Effect of Impurities on the Josephson Current through Helical Metals: Exploiting a Neutrino Paradigm

Pouyan Ghaemi and V. P. Nair

Phys. Rev. Lett. **116**, 037001 — Published 21 January 2016

DOI: 10.1103/PhysRevLett.116.037001

Effect of impurities on the Josephson current through helical metals: Exploiting a neutrino paradigm

Pouyan Ghaemi and V. P. Nair
Physics Department, City College of the City University of New York, New York, NY 10031

In this letter we study the effect of time-reversal symmetric impurities on the Josephson supercurrent through two dimensional helical metals such as on topological insulator surface state. We show that contrary to the usual superconducting-normal metal-superconducting junctions, the suppression of supercurrent in superconducting-helical metal-superconducting junction is mainly due to fluctuations of impurities in the junctions. Our results, which is a condensed matter realization of a part of the MSW effect for neutrinos, shows that the relationship between normal state conductance and critical current of Josephson junctions is significantly modified for Josephson junctions on the surface of topological insulators. We also study the temperature-dependence of supercurrent and present a two fluid model which can explain some of recent experimental results in Josephson junctions on the edge of topological insulators.

The helical electronic states on the boundary of topological insulators (TIs)[1, 2] present unique opportunity to experimentally realize novel quantum condensed matter phenomena such as quantum anti-localization[1–3]. On the other hand, they resemble relativistic massless fermions and provide a platform to realize phenomena previously studied in high energy physics, such as axion electrodynamic [4] and supersymmetry [5, 6], in the condensed matter systems. The possibility of inducing superconductivity in surface states of TIs through proximity effect [7, 8] and theoretical predictions of presence of Majorana zero-mode [9] in Josephson junctions through TI surface states [10, 11] motivated many experimental studies of such Josephson junctions and led to many puzzling results [12–18] which were not observed before in superconducting-normal metal-superconducting (SNS) Josephson junctions [19].

Contrary to the normal metallic phase, in the SNS Josephson junction the supercurrent is carried with no applied field. It is instead generated by the variation of phase of the superconducting condensate. As a result the mechanism through which the impurities in the SNS junction affect the supercurrent is different from their effect on normal state current. An important and non-trivial result in this regard, is that the critical suppercurrent in SNS junctions is proportional to the normal state conductivity [20–23]. It has been shown that in SNS junctions the back-scattering of electron and holes in the normal region is the main mechanism affecting the suppercurrent and these processes leads to proportionality of critical current and normal state conductivity[21]. Owing to the strong correlation between spin and momentum in the helical metals, such as surface states of TIs, back-scattering by non-magnetic impurities is prohibited. As a result the effect of impurities on the supercurrent should be through other mechanism which is not explored before.

In this letter we show that non-magnetic impurities can affect the supercurrent through forward scattering processes which resemble a condensed matter version of a part of the Mikheyev-Smirnov-Wolfenstein (MSW) effect for neutrino oscillations [24–27]. There are two ingredients to MSW effect. First, through interaction of the neutrinos with matter, their phase velocity is modified and they effectively acquire a refractive index. The refractive index for electron-neutrinos is different from that for other flavors due to charged current interactions with matter. The resulting difference in the phase of the wave functions modifies the neutrino mass-eigenstates in matter, and hence the oscillations between different flavors (see supplementary materials[28]). It is crucial for the interaction of Neutrinos with matter to be time-dependent in order to modify their phase velocity. This is to some extent similar to the interaction of light with matter where the time dependence of the interaction leads to change of phase velocity of the light and deviates the index of refraction for different materials from that of the vacuum. We will show that similarly in superconducting-helical metal-superconducting Josephson junctions the forward scattering of helical electronic states by impurity potentials, which are naturally time dependent due to quantum and thermal fluctuations in the locations of the impurities, lead to the modification of the phase velocity of the Andreev states. This effective 'refractive index' for Anreev states means that the optical length of the junction (defined via the phase of the wave function) is larger than the geometric length. The modification of the phase modifies (via matching conditions) the energy eigenvalues of the Andreev states which control the magnitude of the supercurrent. The second part of MSW effec is a resonance effect which can enhance the mixing of flavors in matter, even up to the maximal mixing. For us, we have only one flavor to consider, so the situation is simpler and there is no resonance part.

