Boundary Effective Action for Quantum Hall States

Andrey Gromov, Kristan Jensen, and Alexander G. Abanov

Phys. Rev. Lett. **116**, 126802 — Published 21 March 2016

DOI: 10.1103/PhysRevLett.116.126802

Boundary effective action for quantum Hall states

Andrey Gromov,[1] Kristan Jensen,[2] and Alexander G. Abanov[1,3]

[1] *Department of Physics and Astronomy, Stony Brook University, Stony Brook, NY 11794, USA*
[2] *C.N. Yang Institute for Theoretical Physics, SUNY Stony Brook, Stony Brook, NY 11794, USA*
[3] *Simons Center for Geometry and Physics, Stony Brook University, Stony Brook, NY 11794, USA*

(Dated: February 19, 2016)

We consider quantum Hall states on a space with boundary, focusing on the aspects of the edge physics which are completely determined by the symmetries of the problem. There are four distinct terms of Chern-Simons type that appear in the low-energy effective action of the state. Two of these protect gapless edge modes. They describe Hall conductance and, with some provisions, thermal Hall conductance. The remaining two, including the Wen-Zee term, which contributes to the Hall viscosity, do not protect gapless edge modes but are instead related to local boundary response fixed by symmetries. We highlight some basic features of this response. It follows that the coefficient of the Wen-Zee term can change across an interface without closing a gap or breaking a symmetry.

Introduction. Topology and geometry play an important role in modern condensed matter physics. For example, in quantum Hall systems, the observed quantization and rigidity of the Hall conductance σ_H are most naturally explained using topological arguments [1]. There are several types of topology at play in this example. In particular, the Hall conductance appears as the coefficient in front of a Chern-Simons (CS) term in the bulk low-energy effective action S_{bulk} of the state [2],

$$S_{bulk} = \frac{\sigma_H}{2}\int_{\mathcal{M}} d^3x\ \epsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho + \dots\,, \tag{1}$$

where A_μ is an external electromagnetic gauge field and $\mathcal{M}$ the three-dimensional space-time. Charge conservation then implies that σ_H cannot vary continuously in space or time, and is quantized in a way that depends on the electric charges of quasiparticles.

This CS term has another property: it is gauge-invariant up to a boundary term, and so is invariant on a closed spacetime, but not on a spacetime with a boundary. This non-invariance cannot be cured by adding local boundary terms built from A_μ and its derivatives. Charge conservation together with the existence of the CS term then imply that there is a gapless, non-gauge-invariant edge theory which cancels the non-invariance of the bulk. Namely, the quantum effective edge action S_{edge} living on the spacetime boundary $\partial\mathcal{M}$ obeys

$$\delta_\Lambda S_{edge} = -\delta_\Lambda S_{bulk} = -\frac{\sigma_H}{2}\int_{\partial\mathcal{M}} d^2x\,\Lambda\,\epsilon^{\alpha\beta}\partial_\alpha A_\beta\,, \tag{2}$$

where Λ is the gauge transformation parameter and α, β are boundary indices. This non-invariance of the edge theory is known as an "anomaly," and its cancellation against the variation of a CS term is an example of "anomaly inflow" [3]. The edge depends on the details of the state, including boundary conditions, and is often unknown, but it must possess the anomaly (2) and be gapless so as to make up for the non-invariance of the bulk at arbitrarily low energies.

There are other rigid transport coefficients in quantum Hall states. These are encoded in the dimensionless coefficients of CS terms in the low-energy action of the state [4–11]. The most well-known of these is the Hall viscosity [12] and it is related to the Wen-Zee (WZ) term [4, 13], which we discuss below. This term is not invariant on a spacetime with boundary. One natural question is: does the WZ term protect the existence of gapless edge modes, or instead correspond to some boundary-localized response?

