No-Drag Frame for Anomalous Chiral Fluid

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The no-drag frame for anomalous chiral fluid

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We show that for an anomalous fluid carrying dissipationless chiral magnetic and/or vortical currents there is a frame in which a stationary obstacle experiences no drag, but energy and charge currents do not vanish, resembling superfluidity. However, unlike ordinary superfluid flow, the anomalous chiral currents can transport entropy in this frame. We show that the second law of thermodynamics completely determines the amounts of these anomalous non-dissipative currents in the "no-drag frame" as polynomials in temperature and chemical potential with known anomaly coefficients. These general results are illustrated and confirmed by a calculation in the chiral kinetic theory and quark-gluon plasma at high temperature.

Introduction — The collective dynamics of a chiral (parity-violating) medium associated with quantum anomalies has become a subject of much attention recently. In particular, currents along the direction of an external magnetic field (chiral magnetic effect, or CME) discussed hypothetically earlier [1] have been recently proposed in Ref. [2, 3] as a possible explanation of the charge dependent correlations observed in heavy-ion collisions and of negative magnetoresistance in a Dirac-semimetal [4, 5]. Currents in the direction of rotation axis (chiral vortical effect, or CVE) also discussed in astrophysical context before [6] have been (re)discovered in strong-coupling gauge-gravity calculations [7, 8]. The generality of these effects and their connection to chiral anomaly have been demonstrated in Ref. [9, 10] by applying the constraint of the second law of thermodynamics to the hydrodynamic equations for the anomalous chiral fluid.

One of the manifestations of the anomalous nature of the CME and CVE currents is that these currents are dissipationless and do not lead to entropy production, in contrast, e.g., to the ordinary Ohmic current driven by electric field, but similar to the persistent superfluid currents. We wish to gain further understanding of the non-dissipative nature of the anomalous transport.

How can one distinguish the anomalous CME/CVE currents from the uniform (shearless) inertial motion of the fluid as a whole in the same direction, which also carries energy and charge without generating entropy? To do this one needs to determine the "rest frame" of the "normal" component of the flow. In the Landau's two-fluid picture [11] the superfluid component (the condensate) carries no entropy and one can define the rest frame of the normal component as the frame where the entropy flow vanishes. It is tempting to use the same criterion in the case of the anomalous flows. We shall show that, in general, this would not be correct, i.e., the anomalous currents *can* carry entropy.

We propose that a natural way to define the "rest frame" of the normal component is to insert an impurity, or an obstacle, obstructing the flow as it is done, e.g., in Ref. [12] for a gauge theory plasma with a heavy quark. In general, the flow will exert force on the obstacle and, if the obstacle is free to move, it will accelerate until reaching a certain velocity at which the drag, and thus acceleration, vanishes. One can say that the impurity will then be carried by the flow defining the "no-drag frame" – a natural (physically meaningful) rest frame of the fluid.

We shall present a general and universal argument based on the second law of thermodynamics allowing to determine such a no-drag frame, and thus the magnitude of the energy, charge and entropy currents in it. Of course, the magnitude of the drag force experienced by an impurity depends on the properties of the impurity itself and its interaction with the medium. However, the velocity of the no-drag frame is completely determined by the second law of thermodynamics and is insensitive to the details of impurity-medium interactions, thus representing the intrinsic property of the fluid itself.

For a normal fluid in equilibrium, the no-drag frame, of course, is the Landau frame, and the flows of energy, charge and entropy vanish in it. For a superfluid, the no-drag frame is the frame where the normal component rests, energy and charge are carried by the superfluid component, and there is no entropy current. In contrast, using the second law of thermodynamics we shall show that the anomalous currents not only carry energy and charge but, generally, also entropy in the no-drag frame.