Our results, in addition to presenting this novel paradigm in TIs, can be used to interpret the measurements on TI Josephson junctions which are currently the focus of many experimental studies [12–18]. This general result, which should hold in situations with electrons with strong spin-orbit coupling, impacts their contributions to physical phenomena.

The low-energy effective Hamiltonian of TI surface states reads as $\mathcal{H}_s = v_F\, \boldsymbol{\sigma}\cdot\mathbf{k}$ where v_F is the fermi velocity and $\boldsymbol{\sigma} = (\sigma_x, \sigma_y)$ are the Pauli matrices in the basis $(\psi_\uparrow, \psi_\downarrow)$, with ψ_σ being the electronic state with spin σ localized on the surface of the TI. The low-energy effective Hamiltonian describing a Josephson junction on the surface of the TI, with supercurrent along $\hat{x}$, is given by [29]

$$H = (-iv_F \boldsymbol{\nabla}\cdot\boldsymbol{\sigma} - \mu)\,\tau_3 + \Delta_R(x)\tau_1 + \Delta_I(x)\tau_2, \qquad (1)$$

which acts on the superconducting particle-hole state $\left(\psi_\uparrow, \psi_\downarrow, \psi_\downarrow^\dagger, -\psi_\uparrow^\dagger\right)^T$. Here $\Delta_R(x)$ and $\Delta_I(x)$ are real and imaginary parts, respectively, of the induced superconducting gap $\Delta = \Delta_R + i\,\Delta_I$. As for the matrix structure, σ_i act on physical spin space whereas the τ_i act on the superconducting particle-hole space. As the Hamiltonian in (1) is invariant under translation along $\hat{y}$, the momentum k_y in this direction is conserved. The low-energy Andreev states in the junction thus correspond to $k_y = 0$ and k_x close to the two fermi wave vectors, $k_x = \frac{\mu}{v_F}$ for $\sigma_x = 1$ and $k_x = -\frac{\mu}{v_F}$ for $\sigma_x = -1$. Notice that since σ_x commutes with Hamiltonian (1) we can decouple the low-energy effective Hamiltonians into two independent sectors corresponding to electron-hole states close to right or left fermi points. Here we will focus on one of the effective Hamiltonians, but the other independent one can be similarly studied. Since we are aiming for the effect of temporal fluctuations it is more efficient to use the corresponding $1+1$-dimensional action

$$S = \int dxdt\ \bar{\Phi}(x,t)\left[i\tau_1 D_t + \tau_2 D_x + \tilde{M}(x)\right]\Phi(x,t) \qquad (2)$$

