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Optical conductivity of topological surface states with emergent supersymmetry

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Topological states of electrons present new avenues to explore the rich phenomenology of correlated quantum matter. Topological insulators (TIs) in particular offer an experimental setting to study novel quantum critical points (QCPs) of massless Dirac fermions, which exist on the sample's surface. Here, we obtain exact results for the zero- and finite-temperature optical conductivity at the semimetal-superconductor QCP for these topological surface states. This strongly interacting QCP is described by a scale invariant theory with emergent supersymmetry, which is a unique symmetry mixing bosons and fermions. We show that supersymmetry implies exact relations between the optical conductivity and two otherwise unrelated properties: the shear viscosity and the entanglement entropy. We discuss experimental considerations for the observation of these signatures in TIs.

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Topological insulators [1, 2] allow for the experimental study of new quantum states of matter. The strong spin-orbit coupling in these bulk-insulating materials leads to unique gapless Dirac fermion surface states. These can undergo quantum phase transitions forbidden in non-topological systems, and thus constitute a new platform to study the rich physics of quantum criticality [3, 4]. A considerable challenge in the study of interacting QCPs is to determine their dynamical response—that is, their response at finite frequency ω—both at zero and finite temperature T, such as the optical conductivity $\sigma(\omega, T)$. Here, we focus on the dynamical response of a novel QCP that can appear at the surface of a three-dimensional (3D) topological insulator: it describes the interaction-driven quantum phase transition between a single Dirac cone of electrons and a gapped superconductor [5, 6] (see Fig. 1). As an important step towards observing this transition, recent experiments have reported the discovery of intrinsic superconductivity on the surface of a 3D topological insulator, Sb_2Te_3 [7]. We emphasize that standard 2D (or layered) systems that do not break time-reversal symmetry must have an even number of Dirac cones and thus cannot host this transition. More complex scenarios realizing multiple copies of this QCP can occur via f-wave pairing [8] and pair-density-wave [9] instabilities of spinless Dirac fermions on the 2D honeycomb lattice, or for interacting ultracold atomic gases in optical lattices [10].

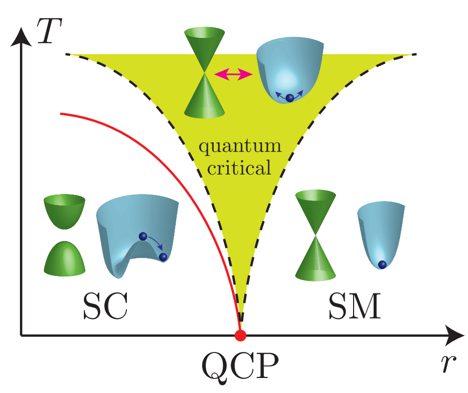

FIG. 1. Phase diagram near the semimetal-superconductor (SM-SC) quantum critical point of Dirac fermions on the surface of a 3D topological insulator. T is the temperature and r is the nonthermal tuning parameter [see Eq. (1)]. The evolution of the Dirac dispersion and Cooper field potential are shown. Supersymmetry emerges at the QCP where it relates the Dirac fermions and the bosonic Cooper pairs.

When the chemical potential is at the Dirac point, a special type of symmetry emerges at the QCP [5, 6, 8, 9, 11]: spacetime supersymmetry (SUSY). SUSY relates bosons and fermions, and has been proposed to exist in extensions of the Standard Model of elementary particle physics, but has not yet been observed. At the QCP of Fig. 1, it emerges naturally by relating the Dirac fermions of the semimetal to the bosonic Cooper pairs of the superconductor. These two become degenerate at the transition and in fact share a deeper relation described by SUSY. We emphasize that this is a consequence of the strong interactions at the QCP, where long-lived excitations (quasiparticles) are destroyed by quantum zero-point fluctuations. We show that even in the presence of such strong interactions, SUSY allows the *exact* determination of the zero-temperature optical conductivity $\sigma(\omega, 0)$ of the topological surface states at the QCP. We are not aware of any known exact result for the dynamical response of a realistic strongly interacting QCP in spatial dimensions higher than one. In addition, SUSY implies that the conductivity directly determines the shear vis-

	Dirac SM-SC	SC-Insulator
σ_∞	$\dfrac{5(16\pi - 9\sqrt{3})}{243\pi} \approx 0.227$	0.226
η_∞	$\sigma_\infty/40 \approx 5.68\times 10^{-3}$	3.68×10^{-3}
λ_{corner}	$\sigma_\infty/20 \approx 0.0113$	0.00737
b'	0	$-0.3(1)$