The goal of this Letter is to answer this question. We consider CS terms consistent with the symmetries of a quantum Hall state, and deduce which correspond to anomalies and which to local boundary terms. We show that Wen-Zee terms belong to the latter category and do not correspond to protected gapless edge states. Nevertheless, they still encode symmetry-protected boundary response, which we discuss below. Our analysis only employs the symmetries of the problem as in e.g. [13–16], and so is robust even when the microscopic system underlying the Hall state is strongly interacting. Another outcome of our analysis is that there are no anomalies in addition to the ones that appear in the relativistic setting.

The setup. We consider gapped systems in two spatial dimensions with a conserved current j^μ and spatial stress tensor T^{ij}, to which we respectively couple an external gauge field A_μ and spatial metric g_{ij}. We assume that the underlying state is rotationally invariant in flat space [17]. Due to the gap, the low-energy effective action S_{bulk} only depends on the external fields (A_μ, g_{ij}) and can be presented as an expansion in gradients thereof.

The total low-energy effective action $S_{eff} = S_{edge} + S_{bulk}$ is invariant under all the symmetries of the underlying theory, including gauge transformations under which A_μ varies as $\delta_\Lambda A_\mu = \partial_\mu\Lambda$. It is also invariant under spatial reparameterizations of space $x^i = x^i(y^j)$, provided that we equip the external fields (A_μ, g_{ij}) with the right transformation properties. We will use these symmetries to constrain the form of both bulk and boundary parts of the effective action.

One can extend the spatial reparameterization invariance to a full space-time invariance by introducing a frame $\beta_a^\mu = (\beta_0^\mu, E_A^\mu)$ and coframe $(\beta^{-1})_\nu^b = ((\beta^{-1})_\nu^0, e_\nu^B)$, which we have separated into temporal and spatial parts. Here μ,ν are spacetime indices, $a,b = 0,1,2$ order the basis, and $A,B = 1,2$ label spatial vectors. (A frame is just a local basis of tangent vectors.) We take the "time vector" to be $\beta_0^\mu = \delta_t^\mu$ and $(\beta^{-1})_\mu^0 = \delta_\mu^t$. The remaining spatial vectors E_A with $A = 1,2$ give a spatial vielbein and the e_μ^A a spatial coframe. From the e_μ^A we construct a spacetime covariant version of g_{ij}, given by $g_{\mu\nu} = \delta_{AB} e_\mu^A e_\nu^B$, which is invariant under local $SO(2)$ rotations which rotate the e_μ^A into each other. We use an $SO(2)$ spin connection for this transformation, $\omega_\mu = \frac{1}{2}\epsilon^A{}_B E_A^\nu D_\mu e_\nu^B$, which characterizes the geometry. Here D_μ is a covariant derivative defined with a connection $\Gamma^\mu{}_{\nu\rho}$ which we describe in the Supplement. Under a local $SO(2)$ rotation θ we have $\omega_\mu \to \omega_\mu + \partial_\mu\theta$, and in general there is nonzero torsion as determined by the Cartan structural equations.

The spatial curvature is related to ω as follows. The curvature constructed from ω is $d\omega$. On a constant-time, or spatial, slice Σ with scalar curvature R we have

$$\int_\Sigma d\omega = \frac{1}{2}\int_\Sigma d^2x\sqrt{g}\,R\,. \tag{3}$$

The microscopic theory (and so also S_{eff}) is invariant under (i.) $U(1)$ gauge transformations, (ii.) coordinate reparameterizations, and (iii.) local $SO(2)$ rotations. The CS terms [18] that can appear in S_{eff} are then [16], in terms of differential forms [19],

$$S_{CS} = \frac{\nu}{4\pi}\int AdA + 2\bar{s}Ad\omega + \overline{s^2}\omega d\omega + \frac{c}{96\pi}\int I_{CS}[\Gamma]\,, \tag{4}$$

where $I_{\rm CS}[\Gamma] = \Gamma^\mu{}_\nu d\Gamma^\nu{}_\mu + \frac{2}{3}\Gamma^\mu{}_\nu\Gamma^\nu{}_\rho\Gamma^\rho{}_\mu$. The second term in (4) is the WZ term, the third is sometimes called the second WZ term, and the last as the gravitational Chern-Simons (gCS) term.