where $\Phi^T(x,t) = \left(\phi_\uparrow(x,t), -\phi_\downarrow^\dagger(x,t)\right)$ and $\phi_\sigma(x,t) = \psi_\sigma(x,t)\, e^{\mp i k_F x}$, with $\sigma =\uparrow,\downarrow$, are the fermionic field operators for excitations close to the right or left Fermi points. $D_\mu = \partial_\mu - i\, e\, \tau_3 A_\mu$ is the covariant derivative. Notice that (2) is of the standard form of a Dirac action $\bar{\Phi}(i\gamma^\mu D_\mu + \tilde{M})\Phi$ if we identify $\gamma^0 = \tau_1$, $\gamma^1 = -i\tau_2$, $\gamma^5 = \tau_3$. Further, in (2), $\bar{\Phi}(x,t) = \Phi^\dagger(x,t)\,\gamma^0 = \Phi^\dagger(x,t)\,\tau_1$ and $\tilde{M}(x) = \Delta_R(x)\tau_0 + i\Delta_I(x)\tau_3$. The Fermi velocity has been set to 1 by scaling x, or equivalently, the momentum k_x. The effect of charged impurities is captured by $A_0 = V(x - a(t))$ where $a(t)$ identifies instantaneous position of the impurity. As we will see below, in order to capture the effect of impurities on the supercurrent in the junction, we should consider the natural fluctuations in the position of the impurity. For small fluctuations $a(t) = a_0 + \xi(t)$, the impurity potential reads as $V(x - a(t)) \approx V(x - a_0) + \partial_x V(x - a_0)\,\xi(t)$. As we will show below, the impurities can only affect the supercurrent as a result of their temporal fluctuations. We would like to note that such treatment of impurities and its effect on superconductivity in normal metals have been considered long before [30]. But here we show that as a result of helical band structure of the surface states of the TIs, the temporal fluctuations are the sole mechanism through which impurities can affect the supercurrent. The action, including the effect of fluctuating impurities, is $S = S_0 + S_{int} + S_{osc}$ with

$$S_0 = \int dxdt\Big[\bar{\Phi}(x,t)\left(i\tau_1\partial_t + \tau_2\partial_x + \tilde{M}(x)\right)\Phi(x,t) - V(x-a_0)\,\bar{\Phi}(x,t)\, i\tau_2\Phi(x,t)\Big]$$

$$S_{int} = \int dxdt\ \partial_x V(x-a_0)\ \bar{\Phi}(x,t)\, i\tau_2\Phi(x,t)\ \xi(t)$$

$$S_{osc} = \frac{1}{2} M_I \int dt\ \bar{\xi}(t)\left(-\frac{\partial^2}{\partial t^2} - \omega^2\right)\xi(t) \qquad (3)$$

Since we are interested in localized impurities, both the impurity potential $V(x-a)$ and the resulting electric field $E(x) = \partial_x V(x-a)$ are localized in space and the dynamics of the impurities are captured by the harmonic oscillator action S_{osc} (M_I and ω are the mass and the harmonic oscillation frequency of the impurities). The electron-impurity coupling is given by S_{int}; it generates a self-energy correction in S_0 (see supplementary materials), leading to an effective action of the form

$$\bar{S} = \int dxdt\ \Big[(1+\Sigma_1(x))\,\bar{\Phi}\left(i\tau_1\partial_t + \tilde{M}(x)\right)\Phi + \bar{\Phi}\,\tau_2\partial_x\Phi + \bar{\Phi}\,\tilde{M}(x)\Sigma_2(x)\,\Phi\Big] + \int dxdt\, V(x-a_0)\,\bar{\Phi}\, i\tau_2\,\Phi \qquad (4)$$

To the lowest nontrivial order in perturbation theory, the self-energies can be calculated as $\Sigma_1(x) \approx \frac{e^2 E^2(x)}{2\pi\omega^2 M_I} R$ and $\Sigma_2(x) \approx \frac{e^2 E^2(x)}{2\pi\omega^2 M_I} R\left[\log(\frac{2\omega}{M_I}) - 1\right]$ where R is the length scale over which $E(x) = -\partial_x V(x-a)$ is non-zero (see supplementary materials). What is important for us is not so much the specific formulae for these self-energies, but that the general form of the effective action is as given in (4), with the self-energies as corrections concentrated around the impurities.

It is important to note that the modification due to Σ_1 does not affect the spatial derivative term for the electrons. This is because the oscillator variable $\xi(t)$ does not have a spatial dependence. Thus although the free fermion action $\bar{\Phi}(i\gamma^\mu\partial_\mu + \tilde{M})\Phi$ has a Lorentz-type symmetry (albeit with v_F in place of the speed of light c) the interactions with impurities, and hence corrections, do not respect this symmetry. As a result the temporal and spatial derivative terms in (4) will be renormalized differently, thus affecting the phase velocity of Andreev states. This is similar to neutrinos or light interacting with matter for which also, the time-dependence of the interactions leads to deviation of their refractive index from that of the vacuum.

