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Phys. Rev. Lett. **116**, 106405 — Published 10 March 2016

DOI: 10.1103/PhysRevLett.116.106405

Tunable Splitting of the Ground-State Degeneracy in Quasi-One-Dimensional Parafermion Systems

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(Dated: February 17, 2016)

Systems with topologically protected ground-state degeneracies are currently of great interest due to their potential applications in quantum computing. In practise this degeneracy is never exact, and the magnitude of the ground-state degeneracy splitting imposes constraints on the timescales over which information is topologically protected. In this Letter we use an instanton approach to evaluate the splitting of topological ground-state degeneracy in quasi-1D systems with parafermion zero modes, in the specific case where parafermions are realized by inducing a superconducting gap in pairs of fractional quantum Hall (FQH) edges. We show that, like 1D topological superconducting wires, this splitting has an oscillatory dependence on the chemical potential, which arises from an intrinsic Berry phase that produces interference between distinct instanton tunneling events. These Berry phases can be mapped to chiral phases in a (dual) quantum clock model using a Fradkin-Kadanoff transformation. Comparing our low-energy spectrum to that of phenomenological parafermion models allows us to evaluate the real and imaginary parts of the hopping integral between adjacent parafermionic zero modes as functions of the chemical potential.

PACS numbers: 74.78.Fk, 73.43.−f, 03.65.Vf, 71.10.Pm

The possibility of creating bound states with exotic non-Abelian statistics at the ends of quasi-1D systems is an exciting prospect that has been developed both theoretically [1–16] and experimentally [17–25] over the past few years. These bound states are known to generate a topological ground-state degeneracy. In arrays of 1D systems, states in the resulting low-energy Hilbert space can be entangled and manipulated by braiding processes [26], which are (at least in principle) robust to noise. Such systems have drawn significant interest due to their potential to realize topological quantum computation [27–31].

Parafermion bound states [6–16], which have k ($k > 2$) topologically degenerate ground states, are particularly promising for such quantum computing applications: In comparison to Majorana bound states ($k = 2$), parafermion bound states allow for a denser (albeit non-universal) set of computational gates, and are believed to be intrinsically more robust to environmental noise [8–10, 32]. Parafermions are significantly more challenging to produce than their Majorana counterparts: Most proposals entail generating them as defects in 2D Abelian FQH states, in ways that have yet to be carried out experimentally. However, their non-Abelian statistics are also more complex than for Majoranas, which makes realizing them a particularly exciting prospect. It is noteworthy that parafermions cannot be realized in strictly 1D wire models according to the results by Refs. [33, 34].

For bound states confined to the endpoints of a system of finite length L, the topological ground-state degeneracy is split by an amount $\Delta E \sim e^{-L/\xi}$, where ξ is proportional to the correlation length in the bulk of the system, causing superpositions of ground states to decohere over time. Though theoretically it is possible to make this splitting as small as required by making L large, there are definite advantages when this splitting can be made small even for modest-length systems. For 1D topological superconducting wires [35, 36] or spin-Hall based superconductors [2, 5], this can be achieved by small adjustments in the appropriate chemical potential, since in addition to the exponential fall-off in L the splitting has an oscillatory dependence on the Fermi momentum via $\Delta E \sim e^{-L/\xi}\cos(k_F L)$. The observed oscillations of the splitting of the zero-bias conductance peaks [20] can be viewed as evidence of Majorana bound states [35–39].

Despite the publicity it has had in Majorana systems [40–43], little attention has been given to the ground-state splitting in parafermion systems. Specifically, one might wonder whether these exhibit an analogue of the oscillatory $\cos(k_F L)$ term. In this Letter, we calculate the splitting of the topological ground-state degeneracy in parafermion platforms obtained by inducing superconductivity or ferromagnetism at certain types of edges in FQH states [8–13]. Following Ref. [36] in the Majorana case, we perform this calculation using a bosonised description of the strongly interacting 1D system, in which the splitting of the ground-state degeneracy is obtained by an instanton calculation in the resulting sine-Gordon model. Interestingly, as in the Majorana case we do find oscillations in the splitting as functions of chemical potential or applied magnetic field. These intriguing oscillations result from a Berry phase term in the generic sine-Gordon action. We also use our calculation to deduce the magnitude and phase of the hopping coefficients that arise most naturally in parafermion chains, such as those studied by Refs. [6, 12, 44–47].

