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A Constraint on Defect and Boundary Renormalization Group Flows

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A conformal field theory (CFT) in dimension $d \geq 3$ coupled to a planar, two-dimensional, conformal defect is characterized in part by a "central charge" b that multiplies the Euler density in the defect's Weyl anomaly. For defect renormalization group flows, under which the bulk remains critical, we use reflection positivity to show that b must decrease or remain constant from ultraviolet to infrared. Our result applies also to a CFT in $d = 3$ flat space with a planar boundary.

Introduction. Monotonicity theorems, such as Zamolodchikov's c-theorem [1], are of fundamental importance in quantum field theory (QFT). They make precise the intuition that the number of degrees of freedom (DOF) should decrease under renormalization group (RG) flow. They therefore place stringent constraints on the low-energy physics of QFTs. For example, they can eliminate the possibility of RG limit cycles, and can eliminate potential low-energy dualities between QFTs (see *e.g.* [2]).

An ideal monotonicity theorem consists of six constraints on an observable X, treated as a function over the space of couplings in the QFT:

1. The value of X at the ultra-violet (UV) fixed point is greater than or equal to its value at the infra-red (IR) fixed point: $X_{\mathrm{UV}} \geq X_{\mathrm{IR}}$ (the "weak" form);
2. X strictly decreases or remains constant along the RG flow (the "strong form");
3. X decreases along a gradient along the RG flow ("strongest form");
4. X is stationary at fixed points (and nowhere else);
5. X is bounded from below;
6. X counts only non-topological DOF.

These are listed roughly in decreasing order of importance: 1 is essential, 2 and 3 are highly desirable, and 4 through 6 are appealing but expendable. Obviously 3 implies 2, and 2 implies 1. While 1, 5 and 6 can be deduced from fixed points alone, 2, 3, and 4 require an "X-function" defined everywhere along the RG flow.

Ideally, the derivation of a monotonicity theorem should be non-perturbative, relying only on generic properties of a "healthy" QFT. To date, the standard assumptions are that the QFT is renormalizable, local, and for Lorentzian QFTs, Poincaré-invariant and unitary, or for Euclidean QFTs, Euclidean-invariant and reflection-positive. The only other, more restrictive, assumption is that RG fixed points are conformal field theories (CFTs).

The gold standard remains Zamolodchikov's c-theorem, for QFTs in dimension $d = 2$ [1]. Zamolodchikov identified X as a particular linear combination of two-point functions of the stress tensor and its trace, called the "c-function," which at fixed points reduces to the central charge c. Zamolodchikov established constraints 2 and 4 using the assumptions above, and constraint 3 within conformal perturbation theory to second order, while reflection positivity implies 5 and c's definition implies 6.

Zamolodchikov's arguments rely crucially on the special form of the stress tensor two-point function in $d = 2$, and are thus difficult to generalize to $d > 2$. Moreover, for a CFT in $d = 2$, a single number, c, fixes the Virasoro algebra, Weyl anomaly, thermal entropy, and more. The same is not true for CFTs in $d > 2$, raising the question of which X to target for a proof.

For even $d > 2$, Cardy targeted a, the coefficient of the Euler density in the Weyl anomaly [3]. By definition, a satisfies constraint 6. In $d = 4$, positivity of energy flux at spatial infinity [4] implies that a satisfies constraint 5. Moreover, in $d = 4$ Jack and Osborn established a strong a-theorem valid to all orders in perturbation theory [5**?** **?**], although their method, based on local Weyl consistency, is difficult to generalize to $d > 4$ [**?**]. Komargodski and Schwimmer provided a non-perturbative argument for the weak form in $d = 4$, $a_{\mathrm{UV}} \geq a_{\mathrm{IR}}$ [6, 7] (see also [8]). Their method, which uses an external scalar field to match UV and IR Weyl anomalies [9], is also difficult to generalize to $d > 4$ [10]. Evidence for an a-theorem in $d = 6$ appears in [11–13].