The general result is that the effect of fluctuating impurities appear as renormalization of the phase velocity of Andreev states and of the size of superconducting gap in the region where the electric field of the impurity potentials are present. The change of phase velocity can be viewed as a "refractive index" for the electron resulting in an additional phase for the wave functions as the effective "optical length". This will be the essence of how the Andreev states are modified.

It is well known that for massless particles, propagating with speed c in vacuum, the primary effect of interactions is to generate a refractive index rather than a mass (which is usually forbidden for symmetry reasons). This effect is also obtained for massive particles in the ultra-relativistic limit. Our argument is that, for the surface states in a TI as well, which have a Lorentzian symmetry (with $c \to v_F$), a refractive index is precisely what we should expect as the primary effect of interactions.

The Andreev states are determined using the effective Schrödinger equation acting in two-component space of superconducting particle-hole space which results from the action in (4):

$$-\left[\left(i\,\bar{v}\,\partial_x + V(x-a_0)\right)\tau_3 + \Delta_R\tau_1 + \Delta_I\tau_2\right]\Psi(x) = E\Psi(x). \tag{5}$$

For constant Δ, the two independent eigenstates are $\Psi^{\pm}_E(x) = e^{iW^{\pm}(x)}\ \eta^{\pm}(E,\Delta)$, where

$$\eta^{+}(E,\Delta)^T = \frac{1}{\sqrt{2E}}\left[\sqrt{E+v_F\kappa}\ ,\ -\Delta/\sqrt{E+v_F\kappa}\right]$$
$$\eta^{-}(E,\Delta)^T = \frac{1}{\sqrt{2E}}\left[-\Delta^*/\sqrt{E+v_F\kappa}\ ,\ \sqrt{E+v_F\kappa}\right]$$

$W^{\pm}(x) = \int_0^x du\ \frac{V(u-a_0)\pm v_F\ \kappa}{\bar{v}(u)}$, $\kappa = \sqrt{E^2-\Delta^2}\,/v_F$ and $\bar{v}(x) = v_F/(1+\Sigma_1(x))$ is the effective phase velocity of the Andreev states .

To model the Josephson Junction, we consider the stepwise variation of $\Delta(x)$ in three regions $x < 0$ (region I), $0 < x < x_p$ (region II) and $x > x_p$ (region III) as shown in Fig. 1. The eigenstates can be expressed in each region as the superposition of $\Psi^{\pm}_E(x)$ as $\Psi_i(x) = A^i\Psi^{+}_E(x) + B^i\Psi^{-}_E(x)$ where $i =$ I, II and III, corresponding to the three regions. We define the transfer matrices $\hat{T}_B(E,\Delta) = \sqrt{\frac{1}{2}+\frac{\kappa v_F}{2E}}\begin{bmatrix} 1 & -\frac{\Delta^*}{E+\kappa v_F} \\ -\frac{\Delta}{E+\kappa v_F} & 1 \end{bmatrix}$ and $\hat{T}_n(x) = e^{i\phi_I(x)}\begin{bmatrix} e^{i\langle k\rangle_x x} & 0 \\ 0 & e^{-i\langle k\rangle_x x}\end{bmatrix}$, where $\langle k\rangle_x = \frac{E}{x}\int_0^x \frac{du}{\bar{v}(u)}$ is the averaged wave vector in the normal (TI) region of the junction and $\phi_I(x) = \int_0^x \frac{V(u-a_0)}{\bar{v}(u)}du$ is the phase resulting from the static impurity.The boundary conditions which determine the spectrum of the states in the junction are $\Psi_{\rm I}(0^-) = \Psi_{\rm II}(0^+)$ and $\Psi_{\rm II}(x_p^-) = \Psi_{\rm III}(x_p^+)$. The S-matrix for the junction must relate the incoming and outgoing states as $\begin{bmatrix} A^{\rm III} \\ B^{\rm I}\end{bmatrix} = \mathcal{S}\begin{bmatrix} A^{\rm I} \\ B^{\rm III}\end{bmatrix}$; this S-matrix can be written in terms of $\hat{T}_J(E,\Delta,\phi_0) = \hat{T}_B(E,\Delta)^{-1}\,\hat{T}_n(x_p)^{-1}\,\hat{T}_B(E,\Delta e^{i\phi_0})$ as