Model of parafermion zero modes.—Several groups [8–13] have suggested that parafermion zero modes can be generated in systems with counterpropagating chiral edges separating two FQH regions with opposite g-factors. The edge of interest consists of one right-moving and one left-moving mode with opposite spin polarizations. Two types of electron tunneling processes can open a gap at this edge: Inducing superconductivity (SC) generates a Cooper-pairing $\Delta(\psi^\dagger_{L,\downarrow}\psi^\dagger_{R,\uparrow} + \text{H.c.})$, while spin backscattering $\mathcal{B}(\psi^\dagger_{L,\downarrow}\psi_{R,\uparrow} + \text{H.c.})$ can be induced by tunnel-coupling the edge to a ferromagnet (FM). Any in-

terface between these different types of induced gaps will host parafermion bound states (Fig. 1).

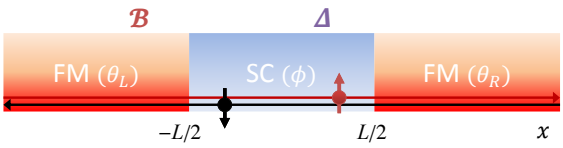

FIG. 1: (color online). Schematic spatial profiles of the proximity-induced gaps $\Delta(x)$ and $\mathcal{B}(x)$ for the FM-SC-FM setup.

To describe this system, it is convenient to bosonise the two edge modes, representing the right- (left-) moving electrons as $\psi^\dagger_{R/L} \sim \frac{1}{\sqrt{2\pi n\varsigma}} e^{-in\varphi_{R/L}}$, where $\varphi_{R/L}$ are chiral bosonic fields, and $1/n$ is the filling fraction of the corresponding FQH bulk regions. Here ς is related to the inverse energy cutoff (E_{cutoff}) of the bosonised theory via $\varsigma \sim v\hbar/E_{\text{cutoff}}$, where v is the velocity of the edge modes.

The two backscattering terms are most simply expressed in the basis $\phi = \frac{1}{2}(\varphi_{R,\uparrow} + \varphi_{L,\downarrow})$ and $\theta = \frac{1}{2}(\varphi_{R,\uparrow} - \varphi_{L,\downarrow})$. These non-chiral fields are related to the charge density ρ_C and spin density ρ_S via $\rho_C = \frac{1}{\pi}\partial_x\theta$ and $\rho_S = \frac{1}{\pi}\partial_x\phi$. In this bosonised basis, the two backscattering terms take the form $\Delta(\psi^\dagger_{L,\downarrow}\psi^\dagger_{R,\uparrow}+\text{H.c.}) \sim \Delta\sin(2n\phi)$ and $\mathcal{B}(\psi^\dagger_{L,\downarrow}\psi_{R,\uparrow}+\text{H.c.}) \sim \mathcal{B}\sin(2n\theta)$, and the 1D parafermion system is described by the following Euclidean sine-Gordon action [8–12, 48–50]:

$$S_E = \int d\tau dx \{ i\hbar\frac{n}{\pi}\partial_x\theta(x,\tau)\partial_\tau\phi(x,\tau) - \frac{\mu(x)}{\pi}\partial_x\theta(x,\tau) + \frac{\hbar nv}{2\pi}(\partial_x\theta(x,\tau))^2 + \frac{\mathcal{B}(x)}{\pi n\varsigma}[\sin(2n\theta(x,\tau))+1] + \frac{\hbar nv}{2\pi}(\partial_x\phi(x,\tau))^2 + \frac{\Delta(x)}{\pi n\varsigma}[\sin(2n\phi(x,\tau))+1]\}. \quad (1)$$

Here μ represents the chemical potential and $\mathcal{B}, \Delta$ are energy gaps induced by spin- and charge-backscattering processes, respectively. The commutation relation $[\phi(x), \theta(x')] = i\frac{\pi}{n}\Theta(x-x')$ dictates that only one of the two possible gapping terms can have a non-vanishing expectation value at a given spatial position. However, if a region where $\Delta \sim \hbar v/\varsigma$ can be sandwiched between two regions where $\mathcal{B} \sim \hbar v/\varsigma$, parafermion bound states arise at the interfaces between them. In the bosonised picture, the resulting topological ground-state degeneracy is manifest in the $2n$ values of ϕ for which the sine term is minimized [7–12].