TABLE I. **Exact results.** Comparison of the conductivity, viscosity, and entanglement entropy at two different QCPs. Left column: exact results obtained in this paper for the Dirac semimetal (SM) to superconductor (SC) QCP with emergent supersymmetry. Right column: known approximate results for the SC to Cooper-pair-insulator QCP. The optical conductivity and dynamical shear viscosity at $T=0$ are $\sigma(\omega,0)=\sigma_\infty e^2/\hbar$ and $\eta(\omega,0)=\eta_\infty\omega^2\hbar$. λ_{corner} determines the entanglement entropy of nearly smooth corners [Eq. (5)]. b' determines a finite-T correction to the optical conductivity of the form $b'\,(ik_BT/\hbar\omega)^3$.

cosity and certain many-body entanglement properties. Our exact findings are summarized in Table I. We begin by describing the low-energy theory of the QCP, and then explain how the emergent SUSY allows the exact determination of various properties such as the optical conductivity. We end by discussing considerations relevant for the experimental observation of these signatures.

The Landau-Ginzburg theory for the quantum phase transition couples a single charge-e Dirac fermion ψ to the charge-$2e$ Cooper pair bosonic field, ϕ,

$$\mathcal{L} = i\bar{\psi}\gamma_\mu\partial_\mu\psi + \frac{1}{2}|\partial_\mu\phi|^2 + \frac{r}{2}|\phi|^2 + \frac{\lambda}{2}|\phi|^4 + h(\phi^*\psi^T i\gamma_2\psi + \text{c.c.}), \quad (1)$$

in imaginary time, where $\bar{\psi}=\psi^\dagger\gamma_0$ and γ_μ, $\mu=0,1,2$ are 2×2 matrices satisfying the Pauli algebra. We note that time-reversal invariance forbids a fermion mass term. The QCP is obtained by tuning r to zero, and the resulting system is strongly correlated because both the quartic coupling λ and the fermion-boson coupling h are relevant at the noninteracting, UV fixed point $\lambda=h=0$. There is a single stable IR fixed point with $\lambda=h^2\neq 0$ [5, 6, 8, 9, 11], at which (1) becomes invariant under SUSY transformations that rotate the Dirac fermion into the boson and vice-versa [12]. In line with the requirement of SUSY, it was shown [5, 8] that the fermion and Cooper pair velocities flow to the same value at low energies, which we henceforth set to unity. As such (1) also displays emergent Lorentz invariance. By virtue of SUSY, the fermion and boson anomalous dimensions are known *exactly* [12]: $\eta_\psi=\eta_\phi=1/3$, a clear indication of the destruction of quasiparticles. The electric current is given by the sum of fermionic and bosonic contributions: $J_\mu=\bar{\psi}\gamma_\mu\psi + i(\phi^*\partial_\mu\phi - \text{c.c.})$.

The QCP (1) has an important purely bosonic analog obtained by omitting the fermions, in which case it describes the superconductor-to-insulator quantum phase transition obtained by localizing Cooper pairs [13]. Part of the interest in this QCP (and its optical conductivity) comes from the fact that it is believed to occur in certain thin-film superconductors [13]. The QCP that we study belongs to a different universality class because it involves fermions, and we shall contrast the two throughout (see Table I).

Exact charge & shear responses: As the system is tuned to the QCP, the optical conductivity depends only on the ratio $\hbar\omega/k_BT$ [14]:

$$\sigma(\omega,T) = \frac{e^2}{\hbar}\,\Phi\left(\frac{\hbar\omega}{k_BT}\right), \quad (2)$$

where $\Phi(x)$ is a dimensionless, universal scaling function that is fully determined by the universality class of the transition. We recall that the conductivity is obtained from the current-current correlator via the Kubo formula, $\sigma=\frac{1}{i\omega}\langle J_x(\omega,\vec{k}=0)J_x(-\omega,\vec{k}=0)\rangle_T$. An important consequence of the scale invariance is that the optical conductivity at $T=0$ is a frequency-independent *constant*: $\sigma(\omega,0)=e^2\,\sigma_\infty/\hbar$, where we have defined $\sigma_\infty=\Phi(\infty)$, and we are working at frequencies lesser than microscopic energy scales such that we are probing the universal response. For QCPs such as the one under consideration, this universal constant determines the charge response of the ground state in a system lacking quasiparticles. We now describe how the emergent SUSY can be used to compute σ_∞ exactly.

