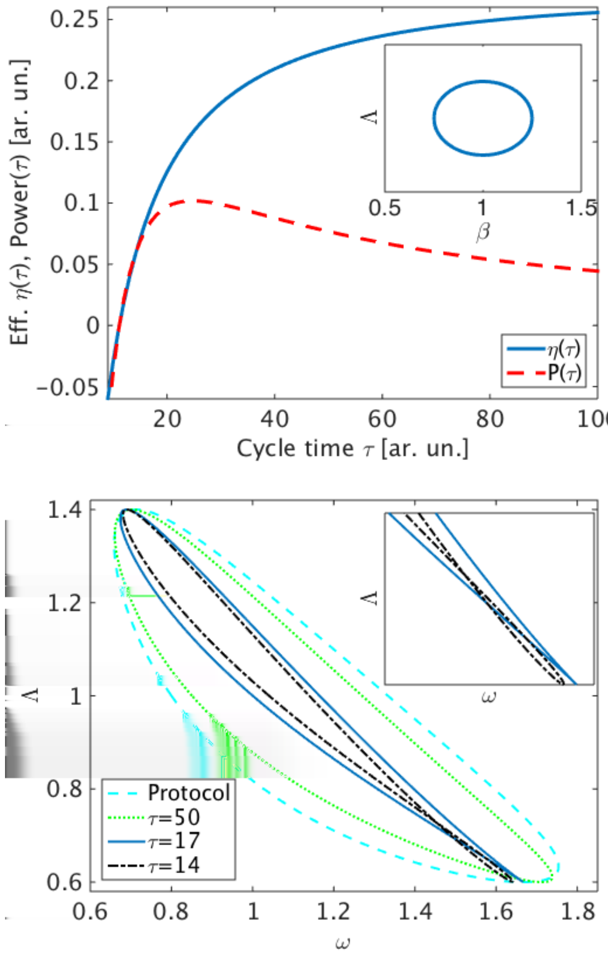

FIG. 1: **Upper panel:** Power and efficiency as a function of the cycle time τ for a circular protocol in the $[\beta, \Lambda]$ control space (inset). This protocol attains its maximal efficiency ($\eta_{max} \approx 0.26$) at the $\tau \to \infty$ limit. The Carnot limit for this protocol is $\eta_C = 0.4$. At high driving rates, the power changed its sign. **Lower panel:** The $[w, \Lambda]$ curves of the above protocol for few values of the cycle time τ. At about $\tau = 17$ the curve develops a cusp, which for even smaller τ evolves into a loop with a negative orientation. The inset shows a blow-up of the cusp and a negatively oriented loop.

$\bar{w}(s) \neq w(s)$ and $\bar{\Lambda}(s) \neq \Lambda(s)$, their corresponding curves, $\Gamma_s^{w,\Lambda}$ and $\Gamma_s^{\bar{w},\bar{\Lambda}}$, trace *the same contour* in the engine's state space, $[w, \Lambda]$, hence they have the same work and efficiency. In this reparameterization,

$$\frac{d\bar{w}}{ds} = \bar{\tau}\left(\bar{\Omega}(s) - \bar{w}(s)\right) \tag{6}$$

where $\bar{\tau} = \frac{\tau}{\int_0^1 \bar{\Lambda}(x)^{-1} dx}$.

The main advantage in the reparametrization is that Eq. (6) is independent of the potential width $\bar{\Lambda}$. Therefore, $\bar{\Lambda}(s)$ can be manipulated without affecting $\bar{w}(s)$, but at the price of rescaling τ.

To avoid cusp generation, we next explain how the extreme points of $\bar{w}(s)$ change with $\bar{\tau}$. Let us denote by s_i^* the values of s at which $\bar{w}(s)$ has an extreme point, namely $\frac{d\bar{w}}{ds}(s_i^*) = 0$. By taking the derivative of Eq.(6) and using $\frac{d\bar{w}}{ds}(s_i^*) = 0$ it follows that

$$\bar{w}(s_i^*) = \bar{\Omega}(s_i^*); \quad \frac{d^2\bar{w}}{ds^2}(s_i^*) = \bar{\tau}\frac{d\bar{\Omega}}{ds}(s_i^*). \tag{7}$$