In the following, we consider the FM-SC-FM heterostructure shown in Fig. 1, on which we take $\mathcal{B} = 0$ ($\Delta, \mu = 0$) in the SC (FM) region $|x| < L/2$ ($|x| > L/2$). In the FM regions $|x| > L/2$, the field θ is therefore pinned to one of the potential minima, and ϕ is strongly fluctuating as required by the commutation relations. As we show in [51], under these conditions the FM regions do not contribute to the ground-state energy splitting, and after integrating out θ we obtain the following effective action for the SC region:

$$S_\phi = \int_{-\frac{T_\tau}{2}}^{\frac{T_\tau}{2}} d\tau \int_{-\frac{L}{2}}^{\frac{L}{2}} dx \{ \frac{\hbar n}{2\pi v}(\partial_\tau\phi(x,\tau))^2 + \frac{\hbar nv}{2\pi}(\partial_x\phi(x,\tau))^2 + \frac{\Delta}{\pi n\varsigma}[\sin(2n\phi(x,\tau))+1] + i\frac{\mu(x)}{\pi v}\partial_\tau\phi(x,\tau)\}. \quad (2)$$

In the ground states of this reduced system the ϕ field is also approximately pinned at one of the $2n$ inequivalent local minima ϕ_{min} of the sine potential; we will take Δ to be sufficiently large that these low-energy states are well separated from the rest of the spectrum. The fluctuation-induced splitting between the $2n$ otherwise degenerate ground states is then determined by the amplitude for tunneling between adjacent local minima.

The last term in Eq. (2) plays the role of a topological Berry phase term $S_{\text{B-p}}$, contributing to the net action only for field configurations which start and end at different values of θ (i.e. only for instantons). It introduces oscillations in the splitting of the ground-state degeneracy as the chemical potential μ is varied. To the best of our knowledge, $S_{\text{B-p}}$ was not included in previous studies of instantons in the bosonised periodic sine-Gordon model, which considered the case $\mu = 0$ [36, 50, 52].

Instanton calculation of level splitting.—For the sine-Gordon model described by Eq. (2), the minima of the potential term are at $\phi_{\text{min}} = -\frac{\pi}{4n} + \frac{j\pi}{n}$ where $j = 0, 1, \ldots, 2n-1$, with the j-th ground state denoted by $|j\rangle$. The classical soliton solution interpolating between vacua at $j\frac{\pi}{n} - \frac{\pi}{4n}$ and $(j\pm1)\frac{\pi}{n} - \frac{\pi}{4n}$ has the form [53–55]:

$$\phi_{sol}(\tau) = -\frac{\pi}{4n} + j\frac{\pi}{n} \pm \frac{2}{n}\arctan[e^{\omega(\tau-\tau_0)}], \quad (3)$$

where $\omega = 2\sqrt{\Delta v/(\hbar\varsigma)} > 0$. To a good approximation, we may neglect spatial variations in the instanton solution due to boundary effects [51].

Following Refs. [52, 56–59], the amplitudes for the one-instanton and one-anti-instanton processes are:

$$\langle j|e^{-H_\phi T_\tau/\hbar}|j\mp1\rangle_{o.i.} = \left(\mathcal{N}e^{-\tilde{\omega}T_\tau/2}\right)\left(\sqrt{\frac{v}{L}}\sqrt{\frac{S_0}{2\pi\hbar}}e^{-S_0/\hbar\mp i\gamma}\right)\sqrt{\omega}T_\tau. \quad (4)$$

Here $S_0 + (-)i\hbar\gamma$ is the effective action of an instanton (anti-instanton) tunneling event (with small quantum corrections omitted [51]), where the imaginary contribution γ stems from $S_{\text{B-p}}$, and $\hbar\tilde{\omega}/2$ stands for the zero-point energy of the 1D harmonic oscillator. In terms of the parameters in Eq. (2), we have $S_0 = \frac{2\hbar\omega L}{n\pi v}$, $\gamma = \frac{\mu L}{\hbar nv}$. As anticipated, the amplitude for tunneling vanishes exponentially with the wire length L provided the bulk is gapped (i.e. $\Delta \neq 0$). More significantly, we now perceive the importance of the Berry phase term, which contributes a different net phase to the amplitude for instanton and anti-instanton processes.

The prefactor $\mathcal{K} = \sqrt{v\omega S_0/(2\pi\hbar L)}$ is the Fredholm determinant describing Gaussian fluctuations about the saddle-point solution (3). This determinant is sensitive to spatial fluctuations in the SC region, and its scaling with L is sensitive

to the choice of boundary conditions (BCs). As shown in [51], for Neumann-type BCs appropriate to the setup shown in Fig. 1, this prefactor is independent of L, in agreement with existing work on Majorana bound states [35, 37, 60]. For the energy splitting in periodic BCs relative to antiperiodic ones, on the other hand, the prefactor contains an extra factor of $\frac{1}{\sqrt{L}}$ [52]. Fig. 2 compares this prediction to numerical values for the energy splitting in the Majorana nanowire for both BCs.