Anomalous hydrodynamics with drag — Let us consider a fluid flowing past a fixed point-like obstacle, or impurity, for example, an infinitely heavy quark, or possibly a lattice of such impurities, or a porous solid medium. The hydrodynamic equations then contain an additional term due to the momentum transfer between the impurities and the fluid:

$$\partial_\mu T^{\mu\nu} = F^{\nu\lambda} J_\lambda - \mathcal{F}^\nu \qquad (1)$$

The local 4-momentum transfer from the fluid to the impurities, i.e., the drag force, per unit volume, $\mathcal{F}^\nu$, depends on the hydrodynamic variables and the 4-velocity of the impurity. We shall show that this dependence is constrained by the second law of thermodynamics up to an overall non-negative coefficient.

To simplify the analysis we shall consider a single-flavor anomalous fluid obeying

$$\partial_\mu J^\mu = CE \cdot B \quad (2)$$

where $E^\mu = F^{\mu\nu}u_\nu$ and $B^\mu = (1/2)\varepsilon^{\mu\nu\alpha\beta}u_\nu F_{\alpha\beta} = \widetilde{F}^{\mu\nu}u_\nu$ as in Ref.[9]. The constitutive equations are given by

$$T^{\mu\nu} \equiv wu^\mu u^\nu + pg^{\mu\nu} + \tilde{T}^{\mu\nu};$$
$$\tilde{T}^{\mu\nu} = (\xi_{T\omega}\omega^\mu + \xi_{TB}B^\mu)u^\nu + (\mu \leftrightarrow \nu), \quad (3)$$

where $w = \varepsilon + p$, and

$$J^\mu \equiv nu^\mu + \tilde{J}^\mu; \qquad \tilde{J}^\mu = \xi_{J\omega}\omega^\mu + \xi_{JB}B^\mu, \quad (4)$$

where $\omega^\mu = (1/2)\varepsilon^{\mu\nu\alpha\beta}u_\nu\partial_\alpha u_\beta$ and ξ's are the anomalous transport (CME and CVE) coefficients. We do not write the usual dissipative terms (viscosity and conductivity/diffusion) because their inclusion will not affect the constraints on anomalous coefficients ξ (this situation is similar to Ref.[9]). In effect, we are considering an equilibrium state of the fluid where all dissipative terms have vanished, except, possibly, for the drag force on a test impurity. This allows us to focus on the novel conceptual issues. Practical applications of our results will require extensions of the analysis to (weakly) non-equilibrium conditions, which we shall defer to further work.

The entropy current is given by

$$S^\mu = su^\mu + \tilde{S}^\mu; \qquad \tilde{S}^\mu = \xi_{S\omega}\omega^\mu + \xi_{SB}B^\mu. \quad (5)$$

where s is the entropy density.

The 4-vector u^μ defines the local reference frame of the fluid (the frame in which $u^\mu = (1, \mathbf{0})$). The well-known freedom of choice of this frame is used to optimize the form of the equations. E.g., we can choose it to be the frame in which $T^{0i} = 0$ (Landau) or $J^i = 0$ (Eckart) or $S^i = 0$ (entropy frame). These three choices coincide for a normal fluid flow *in equilibrium*, as we discussed above, but for the anomalous transport (CME or CVE) these choices are different.

The freedom of choosing the local reference frame allows us, starting from any choice, to redefine the velocity by, e.g.,

$$u^\mu \to u^\mu + \alpha_\omega\omega^\mu + \alpha_B B^\mu, \quad (6)$$

where α_ω and α_B are arbitrary coefficients. Then the anomalous transport coefficients in Eqs. (3), (4) and (5) would change accordingly:

$$\xi_{T\omega} \to \xi_{T\omega} - w\alpha_\omega; \qquad \xi_{TB} \to \xi_{TB} - w\alpha_B; \quad (7a)$$
$$\xi_{J\omega} \to \xi_{J\omega} - n\alpha_\omega; \qquad \xi_{JB} \to \xi_{JB} - n\alpha_B; \quad (7b)$$
$$\xi_{S\omega} \to \xi_{S\omega} - s\alpha_\omega; \qquad \xi_{SB} \to \xi_{SB} - s\alpha_B; \quad (7c)$$

One can, and we shall, use this freedom to go to a frame which is most suitable for a given purpose – in our case, the frame where drag vanishes.