These equations can be interpreted as follows: In the limit $\hat{\tau} \to \infty$, $\bar{w}(s) = \bar{\Omega}(s)$ everywhere. Decreasing $\bar{\tau}$, the maximal points of $\bar{w}(s)$, for which $\frac{d^2\bar{w}}{ds^2}(s_i^*) < 0$, “slide” along the $\bar{\Omega}(s)$ curve down and to the right (where the slope of $\bar{\Omega}(s)$ is negative), and similarly, the minimal points of $\bar{w}(s)$ (where $\frac{d^2\bar{w}}{ds^2}(s_i^*) > 0$) slides up and to the right along the positive slope of $\bar{\Omega}(s)$. Physically, this means that decreasing $\bar{\tau}$ results in a flattening of $\bar{w}(s)$, and an increasing phase-lag between $\bar{w}(s)$ and $\bar{\Omega}(s)$, (see lower panel of Fig. 2). As we show in the SI, it is the phase-lag that deteriorates the power, not the flattening.

Designing a protocol without a power-efficiency trade-off. To avoid the power deterioration, which is the typical behavior for simple protocols in the $[\beta, \Lambda]$ control space, we should design a protocol that does not form singularities. This can be done by choosing $\bar{\Omega}(s) = c_1\bar{\Lambda}(s) + c_2$ for some constants c_1 and c_2, and $\bar{\Lambda}(s)$ that has a single oscillation. In such a protocol, the extreme points of $\bar{w}(s)$ coincide with those of $\bar{\Lambda}(s)$ only at $\bar{\tau} \to \infty$, and no self tangent can be formed (SI). In the $\bar{\tau} \to \infty$ limit, $\Gamma_s^{\bar{w},\bar{\Lambda}}$ bounds no area — the work and efficiency are zero. Decreasing $\bar{\tau}$, the phase-lag between $\bar{\Lambda}(s)$ and $\bar{w}(s)$ ‘inflates’ the area. In Fig.(2) we demonstrate this through the example $\bar{\Lambda}(s) = \bar{\Omega}(s) = 1.5 + \sin(2\pi s)$. As discussed, decreasing $\bar{\tau}$ shifts $\bar{w}(s)$ to the right and its amplitude decreases. In this protocol both the maximal power and maximal efficiency are attained asymptotically at the fast driving limit.

To implement this protocol in an experimental realization as in [21], we need to transform the protocol form $[\bar{w}, \bar{\Lambda}]$ into $[\beta, \Lambda]$. As $\lambda(s)$ is given in terms of $\bar{\Lambda}(s)$ (Eq. 5), this can be done by a straight-forward calculation. The resulting protocol in the $[\beta, \Lambda]$ space is shown in the inset of Fig.(2). The maximal efficiency in this protocol is only $\eta_{max} = 0.687$, compared to the Carnot efficiency calculated from the minimal and maximal temperatures which is $\eta_{Carnot} = 0.96$. This is expected, since in this protocol the engine exchanges heat with many heat baths at intermediate temperatures, hence it cannot attain the Carnot efficiency computed with the extreme temperatures. However, *this protocol does not suffer from a trade-off between power and efficiency.*

With these tools, it is natural to ask: is it possible to design a protocol that achieves the Carnot limit but has non-zero power? As we show in the SI, the answer is no. This is proven by showing that for any piecewise continuous protocol $\Gamma_s^{\beta,\Lambda}$ attaining the Carnot limit at some finite cycle time τ, it is possible to slightly deform the protocol into a different protocol $\tilde{\Gamma}_s^{\beta,\Lambda}$ with the same