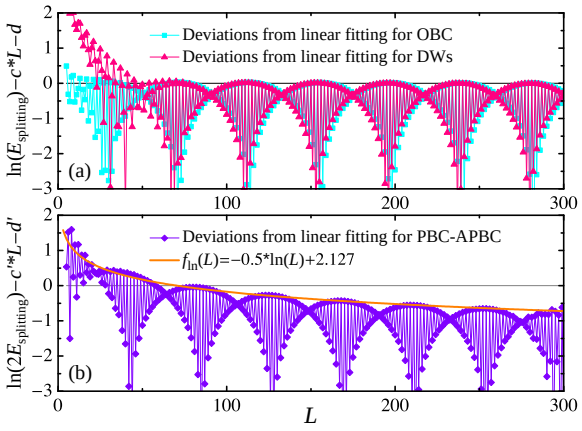

FIG. 2: (color online). Numerical results for the *oscillatory* zero-energy splitting as a function of L in the 1D nanowire models hosting Majorana bound states [1, 3, 4], for (a) open and domain-wall BCs, and (b) periodic versus antiperiodic BCs (PBC-APBC). (Recall that for a Majorana SC ring, the ground state with PBC (APBC) is a parity-even (parity-odd) state.) The term linear in L has been subtracted off in both cases, leaving a result independent of L in (a), and depending logarithmically on L in (b).

Armed with one-instanton solutions, the total instanton contribution is obtained using a dilute instanton gas approximation [57], which gives:

$$\langle j_+|e^{-H_\phi T_\tau/\hbar}|j_-\rangle = \mathcal{N}\int_0^{2\pi}\frac{d\zeta}{2\pi}e^{i\zeta(j_- - j_+)} \times \exp\left[-\frac{\tilde{\omega}}{2}T_\tau + 2\mathcal{K}T_\tau e^{-S_0/\hbar}\cos(\zeta-\gamma)\right]. \quad (5)$$

The cosine term can be viewed as arising from interference between instanton and anti-instanton trajectories, for which $S_{\text{B-p}}$ has the opposite phase.

In the presence of instanton tunneling events, the eigenstates of H_ϕ are therefore Bloch-wave-like states of the form: $|\zeta\rangle \propto \left(\frac{1}{2\pi}\right)^{\frac{1}{2}}\sum_j e^{-i\zeta j}|j\rangle$. Imposing $\langle j_+ \pm 2n|e^{-H_\phi T_\tau/\hbar}|j_- \pm 2n\rangle = \langle j_+|e^{-H_\phi T_\tau/\hbar}|j_-\rangle$ to account for the fact that only $2n$ of these minima are physically distinct forces ζ to take on the $2n$ discrete values $\frac{\pi}{n}q_\phi$, with $q_\phi = 0, 1, \ldots, 2n-1$, which gives the $2n$ states:

$$|q_\phi\rangle \propto \left(\frac{1}{2n}\right)^{\frac{1}{2}}\sum_{j=0}^{2n-1} e^{-i\frac{\pi}{n}q_\phi j}|j\rangle. \quad (6)$$

From Eq. (5), the energies of these states are, up to an overall constant,

$$E(q_\phi) = -\frac{2\hbar\omega}{\pi}\sqrt{\frac{1}{n}}e^{-S_0/\hbar}\cos\left(\frac{\pi}{n}q_\phi - \frac{\mu L}{\hbar n v}\right), \quad (7)$$

where q_ϕ is the generalised charge parity conjugate to the discrete, compact variable ϕ_{min}. This form agrees with the general result of Ref. [61] for the energies of anyon pairs:

$$E(q_\phi) = \sum_{a=0}^{2n-1}\left(\Gamma_a[F_{q_\phi}^{\alpha a a \alpha}]_{\alpha\alpha} + \text{H.c.}\right), \quad (8)$$

where α denotes the parafermion bound states, and F is a part of the topological data. For the case at hand the possible choices of F are described in Ref. [62]; taking $[F_{q_\phi}^{\alpha a a \alpha}]_{\alpha\alpha} = e^{i(\pi/n)[(a\cdot q_\phi) \bmod 2n]}$ and $\Gamma_a = -\hbar\mathcal{K}e^{-S_0/\hbar - i\gamma}\delta_{a,1}$ recovers the form (7) for the energies of these ground states.