The second law and the drag — Combining equations (1)–(5) with the first law $d\varepsilon = Tds + \mu dn$ we can calculate the divergence of the entropy current and use the second law as a constraint [9, 10] to establish the relations (invariant under (7)) between the coefficients:

$$Td\xi_{S\omega} + \mu d\xi_{J\omega} - d\xi_{T\omega} = K_\omega\, dp/w; \quad (8)$$
$$Td\xi_{SB} + \mu d\xi_{JB} - d\xi_{TB} = K_B\, dp/w; \quad (9)$$
$$\xi_{J\omega} - 2(T\xi_{SB} + \mu\xi_{JB} - \xi_{TB}) = -K_\omega\, n/w; \quad (10)$$
$$\xi_{JB} - \mu C = -K_B\, n/w, \quad (11)$$

where we defined the following linear combinations:

$$K_\omega \equiv 2T\xi_{S\omega} + 2\mu\xi_{J\omega} - 3\xi_{T\omega}; \; K_B \equiv T\xi_{SB} + \mu\xi_{JB} - 2\xi_{TB}\,. \quad (12)$$

With that, the heat production rate is given by:

$$T(\partial \cdot S) = \left(u - \frac{K_\omega\omega + K_B B}{w}\right) \cdot \mathcal{F} \quad (13)$$

Note that under transformations in Eqs. (7)

$$K_\omega \to K_\omega + w\alpha_\omega; \qquad K_B \to K_B + w\alpha_B, \quad (14)$$

which is also clear from Eqs. (6), (13) and the fact that the entropy production should not depend on our choice of local reference frame u^μ.

Although the discussion can be continued using arbitrary frame, we find it most convenient to fix the frame now by conditions [13]

$$K_\omega = K_B = 0. \quad (15)$$

With this choice, the only remaining nonzero term in the r.h.s. of Eq.(13) is $u \cdot \mathcal{F}$. The requirement that it be non-negative for an arbitrary fluid flow and heavy quark 4-velocity U fixes vector $\mathcal{F}$ up to an arbitrary, but non-negative, coefficient, $\lambda_\mathcal{F} \geq 0$:

$$\mathcal{F}^\mu = \lambda_\mathcal{F}(u^\mu + U^\mu(u \cdot U)) \quad (16)$$

Both terms in Eq. (16) are needed because fluid cannot do work on a static impurity and thus $\mathcal{F}^0$ must vanish in the frame defined by U, i.e., $U \cdot \mathcal{F} = 0$. One can also arrive at this condition by considering the 4-momentum of a heavy quark $P^\mu = MU^\mu$ and requiring that the transfer of the 4-momentum $\mathcal{F}^\mu \sim dP^\mu/d\tau$ does not violate the mass-shell condition $P \cdot P = M^2$.

The second law of thermodynamics $T(\partial \cdot S) = u \cdot \mathcal{F} = \lambda_\mathcal{F}((u \cdot U)^2 - 1) \geq 0$ only constrains the *sign* of $\lambda_\mathcal{F}$, whose magnitude could be a (local) function of hydrodynamic variables ε, n as well as $u \cdot U$ and coordinates.

Eq. (16) implies that $\mathcal{F} = 0$ when $U = u$, i.e., if the heavy quark (or impurity) is at rest in the reference frame we chose. In other words, the frame where coefficients obey Eq. (15) is the no-drag fame. When both velocities are small $U \approx (1, \boldsymbol{V})$ and $u \approx (1, \boldsymbol{v})$, the drag force is proportional to the relative velocity $\boldsymbol{\mathcal{F}} \approx -\lambda_\mathcal{F}(\boldsymbol{V} - \boldsymbol{v})$, as one would expect, and the heat production rate is $u \cdot \mathcal{F} \approx \lambda_{\mathcal{F}}(\boldsymbol{V}-\boldsymbol{v})^2$.