The dimensionless coefficients $(\nu, \bar{s}, \overline{s^2}, c)$ are known as the filling factor, mean orbital spin per particle, mean orbital spin squared per particle, and chiral central charge. The flat-space Hall conductance is $\sigma_H = \frac{\nu}{2\pi}$, and when the space has curvature R, the Hall viscosity is then $\eta_H = \frac{\bar{s}}{2}\rho + (12\nu\,\mathrm{var}(s) - c)\frac{R}{96\pi}$, with ρ the charge density and $\mathrm{var}(s) \equiv \overline{s^2} - \bar{s}^2$ the orbital spin variance [20] [21]. Quantities $\bar{s}$ and $12\nu\,\mathrm{var}(s) - c$ can be used to distinguish different quantum Hall states at the same filling fraction.

The third and fourth terms in (4) are related as

$$2\omega d\omega + I_{cs}[\Gamma] = \frac{1}{3}(\beta d\beta^{-1})^3\,, \tag{5}$$

where β_a^μ is the frame. The integral of the RHS of Eq. (5) over a closed space-time is proportional to an integer, a "winding number" of the frame over $\mathcal{M}$, so $\overline{s^2}$ and c contribute to the bulk response only through the combination $12\nu\,\mathrm{var}(s) - c$. This combination and $(\nu, \bar{s})$ have been computed for integer quantum Hall states in [11, 22] and for various model fractional quantum Hall states in [20, 23–29].

When the space has a boundary, $\mathrm{var}(s)$ and c can be disentangled. It has been conjectured that the thermal Hall conductance of a quantum Hall state with an edge is given by $\kappa_H = c\frac{\pi}{3}k_B T$ [6]. A similar relation has been shown to hold in any two-dimensional relativistic theory [30].Thus measuring κ_H would determine c, and $\mathrm{var}(s)$ could be deduced from the Hall viscosity.

Boundary terms and anomalies. The CS terms in (4) are no longer invariant when $\mathcal{M}$ has boundary, leaving two possibilities for each CS term: (i) it cannot be made invariant by adding local boundary terms built from the external fields, or (ii.) it can. In the first case, we say that the CS term corresponds to an anomaly of a gapless edge theory, that cancels the non-invariance of the bulk CS term. In the second case, the CS term does not correspond to an anomaly, and so does not protect the existence of gapless edge modes.

Electromagnetic CS term (the first term in (4)) belongs to type (i.). Similarly, in relativistic field theories the gCS term is known to correspond to a boundary diffeomorphism anomaly [31]. This holds true in non-relativistic setup as well. This leaves the WZ terms.

To proceed, we describe the spacetime boundary $\partial\mathcal{M}$ via embedding functions $X^\mu = X^\mu(\sigma^\alpha)$ where $\mu = 0,1,2$ and (σ^0, σ^1) are boundary coordinates. The partial derivatives $\partial_\alpha X^\mu$ are tensors under both reparameterizations of the x^μ and the σ^α. Using the $\partial_\alpha X^\mu$ and the bulk data $(\beta_a^\mu, \omega_\mu)$, we can define the extrinsic curvature of the boundary [32].

To illustrate the basic idea, consider the more familiar case with a time-dependent spatial metric g_{ij}. We consider spatial boundaries whose shape does not change in time[33]. Such a boundary can be parameterized as $X^0 = \sigma^0, X^i = X^i(\sigma^1)$. Given the X^i one can construct tangent and normal vectors t^i and n^i that satisfy

$$n^i n_i = t^i t_i = 1\,, \qquad n_i t^i = 0\,. \tag{6}$$

From this data we can construct an extrinsic curvature one-form K_α as

$$K_\alpha = n_i D_\alpha t^i\,. \tag{7}$$

The one-form K_α can be shown to be related to the spin connection projected to the boundary as

$$\omega_\alpha + K_\alpha = \partial_\alpha\varphi\,, \tag{8}$$

for a locally defined function φ. That is, the extrinsic curvature one-form differs from the spin connection (projected to the boundary) by an $SO(2)$ gauge transformation with boundary value φ.