No Weyl anomaly exists in odd d, making these cases more challenging still. To date, the leading candidate for X is the sphere "free energy" $F \equiv (-1)^{(d-1)/2} \ln Z_{\mathbb{S}^d}$, with $Z_{\mathbb{S}^d}$ the renormalized partition function of a Euclidean CFT on a sphere, $\mathbb{S}^d$ [14, 15]. In $d = 1$, $\mathbb{S}^1$ is the "thermal circle," so F is minus the thermal free energy. Positivity of the heat capacity then immediately implies a strong F-theorem. In $d = 3$, $F \neq 0$ in pure Chern-Simons theory [16], manifestly violating constraint 6. Using a relation between F and disk entanglement entropy (EE) at fixed points [17], Casini and Huerta established a strong F-theorem using strong subadditivity of EE [18]. However, their F-function violates constraint 4 [19]. An alternative is mutual information, which obeys constraint 2 and possibly 5 and 6, but violates 4 [20]. For discussions about F-theorems in $d > 3$, see for example [21, 22].

Another class of monotonicity theorems concern DOF at a boundary or defect. For example, consider a boundary CFT (BCFT), *i.e.* a CFT on a space with a boundary, with conformally-invariant boundary conditions (BC). Under a boundary RG flow, triggered by a relevant operator at the boundary, the bulk remains critical, and the IR fixed point is again a BCFT. For such RG flows in $d = 2$, Affleck and Ludwig proposed a monotonicity theorem for $\ln g \equiv -\ln Z_{\mathbb{HS}^2} + \frac{1}{2}\ln Z_{\mathbb{S}^2}$, with $Z_{\mathbb{HS}^2}$ the BCFT partition function on a hemisphere, $\mathbb{HS}^2$ [23]. Friedan and Konechny established a strongest g-theorem using thermodynamic entropy [24]. The g-theorem applies also to point-like defects, via the folding trick. Conjectures for g-theorems in $d > 2$ appear in [25–28].

In this Letter we establish a weak g-theorem for Euclidean BCFTs in $d = 3$, and for Euclidean defect CFTs (DCFTs) in $d \geq 3$ with a two-dimensional planar defect. Our X is b, the coefficient of the Euler density in the boundary or defect Weyl anomaly. Using the standard assumptions above, we establish $b_{\mathrm{UV}} \geq b_{\mathrm{IR}}$ for boundary or defect RG flows. Our argument is an adaptation of Komargodski's argument for the weak c-theorem [7]. Ultimately, our "b-theorem" is equivalent to the conjectures of [25–27] for two-dimensional defects or boundaries.

The Systems. We begin with local, reflection-positive, parity-invariant Euclidean CFTs in $d \geq 3$. Ultimately we are interested in these CFTs in flat space, but to study their Weyl anomalies we will put them in curved space, unless stated otherwise. We thus introduce an external metric $g_{\mu\nu}$[29]. The CFT's generating functional of renormalized, connected correlators, $W \equiv -\ln Z[g_{\mu\nu}]$, with $Z[g_{\mu\nu}]$ the renormalized partition function, is invariant under coordinate reparameterizations and Weyl transformations, $g_{\mu\nu} \to e^{2\omega} g_{\mu\nu}$ (with ω a real function of space), up to the Weyl anomaly. These invariances imply that the flat-space theory is invariant under the action of the conformal algebra, $\mathfrak{so}(d+1,1)$, generated by infinitesimal rotations, translations, dilatations, and special conformal transformations.

Next we introduce a two-dimensional defect. For example, we can impose BC on CFT fields along a two-dimensional subspace, or introduce fields localized there, with or without couplings to the bulk CFT. Although ultimately we are interested in flat-space CFTs with planar defects, we will put them in curved space, and keep the defect's position arbitrary, unless stated otherwise. We assume the defect preserves locality, reflection positivity [30], parity, and reparameterization and Weyl invariances, up to a Weyl anomaly. The resulting theory is a DCFT. Reparameterization and Weyl invariance imply that the flat-space DCFTs are invariant under the action of the $\mathfrak{so}(d-1,1) \times \mathfrak{so}(d-2)$ subalgebra of $\mathfrak{so}(d+1,1)$ that preserves the planar defect.

