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Anomalous Quasiparticle Symmetries and Non-Abelian Defects on Symmetrically Gapped Surfaces of Weak Topological Insulators

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We show that boundaries of 3D weak topological insulators can become gapped by strong interactions while preserving all symmetries, leading to Abelian surface topological order. The anomalous nature of weak topological insulator surfaces manifests itself in a non-trivial action of symmetries on the quasiparticles; most strikingly, translations change the anyon types in a manner impossible in strictly 2D systems with the same symmetry. As a further consequence, screw dislocations form non-Abelian defects that trap $\mathbb{Z}_4$ parafermion zero modes.

Introduction. Electronic topological insulators [1–5] display numerous exotic properties already at the single-particle level, most famously protected surface metallicity. Much of the richness in these systems emerges from the interplay between symmetry and topology. Recently interactions among surface electrons have been found to further enlarge the possibilities. In a *strong* topological insulator (STI) the surface spectrum for weakly interacting electrons obeying time reversal and charge conservation symmetry features a single Dirac cone. Remarkably, strong interactions can fully gap the STI surface without violating symmetries [6–9] (as anticipated earlier [10]). The symmetrically gapped phases realize non-Abelian topological order and can be viewed as descending from novel gapless states [11–13]. Similar conclusions hold for bosonic topological insulators [14], topological superconductors [15], and topological crystalline insulators [16]. (Not all topological systems, however, admit a symmetric gapped boundary [17].)

We explore for the first time the fate of strongly correlated *weak* topological insulator (WTI) surfaces. A WTI may conveniently be decomposed into a stack of quantum spin Hall (QSH) insulators [1–3] with electrons from the helical edges tunneling between layers; see Fig. 1(a). Provided the system preserves time reversal $\mathcal{T}$, charge conservation, and layer translation symmetry T_y, the *non-interacting* WTI surface hosts two massless Dirac cones at distinct momenta [18]. These systems comprise ideal physically relevant [19] settings where one can controllably explore strong correlation effects that produce surface topological order. The additional symmetries present here compared to the STI surface enrich the topological order that we identify in subtle ways and yield an interesting interplay with lattice defects.

General considerations. Three considerations are useful for anticipating the topological order that emerges when interactions gap the WTI surface without violating these symmetries. First, on very general grounds the topological order must be anomalous, i.e., forbidden in strictly 2D isosymmetric systems. To see this consider the thickened torus of WTI depicted in Fig. 1(b), and gap the interior surface by interactions but leave the exterior gapless. Upon shrinking the torus's thickness a strictly 2D system emerges as in Fig. 1(c). If the gapped surface was non-anomalous, one could simply strip away the topological order, leaving a symmetric 2D system with an 'impossible' band structure [20]—a contradiction.

The second consideration regards a domain wall separating the topologically ordered state from a ferromagnetically gapped surface region. The magnetized region carries a non-zero thermal Hall conductivity and thus the domain wall must host gapless modes. In the STI case the thermal Hall conductivity would be half-integer (in units of $\pi^2 k_B^2 T/3h$), just like the electronic Hall conductivity (in units of e^2/h); this implies that the gapless mode's central charge must also be half-integer, necessitating a non-Abelian topological order. By contrast, the two Dirac cones present for the WTI imply an integer central charge, suggesting an Abelian minimal topological order.

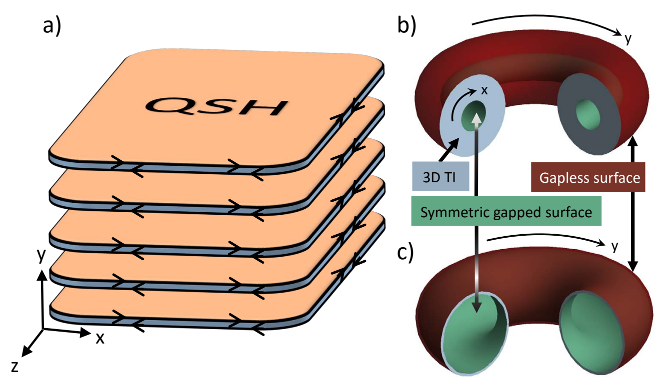

FIG. 1. (color online) (a) Weak topological insulator built from quantum spin Hall layers. Interlayer tunneling yields two symmetry-protected surface Dirac points at momenta $(q_x, q_y) = (0, 0)$ and $(0, \pi)$. (b) Thickened torus of weak topological insulator with symmetric topologically ordered interior and gapless exterior. (c) Two-dimensional limit where the thickness shrinks to zero. The topological order must be anomalous; otherwise one is left with an 'impossible' 2D band structure. This very general argument applies broadly to 3D symmetry-protected topological phases.