$$\mathcal{S} = \frac{1}{\hat{T}_J(E,\Delta,\phi_0)_{11}}\begin{bmatrix} 1 & -\hat{T}_J(E,\Delta,\phi_0)_{12} \\ \hat{T}_J(E,\Delta,\phi_0)_{21} & e^{-2i\phi_I}\end{bmatrix} \tag{6}$$

The supercurrent I in the junction can be derived using the well-known relationship between the Josephson current of the junction and the spectrum [21], namely,

$$I = I_1 + I_2 + I_3$$
$$I_1 = -\frac{e}{\hbar}\sum_n \tanh\left(E_n/2k_BT\right)\frac{dE_n}{d\phi}$$
$$I_2 = -\frac{2ek_BT}{\hbar}\int_\Delta^\infty dE\ \ \ln\left[2\cosh\left(E/2k_BT\right)\right]\frac{\partial\rho(E,\phi)}{\partial\phi}$$
$$I_3 = \frac{e}{\hbar}\frac{d}{d\phi}\int dx|\Delta(x)|^2/g$$

I_1 is the contribution from the discrete spectrum of in-gap states and I_2 is from the continuum of states with energy above the gap with density of states $\rho(E,\phi)$ for the one spin state at each Fermi point. In the third component I_3, g is the interaction constant of BCS theory. I_3 vanishes for the phase independent gap and will be ignored in this letter.

For the states with energy $E < \Delta$, the amplitude of outgoing states vanishes which give $\mathcal{S}\begin{bmatrix} A^{\rm I} \\ B^{\rm III}\end{bmatrix} = 0$. Using the explicit form of $\mathcal{S}$-matrix given in (6), we get the following equation determining the in-gap energies:

$$\cos^{-1}(E_n/\Delta) + \frac{E_n}{v}\mathcal{L} + \frac{\phi_0}{2} = n\,\pi, \qquad n\in\mathbb{Z} \tag{7}$$

where $\mathcal{L} = \int_0^{x_p} dx\ (1+\Sigma_1(x))$ is the effective length of the junction as modified by fluctuations of the impurity. This is the new "optical length" of the junction. To check if the mechanism discussed above can have a sizable impact on the critical current, we now estimate $\Sigma_1(x)$.

Assuming the local potential is due to random charge impurities in the TI, the estimated values of relevant properties are: the density of impurities $\sim 10^{19}\ cm^{-3}$

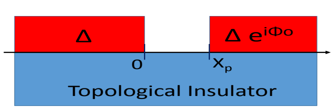

Figure 1. Superconductor-helical metal-superconductor Josephson junction. The superconductivity is induced in the regions of helical metal (TI surface states) below the superconducting electrodes (red regions) with corresponding gap size Δ and the phase 0 for $x < 0$ and ϕ_0 for $x > x_p$. Charged impurities are considered in the region $0 < x < x_p$.

[31], screening length $\sim 20\ nm$ [32], bulk dielectric constant ~ 100 [33], impurity mass $\sim 10^{-27}\ kg$ and oscillation frequency of typical localized impurities $\sim 200\ cm^{-1}$ [34]; this leads to $\Sigma_1 \approx 0.25$, implying a non-negligible impact on the critical current. The phase ϕ_I has cancelled out in (7) confirming that static impurities have no effect on the energy of in-gap Andreev states. The effect of impurities is only through their dynamical fluctuations which leads to the finite self-energy $\Sigma_1(x)$ and modifies Fermi velocity. The effect on the energy eigenvalues is most vividly illustrated by considering states with $E_n \ll \Delta$, in which case we get $E_n \approx (v/2\mathcal{L})\big[2\pi(n+\frac{1}{2}) - \phi_0\big]$. The increase in $\mathcal{L}$ implies that E_n and $\partial E_n/\partial\phi_0$ are decreased relative to the case with no impurities. More generally, defining $\Theta_n = \frac{E_n}{v}\mathcal{L} + \phi_0/2 - n\pi$, the supercurrent associated with each in-gap state reads as