Another option, special to $d = 3$, is that the bulk CFT changes across the defect. Indeed, a BCFT can be viewed as a DCFT with an "empty" CFT on one side of the defect. Our results will thus apply to BCFTs, but we will only explicitly discuss DCFTs, unless stated otherwise.

We are interested in defect RG flows in flat space, meaning flows triggered by a relevant operator at the defect, whose endpoints are flat-space DCFTs with planar defects. For example, consider a DCFT described by a local Lagrangian $\mathcal{L}_{\mathrm{DCFT}} = \mathcal{L}_{\mathrm{CFT}} + \delta^{d-2}\mathcal{L}_{\mathrm{defect}}$, with $\mathcal{L}_{\mathrm{CFT}}$ the bulk CFT's Lagrangian, δ^{d-2} a Dirac delta function which restricts to the defect, and $\mathcal{L}_{\mathrm{defect}}$ representing all defect terms. We trigger a defect RG flow by deforming $\mathcal{L}_{\mathrm{DCFT}} \to \mathcal{L}_{\mathrm{DCFT}} + \delta^{d-2}\lambda\,\mathcal{O}$, with $\mathcal{O}$ a dimension $\Delta_{\mathrm{UV}} < 2$ parity-invariant scalar operator, and λ a dimensionful coupling constant. Such an $\mathcal{O}$ may be built out of defect fields alone, bulk operators evaluated at the defect, or both. For example, we can give masses to defect fields, or change the BC on bulk fields.

Returning to curved space and defects of arbitrary position, let x^μ and σ^a $(a = 1, 2)$ be bulk and defect coordinates. Embedding functions $X^\mu(\sigma^a)$ then describe the defect's position. The defect's induced metric, $\hat{g}_{ab} \equiv g_{\mu\nu}\partial_a X^\mu \partial_b X^\nu$, describes the defect's intrinsic curvature. The bulk covariant derivative, ∇_μ, induces a defect covariant derivative, $\hat{\nabla}_a$. The second fundamental form, $\mathrm{II}^\mu_{ab} \equiv \hat{\nabla}_a \partial_b X^\mu$, describes the defect's extrinsic curvature. More details about the defect's geometry appear in the Supplement.

A key ingredient for us will be the stress tensor, $T^{\mu\nu}$. We define renormalized, connected correlators of $T^{\mu\nu}$, and the "displacement operator," D_μ, as follows. The renormalized partition function Z is a functional of $g_{\mu\nu}$, X^μ, and the set of all marginal or relevant couplings, $\{\lambda\}$. We define one-point functions $\langle T^{\mu\nu}\rangle$ and $\langle D_\mu\rangle$ from the variation of $W \equiv -\ln Z[g_{\mu\nu}, X^\mu, \{\lambda\}]$ with respect to $g_{\mu\nu}$ and X^μ, respectively:

$$\delta W = -\frac{1}{2}\int d^d x\sqrt{g}\,\delta g_{\mu\nu}\langle T^{\mu\nu}_{\mathrm{b}}\rangle - \int d^2\sigma\sqrt{\hat{g}}\left[\frac{1}{2}\delta g_{\mu\nu}\langle T^{\mu\nu}_{\mathrm{d}}\rangle + \delta X^\mu \langle D_\mu\rangle + \ldots\right], \tag{1}$$