In supersymmetric field theories, operators are organized into representations of the SUSY algebra called supermultiplets, the same way spin operators are organized into representations of $SU(2)$. In our case, the electric current J_μ lies in the same supermultiplet as the stress tensor $T_{\mu\nu}$, the so-called supercurrent supermultiplet [15]. Here, supercurrent does not refer to superconductivity but rather to the Noether current associated with SUSY. One associates to each supermultiplet a so-called superfield which contains all the various components of the supermultiplet. The superfield associated with the supercurrent supermultiplet is denoted $\mathcal{J}_\mu$, and is highly constrained by SUSY. Crucially, the two-point correlation function of the supercurrent is entirely fixed up to an overall multiplicative constant [16–18], denoted $\mathcal{C}$. Because $\mathcal{J}_\mu$ contains both the current J_μ and the stress tensor $T_{\mu\nu}$, this implies a relation between their respective two-point correlation functions. This relation in turn implies a nontrivial relation between the universal charge and shear responses at the QCP (1).

In 2D QCPs with emergent Lorentz invariance, the two-point correlation functions of the current and the stress tensor have the power-law forms [19] $\langle J_\mu(x)J_\nu(0)\rangle = C_J\frac{I_{\mu\nu}(x)}{|x|^4}$ and $\langle T_{\mu\nu}(x)T_{\rho\sigma}(0)\rangle = C_T\frac{I_{\mu\nu,\rho\sigma}(x)}{|x|^6}$, where x denotes the spacetime separation, the I's are dimensionless tensors without free parameters [20], and the constants $C_{J,T}$ are universal low-energy properties related to the conductivity and viscosity, respectively, as we shall see below. The above discussion implies that these are both proportional to $\mathcal{C}$ in our SUSY QCP, hence their ratio is fixed. We find that the particular SUSY of (1) imposes $C_J/C_T = 5/3$ [20]. This then leads to a universal ratio between the zero-temperature dynamical shear viscosity and optical conductivity at the strongly interacting QCP (1). The dynamical shear viscosity $\eta(\omega, T)$ is given by the two-point function of the xy-component of the stress tensor [25, 26], and becomes $\eta(\omega, 0) = \eta_\infty\,\omega^2\hbar$ at zero temperature. Fourier transforming from time to frequency, we find $\sigma_\infty = \pi^2 C_J/2$ and $\eta_\infty = \pi^2 C_T/48$, and thus the universal ratio

$$\frac{\sigma_\infty}{\eta_\infty} = 40, \tag{3}$$

which is a nontrivial fingerprint of the emergent SUSY at the QCP of (1). We emphasize that in the absence of SUSY no relation exists in general between the conductivity and shear viscosity of QCPs. In fact, (3) is violated at the superconductor-insulator QCP of Cooper pairs, see Table I.

The emergent SUSY allows the exact calculation of the optical conductivity by geometric methods, and crucially relies on the connection between the conductivity and viscosity (3). First, the shear viscosity coefficient η_∞, or alternatively C_T, can be obtained from the second derivative of the free energy on the squashed three-sphere with respect to the squashing parameter [15]. Remarkably, the free energy on this spacetime geometry can be computed exactly using the so-called SUSY localization technique [27, 28], even if the theory is strongly coupled. For the QCP of interest to us, an integral expression for C_T was recently obtained [29], which can be computed numerically. We were able to evaluate this integral in closed form [20]. We then used the relation (3) to obtain an exact result for the $T = 0$ optical conductivity at the semimetal-superconductor QCP of 2D Dirac fermions:

$$\sigma(\omega, 0) = \frac{5(16\pi - 9\sqrt{3})}{243\pi}\,\frac{e^2}{\hbar} \approx 0.2271\,\frac{e^2}{\hbar}. \tag{4}$$