$$\mathcal{I}_n(\phi_0) = -\frac{e}{\hbar}\frac{\partial E_n}{\partial \phi_0} = \frac{e\,|\Delta|}{2\hbar}\left[\frac{\sin(\Theta_n)}{1+\sin(\Theta_n)\mathcal{L}|\Delta|/v}\right] \quad (8)$$

As a function of Θ_n, this has a maximum at $\Theta_n = \pi/2$, so that the critical current is $\mathcal{I}_{crit} = (e|\Delta|/\hbar)(1+\mathcal{L}|\Delta|/v)^{-1}$. The condition $\Theta_n \approx \pi/2$ is actually obtained for modes of very low energy $E_n \ll |\Delta|$. It is important to note that by that the suppercurrent generated by in gap states decreases by increasing $\mathcal{L}$ which shows that impurities clearly affect the supercurrent.

For the states above the gap, the density of states is given by the Krein-Friedel-Lloyd formula $\rho(E) = \frac{1}{2\pi i}\frac{\partial}{\partial E}(\ln\det\mathcal{S})$ [35]. Using (6) we get

$$\det\mathcal{S} = e^{-i\phi_0}\frac{1-\beta_E^2\cos^2(\Theta_E)}{\left[\sqrt{1-\beta_E^2}\,\cos(\Theta_E) - i\sin(\Theta_E)\right]^2}$$

where $\Theta_E = \frac{E}{v}\mathcal{L} + \phi_0/2$ and $\beta_E = \frac{\Delta}{E}$. The supercurrent due to above-the-gap states then simplifies to

$$I_2 = \frac{e}{2\pi\, i\,\hbar}\left[\int_\Delta^\infty dE\ \tanh\left(E/2k_BT\right)\frac{\partial\left(\ln\det\mathcal{S}\right)}{\partial\phi_0}\right] - \frac{ek_BT}{\pi\hbar}\ln\left[2\cosh\left(\Delta/2k_BT\right)\right] \quad (9)$$

We would like to emphasize two important features of the supercurrent contribution from states with energy above the superconducting gap: 1) For low temperatures $T \ll \Delta$, I_2 is only weakly T-dependent through the temperature dependence of superconducting gap Δ. At higher temperatures, the thermal phase fluctuations can further reduce the supercurrent which is beyond the scope of our letter. 2) I_2 is also only weakly dependent on $\mathcal{L}$, i.e, only weakly sensitive to impurities.

To elucidate the second point, we first note that the second term in (9) is independent of $\mathcal{L}$. For $T \ll \Delta$, $\tanh\left(E/2k_BT\right) \sim 1$. The integrand in the first term in (9) has two types of dependence on E. One is a periodic dependence, with period $\hbar v/\mathcal{L}$ due to $\cos(\Theta_E)$, and the other is a decaying dependence, of the form Δ^2/E^2 for large E. For the effective junction length $\mathcal{L}$ larger than $\hbar v/\Delta$, the oscillatory dependence is much faster than the decay rate and so can be averaged over Θ_E. (This may be viewed as an application of the Riemann-Lebesgue lemma.) As a result, the dependence on $\mathcal{L}$ will be eliminated and I_2 will not be seriously affected by impurities even when fluctuation effects are included.