Integrating over a spatial slice Σ and using Stokes' theorem we obtain the Gauss-Bonnet theorem

$$\frac{1}{2\pi}\left(\int_{\Sigma} d\omega + \int_{\partial\Sigma} K\right) = \chi\,, \tag{9}$$

where χ is the Euler characteristic of Σ, which is also the integer-valued winding number of φ around $\partial\Sigma$.

The crucial point now is that we can use the extrinsic curvature K_α to render the WZ terms invariant by adding

$$S_{WZ,bdy} = \frac{\nu}{4\pi}\int_{\partial\mathcal{M}}\left(2\bar{s}AK + \overline{s^2}\omega K\right)\,, \tag{10}$$

to the effective action. Equivalently, the contributions to effective action

$$S_{WZ,1} \equiv \frac{\nu\bar{s}}{2\pi}\left(\int_{\mathcal{M}} Ad\omega + \int_{\partial\mathcal{M}} AK\right)\,, \tag{11}$$

$$S_{WZ,2} \equiv \frac{\nu\overline{s^2}}{4\pi}\left(\int_{\mathcal{M}} \omega d\omega + \int_{\partial\mathcal{M}} \omega K\right)\,, \tag{12}$$

are invariant with respect to all symmetries of the problem, do not correspond to edge anomalies, and do not necessitate gapless edge modes [34]. This is the main result of this Letter.

Putting the pieces together, we can write the total effective action as a sum

$$S_{eff} = S'_{CS} + S_{WZ,1} + S_{WZ,2} + S_{edge} + \dots\,, \tag{13}$$

where S'_{CS} contains the electromagnetic and gCS terms and the dots refer to additional, invariant bulk terms built from the external fields. The gapless edge theory S_{edge}, varies under gauge transformations and infinitesimal reparameterizations ξ^μ as

$$\delta S_{edge} = -\frac{\nu}{4\pi}\int_{\partial\mathcal{M}} \Lambda\, F - \frac{c}{96\pi}\int_{\partial\mathcal{M}} \partial_\mu \xi^\nu d\Gamma^\mu{}_\nu\,. \tag{14}$$

Lorentz and Galilean invariance. Here we comment on the relation of this work to the literature. We regard the boundary term (10) in a way which mirrors the situation in relativistic Hall states as discussed in [35]. The Riemann curvature can be dualized to the topologically conserved current $\mathcal{R}^\mu = \varepsilon^{\mu\nu\rho}\partial_\nu\omega_\rho$. $\mathcal{R}^\mu$ is the "Euler current," in that its density is proportional to the Euler density R on a spatial slice. The WZ term is just a coupling of A_μ to this conserved current. On a closed space, the "charge" associated with $\mathcal{R}^\mu$ is just the Euler characteristic of the spatial slice, and the conservation of $\mathcal{R}^\mu$ corresponds to the fact that this characteristic is a topological invariant which does not vary in time. On a space with boundary, the Euler characteristic includes an extrinsic boundary term, and so charge conservation mandates that the $A_\mu\mathcal{R}^\mu$ coupling must be supplemented with the extrinsic coupling in (11).

Response. The S'_{CS} and WZ terms (11), (12) lead to certain response functions which are protected by the symmetries as we now discuss.

Because S_{edge} is an a priori unknown, gapless theory, we cannot completely fix the boundary response by the symmetries alone. We proceed by defining correlators of the $U(1)$ current j^μ, spin current s^μ, "stress tensor" T^A_μ, and what we call the displacement operator $\mathcal{D}_\mu$. These are given by functional variations of S_{eff} with respect to $(A_\mu, \omega_\mu, \beta^\mu_A, X^\mu)$ respectively [36]. The symmetries imply that the displacement operator is along the normal vector n^i, and from it we find the external force density $\mathcal{F} = n^i\mathcal{D}_i$ which is required to fix the boundary.