To our knowledge, this is the first exact result for the optical conductivity of a realistic strongly interacting QCP, and will thus serve as a benchmark for the dynamical response of quantum critical systems. We note that (4) is both larger than the Dirac fermion conductivity $\sigma_\infty = \frac{1}{16} = 0.0625$ [19] and the conductivity of the Cooper pair superconductor-insulator QCP $\sigma_\infty = 0.226$ [30–35]. Our result (4) is tantalizingly close to the latter, suggesting that even though the Dirac semimetal-superconductor QCP naively seems to have more conducting degrees of freedom, these interact more strongly. To put our exact result in perspective, we emphasize that σ_∞ for the superconductor-insulator QCP has been the subject of numerous studies [13, 36–38] over the past three decades but was reliably obtained only recently via large-scale quantum Monte Carlo simulations [30–34] and the conformal bootstrap approach [35]. Finally, note that (4) is smaller than the conductivity of the Gaussian fixed point, $\frac{5}{16} = 0.3125$, in agreement with the expectation that strong interactions reduce the charge mobility.

Entanglement entropy: There is currently much interest in the entanglement properties of QCPs [39, 40]. In particular, the ground state entanglement entropy across a spatial region containing a sharp corner with opening angle θ contains a subleading logarithmic term whose coefficient $a(\theta)$ depends only on the universality class of the QCP. This coefficient constitutes a new measure of the gapless degrees of freedom in strongly interacting systems. Recent numerical work has focused on determining $a(\theta)$ for various interacting 2D QCPs, such as the XY and Heisenberg QCPs appearing in theories of quantum magnetism [41–43]. For QCPs with emergent Lorentz invariance, the behavior of $a(\theta)$ near $\theta = \pi$ is determined by the stress-tensor correlation coefficient C_T encountered above [44, 45],

$$a(\theta) \approx \lambda_{\text{corner}}(\pi - \theta)^2, \qquad \lambda_{\text{corner}} = \pi^2 C_T/24. \tag{5}$$

Using our exact result for σ_∞, we obtain an exact result in closed form for the corner coefficient of the semimetal-superconductor QCP occurring on the surface of a topological insulator: $\lambda_{\text{corner}} = \sigma_\infty/20 = \frac{16\pi - 9\sqrt{3}}{972\pi} \approx 0.01136$. Unexpectedly, the optical conductivity at zero temperature entirely determines this property of the entanglement entropy. These two quantities are generally unrelated in the absence of supersymmetry, as can be seen in Table I. We note that an integral expression for λ_{corner} has been given previously [46]. In addition, our result for λ_{corner} leads to an exact lower bound on $a(\theta)$ for *all* opening angles [47]: $a(\theta) \geq (2\sigma_\infty/5)\ln[1/\sin(\theta/2)]$.

Optical conductivity at finite temperature: So far our discussion has centered on $T = 0$ properties. We now study the finite-T optical conductivity. The most reliable statements can be made in the regime $k_B T \ll \hbar\omega$ corresponding to the response at temperatures much lower than the measurement frequency, where one obtains the nontrivial expansion [33]

$$\frac{\sigma(\omega,T)}{e^2/\hbar} = \sigma_\infty + b\left(\frac{ik_BT}{\hbar\omega}\right)^{3-1/\nu} + b'\left(\frac{ik_BT}{\hbar\omega}\right)^3 + \cdots \quad (6)$$

where the dots denote higher powers of $k_BT/\hbar\omega$, corresponding to increasingly small corrections. The dimensionless real coefficients b, b' are universal properties of the QCP, and ν is the correlation-length critical exponent. The structure of (6) follows from simple physical arguments, which we now briefly review. The large frequency expansion follows from the short time expansion of the operator product $J_x(t)J_x(0)$ appearing in the Kubo formula for the conductivity. As $t \to 0$, one can replace the product by a series involving operators of increasing scaling dimensions [33], called the operator product expansion (OPE). The operators that dominate the expansion are the identity, the "mass" operator $|\phi|^2$ that tunes the system to the QCP in the Landau-Ginzburg Lagrangian (1), and the stress tensor $T_{\mu\nu}$. We can thus schematically write $J_xJ_x \sim 1 + |\phi|^2 + T_{\mu\nu} + \cdots$. The coefficients that multiply each operator in the series, omitted in this schematic expansion, are called OPE coefficients. The parameters b, b' are proportional to the OPE coefficients multiplying $|\phi|^2$ and $T_{\mu\nu}$, respectively. The corresponding powers of $k_BT/\hbar\omega$ in (6) are the scaling dimensions of these operators. The dimension of $|\phi|^2$ is $\Delta_r = 3 - 1/\nu$, where the correlation length exponent ν can be estimated via the ϵ expansion, $\nu \approx 0.75$ [11]. A more accurate result is given by the conformal bootstrap, which predicts $\Delta_r = 1.9098(20)$ [48]. In contrast, the stress tensor is conserved and its scaling dimension is not renormalized: $\Delta_T = 3$.