Anomalous coefficients in the no-drag frame — Using four algebraic equations (10), (11) and (12) together with (15) we can express all coefficients in terms of $\xi_{S\omega}$ and ξ_{SB}. Integrating Eq. (9) then gives

$$\xi_{SB} = X_B T \tag{17}$$

with an arbitrary constant X_B. Subsequent integration of Eq. (8) gives

$$\xi_{S\omega} = 2X_B\mu T + X_\omega T^2 \tag{18}$$

with another arbitrary constant X_ω.

Substituting Eqs. (17) and (18) back into the algebraic equations we determine remaining four transport coefficients ξ:

$$\xi_{JB} = C\mu \tag{19}$$

$$\xi_{J\omega} = C\mu^2 + X_B T^2 \tag{20}$$

$$\xi_{TB} = \frac{1}{2}C\mu^2 + \frac{1}{2}X_B T^2 \tag{21}$$

$$\xi_{T\omega} = \frac{2}{3}C\mu^3 + 2X_B\,\mu T^2 + \frac{2}{3}X_\omega T^3 \tag{22}$$

It is remarkable that in the no-drag frame anomalous transport coefficients are *polynomials* in T and μ [14]. The polynomial coefficients have been also found in calculations using Kubo formulas (see [15] and refs. therein), in the calculations in Ref. [16], where the ansatz equivalent to (15) was used to bypass equations of motion, in kinetic theory calculations [17] and in holographic calculations in Ref. [18], where the frame was defined by horizon normal 4-vector. Despite the results suggesting a special nature of the frame, the simple physical significance of it – being the no-drag frame – has not been realized till now.

It has been also suspected that the special frame may be characterized by vanishing entropy flow e.g., [12, 19], in particular, based on the superfluid analogy [20]. This is not true in general, as we shall now discuss.

Entropy flow — It is remarkable, but not unexpected, that, even though the fluid carries energy flow, a static heavy quark (impurity or obstacle) experiences no drag. This reflects dissipationless, persistent nature of the anomalous currents, similar to the superfluid currents.

For comparison, relativistic superfluid hydrodynamics is described by constitutive equations with

$$\tilde{T}^{\mu\nu} = v^2\psi^\mu\psi^\nu,\ \tilde{J}^\mu = -v^2\psi^\mu,\ \tilde{S}^\mu = 0, \tag{23}$$

where $\psi^\mu = \partial^\mu\phi + A^\mu$ and ϕ is the Goldstone field (phase) obeying Josephson equation $u\cdot\psi = \mu$ [21]. Using $d\varepsilon = Tds + \mu dn + v^2\psi\cdot d\psi$ one can again show that $T\partial\cdot S = u\cdot\mathcal{F}$, i.e., superfluid flow does not contribute to drag.

Superfluid flow does not transport entropy (23). On the other hand, the anomalous entropy flow in the no-drag frame is proportional to X_B. When this coefficient is zero, e.g., in Ref. [12], the entropy current is absent. However, this property does not hold more generally, as equations (17) and (18) show.

For a *realistic* magnetic field B, i.e., a dynamic field in an anomaly-free gauge theory, the coefficient X_B, proportional to the mixed gauge-gravity anomaly [22, 23], vanishes and so does the no-drag entropy flow, similar to the superfluid. The notable difference is that the vanishing of the no-drag entropy flow in the CME is tied to such a profound property of quantum field theory as anomaly cancellation. The no-drag entropy flow from the CVE (18) can be nonzero, provided μ is not linked to a dynamic gauge field and the corresponding X_B is nonzero (e.g., for $\mathrm{U}(1)_A$ charge in QCD).

We can verify these general results and better understand the physics involved by using examples where the hydrodynamic behavior is derivable from a microscopic description. We shall consider two such examples: CVE in Lorentz invariant chiral kinetic theory with collisions [24] and CME in a chiral gauge plasma at high temperature.