The $U(1)$ current, spin current, and "stress tensor" have bulk and boundary components. For example, keeping $(\omega_\mu, \beta^\mu_A)$ fixed, j^μ and $\mathcal{D}_\mu$ are defined via

$$\delta S_{eff} = \int_{\mathcal{M}} [d^3x]\, \delta A_\mu j^\mu_{bulk} + \int_{\partial\mathcal{M}} [d^2\sigma]\left(\delta A_\mu j^\mu_{bdy} - \delta X^\mu \mathcal{D}_\mu\right)\,, \tag{15}$$

with $[d^3x] = d^3x\sqrt{g}$ and $[d^2\sigma]$ respectively an invariant bulk volume and boundary area. In other words, the current density is given by

$$j^\mu = j^\mu_{bulk} + j^\mu_{bdy}\delta(x^\perp)\,, \tag{16}$$

with $\delta(x^\perp)$ a delta function with support on $\partial\mathcal{M}$ [37]

All low-energy response functions of these operators are contained in S_{eff}. For illustrative purposes, we focus on the total charge $\mathcal{Q}$, and the contribution of the WZ terms (11), (12) to the total spin $\mathcal{S}$ and force density $\mathcal{F}$ exerted on the boundary. We consider a time-independent state in which the space is curved and threaded with magnetic flux.

The total charge is $\mathcal{Q} = \int_\Sigma d^2x\sqrt{g}\, j^0$, with Σ a spatial slice. From S_{eff} we find from (13)

$$\begin{aligned}\mathcal{Q} =& \frac{\nu}{2\pi}\int_\Sigma F + \frac{\nu\bar{s}}{2\pi}\left(\int_\Sigma d\omega + \int_{\partial\Sigma} K\right) + \mathcal{Q}_{edge}\\ =& \nu N_\Phi + \nu\bar{s}\chi + \mathcal{Q}_{edge}\,,\end{aligned} \tag{17}$$

where N_Φ and χ are the magnetic flux through and Euler characteristic of Σ, and $\mathcal{Q}_{edge}$ is the total charge coming from the edge theory [38]. Here we have used that the local, gauge-invariant terms in the ellipsis of (13) do not contribute to the total charge.

On a closed space, (17) becomes $\mathcal{Q} = \nu N_\Phi + \frac{\nu\bar{s}}{2\pi}\int_\Sigma d\omega = \nu N_\Phi + \nu\bar{s}\chi$. This expression was already known in the FQH literature [4, 39]. Eq. (17) generalizes it to systems with an edge. The effect of the boundary term (10) is to ensure that there is an extrinsic contribution to $\mathcal{Q}$ in such a way that the total charge depends on $\bar{s}$ only through the Euler characteristic χ of the spatial slice.

The total spin $\mathcal{S} = \int_\Sigma d^2x\sqrt{g}\, s^0$ is

$$\mathcal{S} = \nu\bar{s}N_\Phi + \nu\overline{s^2}\chi + \dots\,. \tag{18}$$

The dots indicate contributions from the rest of S_{eff}, including the gCS term. A similar relation has appeared in [40] when space-time is compact. The boundary term (10) gives an extrinsic contribution to $\mathcal{S}$, ensuring that it depends on $\overline{s^2}$ only through χ.

Finally, the external force density $\mathcal{F} = n^i \mathcal{D}_i$ as

$$\mathcal{F} = -\frac{\nu\bar{s}}{2\pi}\left(t^i\partial_i E_{||} + KE_\perp\right) - \frac{\nu\overline{s^2}}{4\pi}\left(t^i\partial_i \mathcal{E}_{||} + K\mathcal{E}_\perp\right) + \dots\,, \tag{19}$$

where again the dots indicate contributions from the rest of S_{eff}. Here $E_{||}$ and $E_\perp$ the electric fields parallel and normal to the boundary (and similarly for the components of "gravi-electric" field $\mathcal{E}_i = \partial_0\omega_i - \partial_i\omega_0$), and $K = t^i K_i$ the geodesic curvature of the boundary.