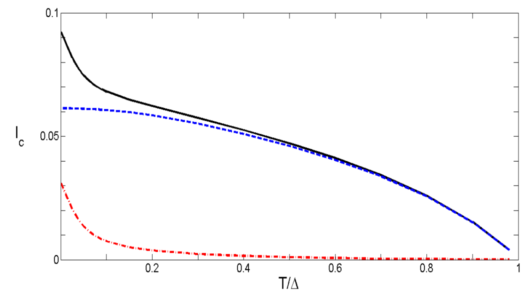

Figure 2. Temperature dependence of critical current (in units of $\frac{e\Delta}{\hbar}$). Black solid curve corresponds to total critical current, the blue dashed curve is the contribution from above-the-gap leaky Andreev states and red dashed-dotted line is the contribution from in-gap Andreev states.

In conclusion we have shown that the supercurrent in Josephson junctions with helical metals, such as on the surface of three-dimensional TIs, is affected by impurities through their temporal fluctuations. However, this applies primarily to the supercurrent generated by in-gap Andreev states. The supercurrent carried by the states above the gap will not be seriously affected by impurities. Based on our results, the supercurrent in the Josephson junctions on the surface of TIs can be interpreted as a superposition of two contributions, one which is strongly temperature-dependent and also sensitive to the impurities in the junction and one which is only weakly temperature-dependent and not sensitive to the impurities, see Fig. 2. Given new advances in controlling the level of disorder in TIs [36, 37], these results will be useful in analyzing many of the experimental results on Josephson junctions made on TIs. For example, our analysis is consistent with the experimental results in [12, 13]; whether different impurities concentration could affect the critical current in Josephson junctions on TI was the main missing ingredient in the theoretical model used to interpret those results. In fact, our work may be considered as further substantiating the interpretation, presented in [12], in terms of two types of supercurrent contributions. Similar behavior have been observed more clearly in samples with very low bulk conduction[12].

We would like to thank J. D. Sau, D. J. Van Harlingnen and A. Bernevig for helpful discussions. This work was supported in part by the U.S. National Science Foundation grant PHY-1213380 and by a PSC-CUNY award (VPN) and NSF EFRI-1542863 (PG).