where g and $\hat{g}$ are the determinants of $g_{\mu\nu}$ and $\hat{g}_{ab}$, respectively, and $\ldots$ indicates possible terms involving derivatives of $\delta g_{\mu\nu}$ normal to the defect. Re-writing the defect's volume as $\int d^2\sigma\sqrt{\hat{g}} = \int d^d x\sqrt{g}\,\delta^{d-2}$, we see from (1) that $\langle T^{\mu\nu}\rangle$ receives distinct bulk and defect contributions (hence the subscripts),

$$\langle T^{\mu\nu}\rangle = \langle T^{\mu\nu}_{\mathrm{b}}\rangle + \delta^{d-2}\langle T^{\mu\nu}_{\mathrm{d}}\rangle + \ldots, \tag{2}$$

where $\ldots$ indicates terms involving normal derivatives of δ^{d-2}, coming from the $\ldots$ in (1). Higher-order variations of W give higher-point correlators, in the usual way.

Reparameterization invariance leads to Ward identities relating $\langle T^{\mu\nu}\rangle$ and $\langle D_\mu\rangle$, which we present in the Supplement (specifically (B6)). However, we only need one fact about the reparameterization Ward identities: the defect stress tensor $T_{\rm d}^{\mu\nu}$ is not conserved. Energy and momentum can flow between bulk and defect, violating conservation of $T_{\rm d}^{\mu\nu}$. As a result, we cannot simply copy Zamolodchikov's derivation of the c-theorem, which relies crucially on conservation of the two-dimensional stress tensor. That is why we turn instead to Komargodski and Schwimmer's method [6, 7], based on Weyl anomaly matching [9].

Weyl Anomaly. CFTs are Weyl-invariant only up to a potential anomaly. That is, W may change under an infinitesimal Weyl variation, $\delta_\omega g_{\mu\nu} = 2\omega g_{\mu\nu}$, $\delta_\omega X^\mu = 0$:

$$\delta_\omega W = -\int d^d x \sqrt{g}\,\omega\,\mathcal{A}, \tag{3}$$

where the local function $\mathcal{A}$ is built out of external fields, such as $g_{\mu\nu}$. Indeed, we will only consider contributions to $\mathcal{A}$ built from $g_{\mu\nu}$ alone. Comparing (3) with (1) leads to the Weyl Ward identity, $\langle T^\mu{}_\mu\rangle = \mathcal{A}$. The general form of $\mathcal{A}$ can be determined by solving the Wess-Zumino (WZ) consistency condition [31], which comes from demanding that two successive Weyl transformations of W commute (the Weyl group is Abelian). For a CFT in even d, solving the WZ consistency condition gives [32]

$$\mathcal{A} = (-1)^{\frac{d}{2}+1}\frac{4a}{d!\,\mathrm{vol}(\mathbb{S}^d)}E_d + \sum_I c_I W_I, \tag{4}$$

with E_d the Euler density and the W_I the Weyl-covariant scalars of weight $-d$. WZ consistency allows total derivatives in (4), which we eliminated using local counterterms. WZ consistency leaves undetermined the "central charges" a and the c_I. For odd d, $\mathcal{A} = 0$ [32].

In a DCFT, $\mathcal{A}$ receives distinct bulk and defect contributions, $\mathcal{A} = \mathcal{A}_{\rm b} + \delta^{d-2}\mathcal{A}_{\rm d}$, where the bulk term $\mathcal{A}_{\rm b}$ takes the form for $\mathcal{A}$ in a CFT, described above. To our knowledge, for the defect term, $\mathcal{A}_{\rm d}$, the WZ consistency condition has been solved in only two cases: for a point-like defect in $d = 2$ [33] and for a two-dimensional defect in $d \geq 3$ [34, 35] (sometimes called the "Graham-Witten" anomaly [36]). We require the latter, which is, using local counterterms to cancel normal derivative terms [34, 35],

$$\mathcal{A}_{\rm d} = \frac{1}{24\pi}\left(b\,\hat{R} + d_1\,\mathring{\mathrm{I\!I}}^\mu_{ab}\mathring{\mathrm{I\!I}}^{ab}_\mu + d_2\,W_{abcd}\hat{g}^{ac}\hat{g}^{bd}\right), \tag{5}$$