Relation to index theorem. There is an intimate connection between quantum anomalies in relativistic field theory and index theorems [41]. It is natural to ask if there is any connection between Hall states and index theorems for manifolds with boundary. Here we illustrate such a connection in the simplest case of non-interacting electrons. Namely, we assume that we have $\mathcal{Q}$ non-interacting electrons and (i) only the lowest Landau level (LLL) is filled and (ii) we apply particular boundary conditions for the bulk electrons. In this system, $\nu = 1$ and $\bar{s} = \frac{1}{2}$, and the LLL states are zero modes of the anti-holomorphic differential operator of momentum $\bar{D}$ on the spatial slice. The number of such zero modes is counted by the Atiyah-Patodi-Singer (APS) index theorem [42] provided that the electrons obey so-called APS boundary conditions. The index of $\bar{D}$ is

$$\mathrm{ind}(\bar{D}) = N_\Phi + \frac{1}{2}\chi + \frac{1}{2}\eta\,, \tag{20}$$

where N_Φ and χ are as above, the "η-invariant" is

$$\eta \equiv \mathrm{sign}\,\bar{D}|_{\partial\Sigma} = \sum \mathrm{sign}\,\lambda\,, \tag{21}$$

where $\bar{D}|_{\partial\Sigma}$ is $\bar{D}$ restricted to the boundary, and the sum runs over eigenmodes of this operator with eigenvalue λ [43]. Note that the index (20) indeed matches our general expression (17) for $\nu = 1, \bar{s} = \frac{1}{2}$, and $\mathcal{Q}_{edge} = \frac{\eta}{2}$.

The total number of electrons $\mathcal{Q}$ is integer, which is guaranteed in (20) by the η-invariant. For example, if the spatial slice is a disk $\chi = 1$, then $\eta = 1 - 2\{N_\Phi\}$, where $\{N_\Phi\}$ is the non-integer part of N_Φ. Then $\mathrm{ind}(\bar{D}) = \mathcal{Q} = \lfloor N_\Phi \rfloor + 1$, indeed giving integer $\mathcal{Q}$.

Singular expansion of charge density. So far our results have been obtained only from the symmetries of the problem. As an application, we derive the singular expansion of the charge density of a flat-space Hall state. From S_{eff} we obtain the charge density $\rho = j^0$

$$\rho = \frac{\nu B}{2\pi}\theta(\Sigma) + \left(\frac{\nu\bar{s}}{2\pi}K + j^0_{bdy}\right)\delta(\partial\Sigma) + \frac{\zeta}{2\pi}\partial_n\delta(\partial\Sigma) + \dots\,. \tag{22}$$

Here $\partial_n\delta(\partial\Sigma)$ denotes the normal derivative of the delta function on the boundary of the system. The first term of (22) comes from (1), the second from the boundary part of the first WZ term (11) and j^0_{bdy} (defined in (15)) depends on the non-universal details of S_{edge}. The third comes from two invariant, higher order terms in S_{eff},

$$\frac{\sigma_H^{(2)}}{2\pi}\int_{\mathcal{M}}[d^3x]\,BD^iE_i\,,\quad \frac{\xi}{2\pi}\int_{\partial\mathcal{M}}[d^2\sigma]\,n^iE_i\,, \tag{23}$$

with $\zeta = \sigma_H^{(2)} + \xi$. Here $\sigma_H^{(2)}$ is the $O(k^2)$ correction to the Hall conductivity, and ξ is a dimensionless parameter related to the total dipole moment at the edge. The latter is an arbitrary parameter that changes the edge dipole moment without affecting the (bulk) Hall viscosity. The coefficient ζ is relevant for the so-called "overshoot" phenomenon [44] and for the Laughlin function is related to the Hall viscosity. When the underlying system is Galilean-invariant, $\sigma_H^{(2)}$ gets a contribution from the Hall viscosity [13], thus relating the "overshoot" with η_H.