[1] M. Z. Hasan and C. L. Kane, Rev. Mod. Phys. **82**, 3045 (2010).
[2] X.-L. Qi and S.-C. Zhang, Rev. Mod. Phys. **83**, 1057 (2011).
[3] M. Joel, Nature Physics **5**, 378 (2009).
[4] X.-L. Qi, T. L. Hughes, and S.-C. Zhang, Phys. Rev. B **78**, 195424 (2008).
[5] P. Ponte and S.-S. Lee, New Journsl of Physics **16**, 013044 (2014).
[6] T. Grover, D. N. Sheng, and A. Vishwanath, Science **344**, 280 (2014).
[7] T. Yilmaz, I. Pletikosić, A. P. Weber, J. T. Sadowski, G. D. Gu, A. N. Caruso, B. Sinkovic, and T. Valla, Phys. Rev. Lett. **113**, 067003 (2014).
[8] N. Bansal, Y. S. Kim, E. Edrey, M. Brahlek, Y. Horibe, K. Iida, M. Tanimura, G. Li, T. Feng, H. Lee, T. Gustafsson, E. Andrei, and S. Oh, Thin Solid Films **520**, 224 (2011).
[9] W. Frank, Nature Physics **5**, 614 (2009).
[10] L. Fu and C. L. Kane, Phys. Rev. Lett. **100**, 096407 (2008).
[11] C. T. Olund and E. Zhao, Phys. Rev. B **86**, 214515 (2012).
[12] C. Kurter, A. D. K. Finck, P. Ghaemi, Y. S. Hor, and D. J. Van Harlingen, Phys. Rev. B **90**, 014501 (2014).
[13] V. Orlyanchik, M. P. Stehno, C. D. Nugroho, P. Ghaemi, M. Brahlek, N. Koirala, S. Oh, and D. J. V. Harlingen, "Signature of a topological phase transition in the josephson supercurrent through a topological insulator," ArXiv:1309.0163.
[14] M. Veldhorst, M. Snelder, M. Hoek, T. Gang, V. K. Guduru, X. L. Wang, U. Zeitler, W. G. van der Wiel, A. A. Golubov, H. Hilgenkamp, and A. Brinkman, Nature Materials **11**, 417 (2012).
[15] A. D. K. Finck, C. Kurter, Y. S. Hor, and D. J. Van Harlingen, Phys. Rev. X **4**, 041022 (2014).
[16] B. Sacepe, J. Oostinga, J. Li, A. Ubaldini, N. Couto, E. Giannini, and A. Morpurgo, Nat. Commun. **2**, 575 (2011).
[17] S. Cho, B. Dellabetta, A. Yang, J. Schneeloch, Z. Xu, T. Valla, G. Gu, M. J. Gilbert, and N. Mason, Nat. Comm. **4**, 1689 (2013).
[18] J. R. Williams, A. J. Bestwick, P. Gallagher, S. S. Hong, Y. Cui, A. S. Bleich, J. G. Analytis, I. R. Fisher, and D. Goldhaber-Gordon, Phys. Rev. Lett. **109**, 056803 (2012).
[19] A. A. Golubov, M. Y. Kupriyanov, and E. Il'ichev, Rev. Mod. Phys. **76**, 411 (2004).
[20] I. O. Kulik, Sov, Phys. JETP **30**, 944 (1970).
[21] C. W. J. Beenakker, Phys. Rev. Lett. **67**, 3836 (1991).
[22] P. F. Bagwell, Phys. Rev. B **46**, 12573 (1992).
[23] B. J. van Wees, H. van Houten, C. W. J. Beenakker, J. G. Williamson, L. P. Kouwenhoven, D. van der Marel, and C. T. Foxon, Phys. Rev. Lett. **60**, 848 (1988).
[24] L. Wolfenstein, Phys. Rev. D **17**, 2369 (1978).
[25] S. Mikheyev and A. Smirnov, Sov. J. Nucl. Phys. **42**, 913 (1985).
[26] S. Mikheyev and A. Smirnov, Nuovo Cimento **9C**, 17 (1986).
[27] P. Langacker, J. P. Leveille, and J. Sheiman, Phys. Rev. D **27**, 1228 (1983).
[28] See Supplemental Material [url], which includes Refs. [38, 39].
[29] P. Hosur, P. Ghaemi, R. S. K. Mong, and A. Vishwanath, Phys. Rev. Lett. **107**, 097001 (2011).
[30] A. Zhernov, Y. A. Malov, and G. K. Panova, Sov, Phys. JETP **42**, 131 (1975).
[31] B. Skinner and B. I. Shklovskii, Phys. Rev. B **87**, 075454 (2013).
[32] H. Beidenkopf, P. Roushan, J. Seo, L. Gorman, I. Drozdov, Y. S. Hor, R. J. Cava, and A. Yazdani, Nature Physics **7**, 939 (2011).
[33] D. Culcer, E. H. Hwang, T. D. Stanescu, and S. Das Sarma, Phys. Rev. B **82**, 155457 (2010).
[34] A. S. Barker and A. J. Sievers, Rev. Mod. Phys. **47**, S1 (1975).
[35] W. Thirring, *Quantum Mechanics of Atoms and Molecules: A Course in Mathematical Physics*, Vol. 3 (Springer, 1979).
[36] M. Brahlek, Y. S. Kim, N. Bansal, E. Edrey, and S. Oh, Appl. Phys. Lett. **99**, 012109 (2011).
[37] A. R. Valdes, L. Wu, A. V. Stier, L. S. Bilbro, M. Brahlek, N. Bansal, S. Oh, and N. P. Armitage, J. Appl. Phys. **113**, 153702 (2013).
[38] G. Fogli *et al.*, Phys. Lett. B **583**, 149 (2004).
[39] G. Fogli and E. Lisi, New J. Phys. **6**, 139 (2004).