It is instructive to check Eq. (7) for the case of Majorana fermions ($n = 1$), where the splitting can be calculated directly from a quadratic fermion Hamiltonian [5, 35, 40, 41]. The relevant calculation in the quantum Hall systems described here can be carried out as for the nanowire case [35, 37] – see Refs. [51, 60]. The result is

$$E_q = -C\frac{\Delta\mathcal{B}}{\Delta+\mathcal{B}}e^{-\frac{\Delta L}{\hbar v}}\cos\left(\pi q - \frac{\mu L}{\hbar v}\right), \quad q = 0, 1, \quad (9)$$

where C is a constant of order unity. The coefficient of the decaying exponential differs from Eq. (7), since for $n = 1$, $S_0/\hbar = \left(\frac{\Delta L}{\hbar v}\right)\frac{4}{\pi}\sqrt{\hbar v/(\Delta\varsigma)}$. However, in the instanton calculation of the exponential term we neglect all modes above the gap set by Δ; hence the cutoff energy $\hbar v/\varsigma$ should not be much larger than Δ. Further, the tunneling process requires the presence of virtual fluctuations up to an energy of approximately Δ, so the cutoff energy should also not be much smaller than Δ. The factor $\frac{4}{\pi}\sqrt{\hbar v/(\Delta\varsigma)}$, which parameterizes the difference in the exponential decay lengths from the two calculations, is therefore a constant of order 1. Parallel reasoning applies to the comparison of the prefactors. Remarkably, the argument of the cosine term agrees exactly with our instanton calculation, suggesting that this oscillatory dependence on μ is insensitive to the BCs and to the various approximations being made.

We note that the preceding analysis also applies to the case $\mathcal{B} > 0, \Delta = 0$ by taking $\phi \rightleftharpoons \theta$ and replacing the chemical potential term with a magnetic field term of the form $-\frac{\hbar}{\pi}\partial_x\phi$. Therefore the oscillatory dependence of the ground-state energy splitting on chemical potential or magnetic field is a relatively ubiquitous feature of parafermion zero modes.

Hopping in parafermion chains.—One interesting application of our calculation is that it allows us to infer the phase of intrawire parafermion hopping terms. This is of particular interest as chains of coupled parafermions can be used to generate even more exotic topological phases [12, 46].

In the setup we consider, the parafermion bound states can be described by operators $\alpha_L^{(\dagger)}, \alpha_R^{(\dagger)}$ which annihilate (create)

parafermion zero modes at the left and right endpoints of the SC region, respectively, and satisfy the relations

$$\alpha_{L/R}^{2n} = 1,\ \alpha_{L/R}^{\dagger} = \alpha_{L/R}^{2n-1},\ \text{and}\ \alpha_L\alpha_R = \alpha_R\alpha_L e^{i\frac{\pi}{n}}, \quad (10)$$

which are sufficient to ensure that these bound states have "parafermionic" non-Abelian statistics [7–11].

In terms of the bosonised fields θ and ϕ, we have $\alpha_L^\dagger\alpha_R = e^{i\frac{\pi}{n}(q-1/2)}$, where $q = \frac{n}{\pi}(\theta(\frac{L}{2}) - \theta(-\frac{L}{2}))$ is the total charge in the SC segment modulo 2 [8–11]. From the commutation relation between ϕ and θ, it follows that

$$(\alpha_R^\dagger\alpha_L)\phi(\alpha_L^\dagger\alpha_R) = \phi + \frac{\pi}{n},\ (\alpha_L^\dagger\alpha_R)\phi(\alpha_R^\dagger\alpha_L) = \phi - \frac{\pi}{n},$$

which carrying out precisely the tunneling processes whose matrix elements we have just evaluated.

The low-energy Hamiltonian describing the parafermion tunneling between the two endpoints is therefore

$$H_A = t\alpha_L^\dagger\alpha_R + t^*\alpha_R^\dagger\alpha_L.$$

Its eigenstates are labeled by an integer $q = 0, 1, 2, \ldots, 2n-1$, and satisfy [8–11]

$$\alpha_L^\dagger\alpha_R|q\rangle = -e^{i\frac{\pi}{n}(q-\frac{1}{2})}|q\rangle,\ \alpha_R^\dagger\alpha_L|q\rangle = -e^{-i\frac{\pi}{n}(q-\frac{1}{2})}|q\rangle. \quad (11)$$