with $\hat{R}$ the Ricci scalar of $\hat{g}_{ab}$, $\mathring{\mathrm{I\!I}}^\mu_{ab}$ the traceless part of $\mathrm{I\!I}^\mu_{ab}$, and W_{abcd} the pullback of the bulk Weyl tensor. WZ consistency leaves undetermined the "defect central charges" b, d_1, and d_2. (The Weyl tensor vanishes identically in $d = 3$, so d_2 exists only for $d \geq 4$.)

Under a Weyl transformation, $\sqrt{\hat{g}}\,\hat{R}$ transforms by a total derivative (that term is type A in the classification of [32]), while $\sqrt{\hat{g}}\,\mathring{\mathrm{I\!I}}^\mu_{ab}\mathring{\mathrm{I\!I}}^{ab}_\mu$ and $\sqrt{\hat{g}}\,W_{ab}{}^{ab}$ are each Weyl-invariant (type B). Our b is thus analogous to a, which obeys the c- or a-theorem in $d = 2$ or 4, respectively, while d_1 and d_2 are analogous to the c_I.

Monotonicity of b. We will now argue that $b_{\rm UV} \geq b_{\rm IR}$ for defect RG flows, using Komargodski and Schwimmer's method [6]. In particular, we will closely follow Komargodski's argument for the weak c-theorem [7].

Explicit breaking of Weyl invariance implies $\langle T^\mu{}_\mu\rangle \neq \mathcal{A}$. In flat space with a planar defect, $\mathcal{A} = 0$, so explicit breaking of Weyl invariance implies $\langle T^\mu{}_\mu\rangle \neq 0$. For a defect RG flow, that occurs only at the defect: $\langle (T_{\rm d})^\mu{}_\mu\rangle \neq 0$, while $\langle (T_{\rm b})^\mu{}_\mu\rangle = 0$, up to contact terms at the defect [24]. In curved space of even d and/or for a curved defect, generically $\mathcal{A} \neq 0$. In that case, for a defect RG flow, explicit breaking of Weyl invariance only at the defect may lead to different defect central charges in the UV and IR, while bulk central charges will remain unchanged: $\mathcal{A}_{\rm d}^{\rm UV} \neq \mathcal{A}_{\rm d}^{\rm IR}$ while $\mathcal{A}_{\rm b}^{\rm UV} = \mathcal{A}_{\rm b}^{\rm IR}$.

However, we can undo explicit breaking of Weyl invariance by treating every relevant coupling λ as a "spurion." That is, we promote λ to a function of defect coordinates, $\lambda \to \lambda(\sigma^a)$, and then endow $\lambda(\sigma^a)$ with a non-trivial Weyl transformation to restore Weyl invariance, up to the anomaly, leading to a modified Weyl Ward identity. Concretely, for a DCFT with a Lagrangian deformed as $\mathcal{L}_{\rm DCFT} \to \mathcal{L}_{\rm DCFT} + \delta^{d-2}\lambda\mathcal{O}$, as described above, we take $\lambda \to \lambda(\sigma^a)$, and under $g_{\mu\nu} \to e^{2\omega}g_{\mu\nu}$ we demand $\lambda(\sigma^a) \to e^{(\Delta_{\rm UV}-2)\omega}\lambda(\sigma^a)$. Following [6, 7], we will implement such a spurionic Weyl invariance using a non-dynamical, external scalar field, τ. Specifically, we re-define $\lambda \to \lambda' e^{(\Delta_{\rm UV}-2)\tau}$, and under $g_{\mu\nu} \to e^{2\omega}g_{\mu\nu}$ we demand $\tau \to \tau + \omega$ and $\lambda' \to \lambda'$.