For simplicity we take Σ to be a flat disk of radius R. Then (22) becomes

$$\begin{aligned}\rho =& \frac{\nu B}{2\pi}\Theta(R-r) + \left(\frac{\nu\bar{s}}{\pi} + 2Rj^0_{bdy}\right)\delta(r^2-R^2)\\ &+ \frac{\zeta}{2\pi}R^2\delta'(r^2-R^2) + \dots\,.\end{aligned} \tag{24}$$

Specifying for Laughlin's state with $\nu = \frac{1}{2n+1}$ and $\nu\bar{s} = \frac{1}{2}$, this matches the singular expansion obtained by Wiegmann and Zabrodin [45] directly from the Laughlin's wave function for $\zeta = 1 - 2\nu$ and $j^0_{bdy} = -\frac{\nu\bar{s}}{2\pi R}$ [46]. One can also match for an infinitesimally different definition of the radius R, in which case ζ is unchanged but $j^0_{bdy} = 0$.

Conclusions. Using effective field theory and symmetries on a space with boundary, we have made a systematic study of the Chern-Simons terms (4) that appear in the low-energy effective action of quantum Hall states.

The main result is that the WZ terms are not Chern-Simons terms per se, but rather the couplings of A_μ and the spin connection ω_μ to a topologically conserved but non-trivial "Euler current." On a space with boundary, these bulk couplings must be supplemented with boundary couplings between A_μ and the spin connection ω_μ to the extrinsic curvature of the edge.

An immediate corollary to our result is that the coefficients of the WZ terms, $\bar{s}$ and $\overline{s^2}$, can jump across an interface without closing a gap or breaking the symmetries of the problem, namely $U(1)$ gauge invariance, coordinate reparameterizations, or local $SO(2)$ invariance.

We are pleased to thank T. Can, A. Cappelli, S. Golkar, G. Monteiro, A. Kapustin and P. Wiegmann for useful discussions and comments. This work was supported in part by the NSF under grants PHY-0969739 and DMR-1206790.

Note added: After this work was completed, the authors of [47] have privately informed us that the results of this Letter are consistent with theirs.