Examples — The Lorentz invariant chiral kinetic equation is given by

$$\partial\cdot j = \mathcal{C}[f] \tag{24}$$

where j^μ is the covariant phase-space particle number current and $\mathcal{C}$ is the collision rate for a given distribution function f (see Ref. [24] for details). A uniformly rotating equilibrium solution to this equation can be written as $f = (e^g+1)^{-1}$ where [24, 25]

$$g = \beta u\cdot p + \frac{1}{2}\Omega_{\mu\nu}S^{\mu\nu} - \beta\mu q. \tag{25}$$

The property which ensures the detailed balance and vanishing of $\mathcal{C}[f]$ is that g is a linear combination of quantities *conserved* in each collision: 4-momentum p^μ, angular momentum $S^{\mu\nu}$ and particle number (charge) q, where βu_μ, $\Omega_{\mu\nu}$ and $\beta\mu$ are the coefficients [24, 25].

Let us now show that for the solution given by Eq. (25) drag would be absent in the frame given by u^μ. To describe an impurity we need to add another term into the kinetic equation describing collisions of the particles with the impurity:

$$\partial\cdot j = \mathcal{C} + \mathcal{C}_U; \qquad \mathcal{C}_U = \int_{AB} C_{AB} \tag{26}$$

$$C_{AB} = W_{B\to A} - W_{A\to B} \tag{27}$$

$$W_{A\to B}[\bar{f}] = |M_2(E,\theta)|^2\,(2\pi)\,\delta(p_A\cdot U - p_B\cdot U)\ \bar{f}_A(1-\bar{f}_B), \tag{28}$$

where the matrix element is a function of two independent Lorentz invariants: the energy $E = p_A \cdot U = p_B \cdot U$, and the scattering angle θ, or $p_A \cdot p_B = E^2(1-\cos\theta)$ in the frame U. The distribution function $\bar{f}$ is also evaluated in frame U. The collisions in $\mathcal{C}_U$ do *not* conserve 3-momentum, unlike those in $\mathcal{C}$. However, collisions are elastic and the energy *is* conserved in the frame U where the impurity is at rest.

The drag force is given by the rate of momentum transfer from the colliding particles to the impurity:

$$F^\mu = \int_{AB} C_{AB}\,(p_A - p_B)^\mu \tag{29}$$

The energy conservation in Eq. (28) ensures that $U \cdot F = 0$. However, even for an equilibrium solution in Eq. (25) with generic u^μ the 3-momentum transfer need not vanish – the impurity will experience a drag, as expected. Correspondingly, the distribution function obtained from Eq. (25) will not satisfy the detailed balance condition $W_{B\to A} = W_{A\to B}$ because such g is not a linear combination of *conserved* quantities – the 3-momentum in frame U is not conserved in Eq. (28). Thus the solution to the kinetic equation (24) given by Eq. (25) will not solve the kinetic equation (26). In other words, the impurity will disturb the flow.

However, when $u = U$, the component of momentum appearing in Eq. (25) $u\cdot p = U\cdot p$ (the energy in frame U) *is* conserved according to Eq. (28). Thus detailed balance will be satisfied when $U = u$, i.e., $C_{AB} = 0$. This then ensures that $\mathcal{C}_U = 0$, i.e., the distribution in Eq. (25) with $u = U$ is still an equilibrium solution, and that the drag force in Eq. (29) vanishes even though particles do scatter off impurity.

The CVE transport coefficients for the distribution in Eq. (25) have been calculated in Ref.[24]:

$$\xi_{J\omega} = \frac{\mu^2}{4\pi^2} + \frac{T^2}{12}\,; \tag{30}$$

$$\xi_{T\omega} = \frac{\mu^3}{6\pi^2} + \frac{\mu T^2}{6}\,; \tag{31}$$

$$\xi_{S\omega} = \frac{\mu T}{6}\,. \tag{32}$$

In agreement with Eqs. (20), (22) and (18) they are polynomials in μ and T, with $C = 1/(4\pi^2)$, $X_B = 1/12$ and $X_\omega = 0$.