The renormalized partition function Z is now a functional of $g_{\mu\nu}$, X^μ, and τ, as well as the set of couplings $\{\lambda'\}$. We define $\mathcal{T}$ as the operator conjugate to τ,

$$\langle \mathcal{T}\rangle \equiv \frac{1}{\sqrt{\hat{g}}}\frac{\delta W}{\delta\tau}. \tag{6}$$

Under an infinitesimal Weyl variation, δW takes the form in (1), with $\delta_\omega g_{\mu\nu} = 2\omega g_{\mu\nu}$, $\delta_\omega X^\mu = 0$, and now an "extra" term $\int d^2\sigma\sqrt{\hat{g}}\langle\mathcal{T}\rangle\delta\tau$ with $\delta\tau = \omega$. From (3) we thus find

$$\langle T^\mu{}_\mu\rangle - \delta^{d-2}\langle\mathcal{T}\rangle = \mathcal{A}, \tag{7}$$

so that the Weyl Ward identity is unmodified in the bulk, $\langle (T_{\rm b})^\mu{}_\mu\rangle = \mathcal{A}_{\rm b}$, but modified at the defect. In flat space with a planar defect, where $\mathcal{A} = 0$, (7) says that $\langle\mathcal{T}\rangle$ cancels $\langle (T_{\rm d})^\mu{}_\mu\rangle \neq 0$ and any contact terms in $\langle (T_{\rm b})^\mu{}_\mu\rangle$, and thus restores Weyl invariance, as advertised (τ is a "conformal compensator"). In curved space of even d and/or with a curved defect, where generically $\mathcal{A} \neq 0$, (7) says that $\langle\mathcal{T}\rangle$ acts to maintain $\mathcal{A}$'s UV value at all scales, including in particular the value at the defect. In other words, τ must account for the difference $\mathcal{A}_{\rm d}^{\rm UV} - \mathcal{A}_{\rm d}^{\rm IR} \neq 0$. This is Weyl anomaly matching [9].

In flat space with a planar defect, the result $\langle T^\mu{}_\mu\rangle = \delta^{d-2}\langle\mathcal{T}\rangle$ shows that τ becomes conjugate to $(T_{\rm d})^\mu{}_\mu$ plus contact terms in $(T_{\rm b})^\mu{}_\mu$. As a result, $\langle\mathcal{T}(\sigma)\mathcal{T}(0)\rangle$ has the same long-distance behavior as the two-point function of $T^\mu_{\ \mu}$ in a $d=2$ flat-space QFT,

$$\langle \mathcal{T}(\sigma)\mathcal{T}(0)\rangle \propto \frac{1}{|\sigma|^{2\Delta_{\rm IR}}}, \qquad (8)$$

where $\Delta_{\rm IR} > 2$ is the dimension of the leading irrelevant deformation at the defect of the IR DCFT. The defect's planar symmetry and (8) together imply that the most general form for $\langle \mathcal{T}(\sigma)\mathcal{T}(0)\rangle$'s Fourier transform is, for small momentum k along the defect,

$$\langle \mathcal{T}(k)\mathcal{T}(-k)\rangle = \alpha_0 + \alpha_2 k^2 + \mathcal{O}(k^{2\Delta_{\rm IR}-2}), \qquad (9)$$

where α_0 and α_2 are constants that can depend on $\{\lambda'\}$, and the $\mathcal{O}(k^{2\Delta_{\rm IR}-2})$ terms arise from (8). For small k the "soft" $\mathcal{O}(k^{2\Delta_{\rm IR}-2})$ terms are sub-leading compared to the contact terms α_1 and $\alpha_2 k^2$, because $2\Delta_{\rm IR}-2>2$. Similar statements apply for higher-point correlators of $\mathcal{T}$ with itself and with $T^{\mu\nu}_{\rm d}$. The IR DCFT's effects on $\mathcal{T}$'s correlators are thus suppressed at small k, or equivalently, in the IR τ decouples from the IR DCFT.