Turning to the coefficients in Eq. (6), SUSY does not impose any constraints on b. However, in the case of b' SUSY leads to the strong result:

$$b' = 0\,. \quad (7)$$

To understand this result, recall that $b' \propto \gamma$, where γ is an OPE coefficient multiplying the stress tensor. This latter coefficient can be determined from the three-point correlation function $\langle T_{\mu\nu}J_\lambda J_\rho\rangle$ at zero temperature [33]. To see if SUSY constrains γ, we use a recent result for the general form of the three-point correlation function $\langle \mathcal{J}_\nu\mathcal{J}_\lambda\mathcal{J}_\rho\rangle$ of the supercurrent [18]. While the precise form of this function is fairly complicated, its crucial feature is that it is characterized by a single overall constant, analogously to the two-point correlation function of the supercurrent. By extracting the $\langle TJJ\rangle$ component of the three-point correlation function of the supercurrent, we find that γ and thus b' vanish identically [20]. As shown in Table I, this is not the case at the superconductor-insulator QCP of Cooper pairs [33], as expected in the absence of emergent SUSY.

Sum rules: From the point of view of the frequency dependence, the finite-temperature results we have given so far for the optical conductivity correspond to the high-frequency regime $\hbar\omega \gg k_BT$. In fact, we have sufficient information about the QCP to go even further and constrain the integral of the finite-temperature optical conductivity over *all* frequencies by way of a sum rule [33, 49, 50]:

$$\int_0^\infty d\omega\,\left(\mathrm{Re}\,\sigma(\omega,T) - \sigma_\infty e^2/\hbar\right) = 0\,. \quad (8)$$

A dual sum rule obtained by replacing σ with $1/\sigma$ also holds [50]. The key point is that the integrand must decay sufficiently fast at high frequencies. This is the case here, since in that limit the integrand scales as $(T/\omega)^{3-1/\nu}$ [Eq. (6)], and we know that $\nu > 1/2$ [48].

Experimental realizations: Recent experiments suggest that intrinsic (as opposed to proximity-induced) superconductivity may have been observed on the surface of the 3D topological insulator Sb_2Te_3 [7]. Scanning tunneling microscopy data suggests an inhomogeneous distribution of local critical temperatures $T_c(\boldsymbol{r})$ as high as 60 K, with global phase coherence achieved only at a much lower $\sim$ 9 K. The QCP discussed here remains stable against quenched disorder in T_c, assuming it is short-ranged, only if the Harris criterion $\nu d > 2$ is satisfied, where $d = 2$ is the spatial dimension and ν is the correlation length exponent of the clean QCP [51]. Using the conformal bootstrap result quoted earlier, one obtains $\nu \approx 0.917$, implying that the QCP is compromised by this type of disorder. Signatures of the clean QCP will nevertheless be observable above the crossover temperature $k_BT^* \sim \Lambda W^{1/(2/\nu-d)} \sim \Lambda W^{5.5}$ where Λ is a high-energy cutoff that can be taken as the bulk gap of the topological insulator and W is some dimensionless measure of the disorder strength [52]. Given the high power of W, one expects that the $\hbar\omega \gg k_BT > k_BT^*$ regime—in which the results discussed here hold—will be reachable in the near future in samples with moderate amounts of disorder.

Discussion & outlook: We have analyzed the dynamical response properties of a strongly interacting QCP occurring on the surface of a 3D topological insulator between the gapless Dirac surface state and a gapped surface superconductor. The emergence of SUSY in the low-energy limit at this QCP allowed us to deduce exact results for the dynamical response of the system in closed form, as summarized in Table I. We found that the zero-temperature optical conductivity and dynamical shear viscosity coefficient are frequency-independent, proportional to each other, and given by a simple irrational number, Eq. (4). We further made exact statements concerning the finite-temperature optical conductivity, including high-frequency asymptotics and sum rules. It is natural to ask if other properties of this QCP can be deduced from SUSY, such as the entanglement Rényi entropies of corners [53]. More broadly, it would be worthwhile to investigate other QCPs with emergent SUSY in both two and three spatial dimensions.

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