[1] D. J. Thouless, M. Kohmoto, M. P. Nightingale, and M. den Nijs, Phys. Rev. Lett. **49**, 405 (1982).
[2] Here an below we work in units $e = \hbar = 1$.
[3] C. G. Callan Jr and J. A. Harvey, Nuclear Physics B **250**, 427 (1985).
[4] X. Wen and A. Zee, Phys. Rev. Lett. **69**, 953 (1992).
[5] J. Fröhlich and U. M. Studer, Rev. Mod. Phys. **65**, 733 (1993).
[6] N. Read and D. Green, Physical Review B **61**, 10267 (2000).
[7] N. Read, Phys Rev B **79**, 045308 (2009).
[8] F. D. M. Haldane, arXiv:0906.1854 (2009).
[9] N. Read and E. H. Rezayi, Phys Rev B **84**, 085316 (2011).
[10] B. Bradlyn, M. Goldstein, and N. Read, Physical Review B **86**, 245309 (2012).
[11] A. G. Abanov and A. Gromov, Phys. Rev. B **90**, 014435 (2014).
[12] J. E. Avron, R. Seiler, and P. G. Zograf, Phys Rev Lett **75**, 697 (1995).
[13] C. Hoyos and D. T. Son, Phys. Rev. Lett. **108**, 066805 (2012).
[14] A. Gromov and A. G. Abanov, Phys. Rev. Lett. **113**, 266802 (2014).
[15] A. Gromov and A. G. Abanov, Phys. Rev. Lett. **114**, 016802 (2015).
[16] B. Bradlyn and N. Read, Phys. Rev. B **91**, 125303 (2015).
[17] The precise statement is that we consider systems which depend on only one spatial metric, as compared with [8]. The generalization of this work to systems with more than one spatial metric is immediate.
[18] S. Deser, R. Jackiw, and S. Templeton, Annals Phys. **140**, 372 (1982).
[19] Here AdA is short for $\epsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho$.
[20] A. Gromov, G. Y. Cho, Y. You, A. G. Abanov, and E. Fradkin, Phys. Rev. Lett. **114**, 016805 (2015).
[21] The combination var(s) has been found to vanish for "conformal block states" [27].
[22] M. R. Douglas and S. Klevtsov, Communications in Mathematical Physics **293**, 205 (2010).
[23] T. Can, M. Laskin, and P. Wiegmann, arXiv:1412.8716 (2014).
[24] T. Can, M. Laskin, and P. Wiegmann, Physical Review Letters **113**, 046803 (2014).
[25] T. Can, M. Laskin, and P. Wiegmann, arXiv:1411.3105 (2014).
[26] G. Y. Cho, Y. You, and E. Fradkin, Phys. Rev. B **90**, 115139 (2014).
[27] B. Bradlyn and N. Read, Phys. Rev. B **91**, 165306 (2015).
[28] F. Ferrari and S. Klevtsov, Journal of High Energy Physics 86 **2014** (2014).
[29] S. Klevtsov and P. Wiegmann, arXiv:1504.07198 (2015).
[30] K. Jensen, R. Loganayagam, and A. Yarom, JHEP **1302**, 088 (2013).
[31] L. Alvarez-Gaume, S. Della Pietra, and G. Moore, Annals of Physics **163**, 288 (1985).
[32] See the Supplement for the details.
[33] More general treatment requires use of the Newton-Cartan geometry [16, 48–51] and is presented in the Supplementary Material.
[34] This statement was anticipated in [27].
[35] S. Golkar, M. M. Roberts, and D. T. Son, arXiv:1407.7540 (2014).
[36] Note that a variation of ω_μ at fixed β^μ_A is a variation of spatial torsion at fixed spatial metric.
[37] In principle, the boundary term in δS_{eff} contains additional terms involving normal derivatives of δA_μ. Those terms are not relevant for the rest of the Letter.
[38] More precisely, $\mathcal{Q}_{edge}$ is the gauge-invariant edge charge, which receives contributions both from S_{edge} and (1).
[39] F. D. M. Haldane, Phys. Rev. Lett. **51**, 605 (1983).
[40] T. L. Hughes, R. G. Leigh, and O. Parrikar, Phys. Rev. D **88**, 025040 (2013).
[41] L. Alvarez-Gaume and P. Ginsparg, Annals of Physics **161**, 423 (1985).
[42] M. F. Atiyah, V. Patodi, and I. Singer, in *Mathematical Proceedings of the Cambridge Philosophical Society*, Vol. 77 (Cambridge Univ Press, 1975) pp. 43–69.
[43] T. Eguchi, P. B. Gilkey, and A. J. Hanson, Physics reports **66**, 213 (1980).
[44] T. Can, P. Forrester, G. Téllez, and P. Wiegmann, Physical Review B **89**, 235137 (2014).
[45] A. Zabrodin and P. Wiegmann, J Phys A-Math Gen **39**, 8933 (2006).
[46] The value of ζ depends on the boundary conditions of the problem. For Laughlin's droplet made out of N particles on an infinite plane one can obtain ζ by matching (24) to the exact sum rule $\langle \sum_{i=1}^N |z_i|^2 \rangle = l^2 N(\nu^{-1}N + 2 - \nu^{-1})$.
[47] S. Moroz, C. Hoyos, and L. Radzihovsky, Physical Review B **91**, 195409 (2015).
[48] M. H. Christensen, J. Hartong, N. A. Obers, and B. Rollier, Phys. Rev. D **89**, 061901 (2014).
[49] D. T. Son, arXiv:1306.0638 (2013).
[50] M. Geracie, K. Prabhu, and M. M. Roberts, (2015), arXiv:1503.02682.
[51] K. Jensen, arXiv:1408.6855 (2014).