That decoupling will persist to $g_{\mu\nu}\neq\delta_{\mu\nu}$, and will be explicit in the low-energy Wilsonian effective action:

$$S_{\rm eff} = S^{\rm IR}_{\rm DCFT} + S_\tau + \mathcal{O}(\partial^{2\Delta_{\rm IR}-2}), \qquad (10)$$

where $S^{\rm IR}_{\rm DCFT}$ is the IR DCFT's effective action, S_τ is τ's effective action, up to two derivatives, and $\mathcal{O}(\partial^{2\Delta_{\rm IR}-2})$ represents τ's soft couplings to the IR DCFT. All terms in (10) are functionals of $g_{\mu\nu}$, X^μ, and τ, except $S^{\rm IR}_{\rm DCFT}$, which does not depend on τ because of the decoupling.

Since τ has support only at the defect, S_τ consists of terms only at the defect. Under an infinitesimal Weyl variation, $\delta_\omega S^{\rm IR}_{\rm DCFT}$ produces the IR Weyl anomaly, $\mathcal{A}^{\rm IR} = \mathcal{A}^{\rm UV}_{\rm b} + \mathcal{A}^{\rm IR}_{\rm d}$, so for Weyl anomaly matching S_τ must include WZ terms, $S_{\rm WZ}$, such that $\delta_\omega S_{\rm WZ}$ produces $\mathcal{A}^{\rm UV}_{\rm d} - \mathcal{A}^{\rm IR}_{\rm d}$. Together with locality and reparameterization invariance, that fixes S_τ's form (superscripts count derivatives of τ) [7]:

$$S_\tau \equiv S^{(0)} + S^{(0)}_{\rm WZ} + S^{(2)}_{\rm WZ}, \qquad (11)$$

$$S^{(0)} \equiv \int d^2\sigma\sqrt{\hat{g}}\left\{-\frac{\beta_0}{4}e^{-2\tau} + \beta_1\hat{R} + \beta_2\mathring{\rm II}^2 + \beta_3 W_{ab}{}^{ab}\right\},$$

$$S^{(0)}_{\rm WZ} \equiv -\frac{1}{24\pi}\int d^2\sigma\sqrt{\hat{g}}\,\tau\left\{\Delta b\hat{R} + \Delta d_1\mathring{\rm II}^2 + \Delta d_2 W_{ab}{}^{ab}\right\},$$

$$S^{(2)}_{\rm WZ} \equiv \frac{\Delta b}{24\pi}\int d^2\sigma\sqrt{\hat{g}}\,\partial_a\tau\partial^a\tau,$$

where $\beta_0,\ldots,\beta_3$ are constants that can depend on $\{\lambda'\}$, while $\Delta b \equiv b_{\rm UV} - b_{\rm IR}$, and similarly for Δd_1 and Δd_2.

In (11), if we set $g_{\mu\nu}=\delta_{\mu\nu}$, Fourier transform, compute $\langle \mathcal{T}(k)\mathcal{T}(-k)\rangle$, and compare to (9), then we find $\beta_0=\alpha_0$ and $\Delta b = -12\pi\alpha_2$. The latter result provides a flat-space definition of Δb, and after a Fourier transform back to position space, implies a sum rule [7]

$$b_{\rm UV} - b_{\rm IR} = 3\pi\int d^2\sigma|\sigma|^2\langle \mathcal{T}(\sigma)\mathcal{T}(0)\rangle. \qquad (12)$$

The integral in (12) is finite by power counting, plus no counterterms exist that can contribute to the right-hand-side of (12). Demanding reflection positivity in (12), $\langle \mathcal{T}(\sigma)\mathcal{T}(0)\rangle \geq 0$, thus leads to our main result,

$$b_{\rm UV} \geq b_{\rm IR}. \qquad (13)$$

For a marginally relevant deformation, $\langle \mathcal{T}(\sigma)\mathcal{T}(0)\rangle$ behaves at small $|\sigma|$ as $(\ln|\sigma|)/\sigma^4$. However, the integral in (12) still converges, so again we find (13) [7, 8].