As another example, we can consider CME in a nonabelian chiral quark-gluon plasma at high temperature. It is known that anomalous flows in the frame used to define canonical thermal density matrix $e^{-\beta\hat{H}}$ agree with Eqs. (30) in non-interacting limit [15], and we expect the same even with interactions. To check the value of the drag, we can take the 1-gluon scattering rate with momentum transfer $\boldsymbol{q}$ in this frame given by [26]

$$\mathcal{R}(\boldsymbol{q}) = \frac{d\Gamma}{d^3\boldsymbol{q}} = \frac{\alpha_s T}{2\pi^2} C_R \lim_{q^0\to 0} \frac{\rho_L(q^0,\boldsymbol{q})}{q^0}\,, \tag{33}$$

where C_R is the color Casimir and $\rho_L(q) = -2\,\mathrm{Im}\,G_R^{00}(q)$ is the longitudinal gluon spectral density. Because G_R^{00} is real in coordinate space, the Fourier transform satisfies $\rho_L(q^0,\boldsymbol{q}) = -\rho_L(-q^0,-\boldsymbol{q})$. Since also $\rho_L(q^0,\boldsymbol{q}) \to q^0 f(\boldsymbol{q})$ for $q^0 \to 0$, we must have $\mathcal{R}(\boldsymbol{q}) = \mathcal{R}(-\boldsymbol{q})$, and the drag force on the heavy quark vanishes

$$\boldsymbol{F} = \int_{\boldsymbol{q}} \boldsymbol{q}\,\mathcal{R}(\boldsymbol{q}) = 0 \tag{34}$$

independently of the external magnetic field.

Conclusions — We observed that the second law of thermodynamics constrains the dependence of the drag force on the velocity (up to an overall positive coefficient determined by the microscopic details of the interaction between the obstacle and the fluid). In equilibrium, the drag vanishes when the obstacle is at rest in a certain frame associated with the fluid – the no-drag frame. Conversely, for a fluid flowing through a pipe, the drag on the walls will dissipate the normal flows until they vanish in the frame where the walls are at rest. For a normal fluid in equilibrium all conserved currents (energy, charge, entropy) vanish in the no-drag frame.

However, for a fluid carrying anomalous CME or CVE currents, the no-drag frame is characterized by certain *non-zero* values of the anomalous currents proportional to the magnetic field or vorticity with coefficients which we found to be universally given by Eqs. (17)–(22).

In other words, the second law of thermodynamics requires the no-drag frame to move relative to the rest frame of the fluid as a whole ($T^{0i} = 0$, or Landau, frame), with velocity $\boldsymbol{v} = -(\xi_{TB}\boldsymbol{B} + \xi_{T\omega}\boldsymbol{\omega})/w$. This may have interesting consequences. E.g., a magnetic field, via CME, can create a drag ($\boldsymbol{F} = \lambda\boldsymbol{v}$) on a static impurity, or a heavy quark in heavy-ion collisions [12].

It is useful to recall the Landau's two-fluid picture of superfluidity [11] in which the normal component of the flow exerts drag and generates entropy moving past an obstacle, while the superfluid component, as the name implies, flows with no drag. The no-drag frame is the frame where the normal component rests, while the superfluid component can transport energy and charge (23). Similarly, the anomalous CME and CVE flows of energy and charge do not vanish in the no-drag frame. In this sense, anomalous flows are also “superfluid”.

This analogy breaks down when we consider the entropy flow. The superfluid flow (proper) carries no entropy (23). In contrast, anomalous currents can carry entropy without drag, according to Eqs. (5), (17), (18). It would be interesting to explore potential consequences (e.g., in a thermomechanical effect) of this novel and unusual fact. Though realistic applications are constrained by anomaly cancellation conditions, the relation of this phenomenon to gauge-gravity anomaly is intriguing.

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