Note that Eqs. (6) and (11) together also fix the phase associated with the action of the parafermion hopping term on the eigenstates of ϕ: $\alpha_L^\dagger\alpha_R|j\rangle = -e^{-i\frac{\pi}{2n}}|j+1\rangle$, $\alpha_R^\dagger\alpha_L|j\rangle = -e^{+i\frac{\pi}{2n}}|j-1\rangle$. The corresponding energies—which are precisely the energies that we have just obtained with our instanton calculation—are:

$$E(q) = -2\sqrt{t_\Re^2 + t_\Im^2}\cos\left[\frac{\pi}{n}\left(q - \frac{1}{2}\right) + \vartheta\right], \quad (12)$$

where we have defined $\tan\vartheta = t_\Im/t_\Re$.

Comparing Eqs. (7) and (12) allows us to constrain $t_\Re$ and $t_\Im$. For Majorana fermions (i.e. $n = 1$), there is an additional constraint: Since $\alpha_{R/L}^\dagger = \alpha_{R/L}$, the two terms in Eq. (11) are not independent. This forces $t_\Re = 0$ (i.e. $\vartheta = \pm\pi/2$), and $t_\Im = \pm\hbar\mathcal{K}e^{-S_0/\hbar}\cos[\mu L/(\hbar n v)]$. For $n > 1$ there is no such a constraint; however, in these cases matching the eigenvalues of both H and the operator $\alpha_L^\dagger\alpha_R = e^{i\left(\theta(\frac{L}{2}) - \theta(-\frac{L}{2}) - \pi/(2n)\right)}$ fixes ϑ, such that

$$t_\Re = \pm\hbar\mathcal{K}e^{-S_0/\hbar}\cos[\pi/(2n) - \mu L/(\hbar n v)], \quad (13)$$

$$t_\Im = \pm\hbar\mathcal{K}e^{-S_0/\hbar}\sin[\pi/(2n) - \mu L/(\hbar n v)]. \quad (14)$$

Using the analogous approach for an SC-FM-SC system (with $\mu \Rightarrow h, \Delta \Rightarrow \mathcal{B}$) gives the analogous conclusion.

It is worth stressing that even at vanishing μ (or h), for $n > 1$ the hopping parameter t is complex with $\vartheta = \pi/(2n)$. This suggests that the proposal for universal quantum computing by manufacturing Fibonacci anyons in coupled parafermion chains [12, 44, 46, 47] is better achieved in systems without finite chemical potential or magnetic field. More specifically, the μ-dependent contribution to ϑ in Eqs. (13) and (14) corresponds to a *chiral* phase [63–66] in the quantum clock model. To be concrete, a system of $2N$ tunnel-coupled parafermion zero modes is dual (via the Fradkin-Kadanoff mapping [6, 47, 67]) to an N-site chiral quantum clock chain, where with appropriate conventions hopping across a SC (FM) region maps to the transverse field (ferromagnetic clock) coupling. Under duality, the phases of the parafermion hopping terms t map to a chiral phase of $\pm e^{i\frac{hL}{\hbar n v}}$ for the ferromagnetic clock coupling, as well as a chiral phase of $\pm e^{i\frac{\mu L}{\hbar n v}}$ for the transverse field term. Notice that the oscillatory dependence of the ground-state degeneracy splitting on the chiral phases and the system's size has been observed numerically for these chiral clock systems [47] recently.

In summary, our non-perturbative calculation shows that it is possible, in principle, to tune the magnitude of the ground-state splitting in parafermion systems (as well as the phase of the parafermion hopping parameter, for $n > 1$) by means of a chemical potential or an external magnetic field due to interference between distinct instanton trajectories resulting from a topological term in the effective action. Because the period of the resulting oscillations is given by $\mu L/(\hbar n v)$ (or $hL/(\hbar n v)$), this splitting can be fine-tuned with relatively small changes in μ (or h). As for Majoranas, we anticipate that this fact will be both of practical use to achieve quantum-coherent systems, and a potential signature of the existence of parafermions in these systems. Finally, our results might also be applicable to the spin-unpolarized $\nu = 2/3$ FQH heterostructures proposed by Refs. [12, 13].

Acknowledgements: We thank J. Alicea, C. L. Kane, S. Ryu, M. D. Schulz, Q. Shi, and M. Zudov for useful discussions. We also thank the anonymous Referees whose suggestions and questions improve the manuscript a lot. FJB is supported by NSF-DMR 1352271 and by the Sloan foundation FG-2015-65927.

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