Tests. We test our result (13) in three examples.

First is the free scalar BCFT in $d=3$, with a Neumann BC. A defect mass term triggers a defect RG flow to a Dirichlet BC. In the Supplement, we compute $b=1/16$ for the Neumann BC (correcting a result of [25]) and $b=-1/16$ for the Dirichlet BC, so indeed $b_{\rm UV} > b_{\rm IR}$. The result $b<0$ for the Dirichlet BC raises the question of whether b is bounded from below (constraint 5).

Second is a DCFT deformed by a weakly relevant defect operator $\mathcal{O}$ of dimension $\Delta_{\rm UV} = 2-\varepsilon$ with $\varepsilon \ll 1$. The change in b can be computed using defect conformal perturbation theory, which involves correlation functions of $\mathcal{O}$ in the undeformed DCFT. The DCFT's $SO(3,1)$ symmetry guarantees that these correlators have the same form as those of weakly relevant scalar operators in a $d=2$ CFT, hence the calculation becomes identical to that for the change in c using ordinary conformal perturbation theory in $d=2$ [37]. Assuming the $\mathcal{O}\mathcal{O}\to\mathcal{O}$ OPE coefficient C is positive, we thus find $b_{\rm UV} - b_{\rm IR} = \frac{\varepsilon^3}{C^2} \geq 0$ [37].

Third is the $\mathcal{N}=6$ supersymmetric (SUSY), strongly-coupled $U(N)_k \times U(N)_{-k}$ Chern-Simons matter theory [38] with N and $N/k^5 \gg 1$, coupled to N_f bi-fundamental hypermultiplet flavor fields at a two-dimensional defect, preserving $\mathcal{N}=(3,3)$ SUSY, with $N_f \ll N$ [39]. That DCFT is holographically dual to $d=11$ supergravity on $d=4$ Anti-de Sitter space, AdS_4, times $\mathbb{S}^7/\mathbb{Z}_k$, with N units of four-form flux, plus N_f probe M5-branes along $AdS_3\times\mathbb{S}^3/\mathbb{Z}_k$. Graham and Witten's holographic result [36] gives $b=\frac{3}{2}NN_f$. A SUSY mass for ΔN_f of the hypermultiplets triggers a defect RG flow to the same DCFT, but now with $N_f - \Delta N_f$ hypermultiplets, hence $b_{\rm UV} > b_{\rm IR}$.

Discussion. Our result (13) can be viewed either as a higher-dimensional g-theorem, or as a generalization of the weak c-theorem to include coupling to a higher-dimensional CFT. Indeed, the g-theorem itself can be viewed as a monotonicity theorem for a $d=1$ QFT with an RG flow coupled to a $d=2$ CFT. A natural question is whether every monotonicity theorem survives coupling to a higher-dimensional CFT.

Other natural questions arise from further comparisons to existing monotonicity theorems. For example, the strong c- and F-theorems can be established using strong sub-additivity of EE [18, 20, 40, 41]. Can we establish a strong(est) b-theorem, for example using EE?

In $d = 2$, the g-theorem can be violated by a bulk RG flow [42]. Can a bulk RG flow violate the b-theorem?

Our result may have implications for many theoretical and experimental systems. One example is a graphene nanoribbon, which at low energy is described by a $d = 3$ CFT (free massless Dirac fermions) [43] on a space with a boundary. Another example is the critical Ising model in $d \geq 3$ with a planar defect, or in $d = 3$ with a boundary. Although we assumed parity invariance, our result (13) is straightforward to generalize to parity-violating theories, and hence may have implications for quantum Hall systems. More abstractly, in string and M-theory, brane intersections can give rise to various DCFTs and BCFTs in $d \geq 3$. What consequences our result may have for all of these systems deserves exploration.[44]

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