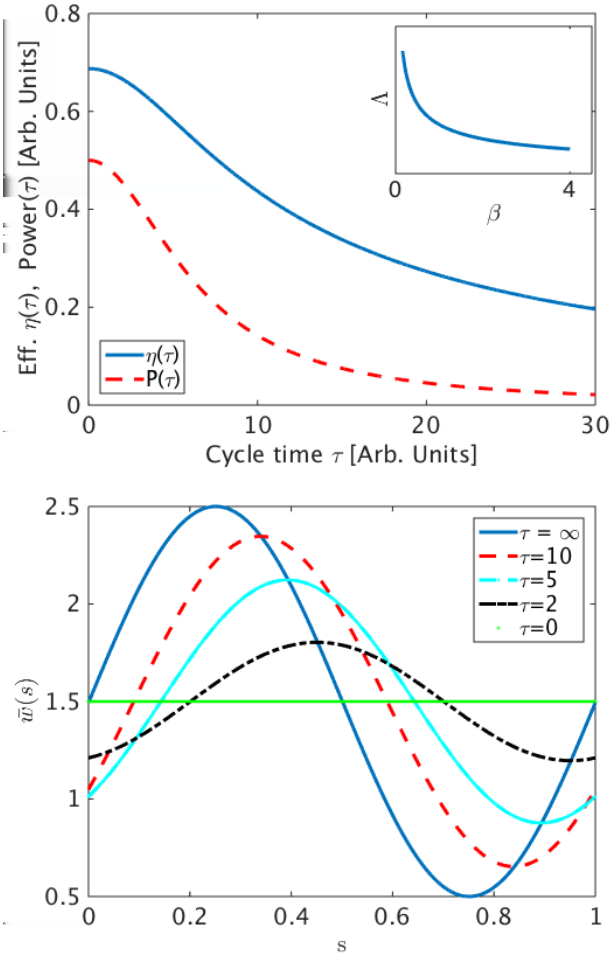

FIG. 2: **Upper panel:** Power and efficiency as a function of the cycle time τ. The protocol is given by $\bar{\Omega}(s) = \bar{\Lambda}(s) = 1.5 + \sin(2\pi s)$. The inset shows the protocol in the $[\beta, \Lambda]$ control space. The protocol traverses the line back and forth, without covering any area. **Lower panel:** $\bar{w}(s)$ for various values of τ. When $\tau = \infty$, $\bar{w}(s) = \bar{\Omega}(s)$. As can be seen, the maximum of $\bar{w}(s)$ "slides" down the negative slope when τ decreases, and the minimum if $\bar{\omega}(s)$ "slides up" on the positive slope. Overall, $\bar{w}(s)$ shifts to the right and its amplitude decreases with decreasing τ.

$\beta(s)$ (hence the same Carnot bound), such that the efficiency of $\tilde{\Gamma}_s^{\beta,\Lambda}$ is strictly larger then that of $\Gamma_s^{\beta,\Lambda}$, hence larger then the Carnot limit. This would violate the second law.

Applicability to other models. The above analysis was made possible due to three properties: (i) The interpretation of the work and heat as areas; (ii) The 2D nature of state space, which enabled a simple characterization of all the possible ways in which the area can change its orientation, and (iii) The ability to separately control $\bar{\Lambda}(s)$ without influencing the other state parameter, $\bar{w}(s)$. What class of models share these properties? A complete answer to this question is yet unknown, however, as shown in the SI, in addition to the example given above, this class contains also any 2-levels Markovian model, which are commonly used to study heat engines [23].

Acknowledgments We would like to thank C. Jarzynski for useful discussions and Y. Sagi, S. Rahav, S. Kotler, O. Hirshberg, V. Alexandrov, J. Hopfield for discussion and comments. O.R. acknowledges the financial support from the James S. McDonnell foundation. R.P. would like to thank S. Christen and A. Levine for their help and support. Financial support from Fullbright foundation, Eric and Wendy Schmidt fund and the Janssen Fellowship are greatfully acknowledged. Y.S. acknowledge financial support from the U.S. Army Research Office under contract number W911NF-13-1-0390.

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[27] A quote from [22]: "A couple of very prolific groups are continuing the old tradition and are analyzing a series of named cycles... ...The tradeoff between power and efficiency is essentially the same for all."
[28] To establish the geometrical picture we assume that both Λ and β are twice differentiable.
[29] For simplicity, we choose the mobility to be one half.
[30] Note that the term $\Theta\left[\Lambda\frac{dw}{ds}\right]$ is not geometrical, since it is not parametrization independent. However, if we consider only re-parameterizations $s \to \lambda$ in which $\lambda(s)$ is a monotonically increasing function, then the geometric interpretation can be applied.
[31] This definition is not the only possible definition, see e.g. [11]. However the geometric picture associate with it is a notable advantage. Moreover, other definitions of efficiency show the same qualitative behavior.