The third consideration results from viewing the WTI as a stack of QSH insulators. Any finite stack may be viewed as two dimensional, with an even-odd effect: the system forms a 2D topological insulator with an odd number of layers but

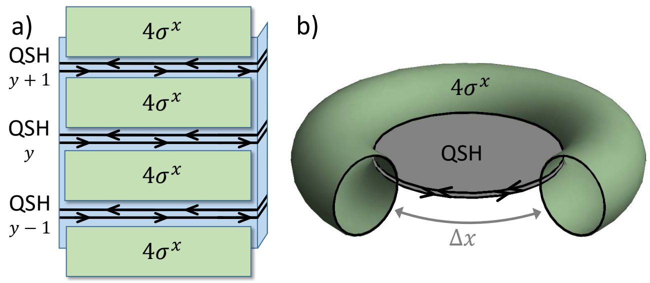

FIG. 2. (color online) (a) Weak topological insulator surface dressed with 2D topologically ordered 'plates'. (b) Limit of a single QSH layer and setup for discussing weak symmetry breaking.

a trivial 2D insulator otherwise. Since the 2D topological insulator edge cannot be gapped without breaking $\mathcal{T}$ or charge conservation, this even-odd effect should also appear when interactions gap the stack's surface to form topological order in the limit of infinitely many layers.

Gapping procedure. We now put this discussion on firmer footing. To facilitate gapping the WTI we imagine patterning the surface with 2D topologically ordered 'plates' that respect the same symmetries as the WTI surface. In the decorated structure the plates simply bridge adjacent QSH layers as shown in Figure 2(a). Crucially, this does not affect the bulk of the WTI, which endows the surface with exactly the same anomaly as in the absence of the plates. Consequently, any phase accessed in this way can equally well arise without such decoration. Similar approaches have been used in Refs. [11, 21] for STI and topological superconductor surfaces.

The decorated structure contains interfaces (enumerated by the integer y) where a helical QSH mode meets two sets of gapless edge states from the adjacent plates, one from above and one from below. We judiciously select the plates such that (i) local interactions within a given interface can remove all gapless modes without breaking any symmetries and (ii) the surface topological order with minimal degeneracy on a torus appears. Note that time-reversal symmetry constrains the latter degeneracy to be the square of an integer [22].

The interfaces to be gapped are conveniently described within the standard K-matrix formalism [23] by a matrix K and charge vector Q, which specify the statistics and charges of low-energy fields, along with a vector X that distinguishes Kramers singlets from doublets [22]; see Supplementary Material [24] for a brief review. More precisely, we have

$$K = \begin{pmatrix} K_h & 0 & 0 \\ 0 & K_p & 0 \\ 0 & 0 & -K_p \end{pmatrix}, \quad Q = \begin{pmatrix} q_h \\ q_p \\ q_p \end{pmatrix}, \quad X = \begin{pmatrix} \chi_h \\ \chi_p \\ \chi_p \end{pmatrix} \quad (1)$$

where the 'h' and 'p' subscripts indicate quantities for the helical QSH modes and plates, respectively. For the QSH sector $K_h = \sigma^z$ (here and below σ^a denote Pauli matrices), $q_h = (1,1)$, and $\chi_h = (0,1)$. For the plates, time reversal demands an even-dimensional K_p. We assume the smallest two-dimensional K_p, which can be either fermionic or bosonic. We focus on the latter since we find that the fermionic case does not permit time-reversal-invariant gapping of the interface. The bosonic case allows two distinct possibilities: (i) $K_p = m\sigma^x$, $q_p = (0,2)$, $\chi_p = (r,0)$ or (ii) $K_p = m\sigma^z$, $q_p = (2,2)$, $\chi_p = (r,0)$ with m an even integer and $r = 0$ or 1. Either possibility yields a minimal charge excitation of $e^* = 2/m$. By the criterion of Ref. [22] the interface may be symmetrically gapped when $\frac{1}{e^*}\chi^T K^{-1} Q$ is even. It follows that the smallest possible value of m is four, and that the value of r does not affect the interface's gappability. Hereafter we set $r = 0$ for concreteness and focus on $K_p = 4\sigma_x$; the gapped phase obtained with this choice simply relates to STI surface topological order [24].

To specify the gap-opening interactions we introduce low-energy fields describing a given interface y. Right/left-moving QSH electron operators are $\psi_{R/L,y} \equiv e^{i\varphi_{R/L,y}}$. We use subscripts $+$ and $-$ to denote fields from the adjacent upper and lower plates. Operators $a_{\pm,y} \equiv e^{i\phi_{a\pm,y}}$ and $d_{\pm,y} \equiv e^{i\phi_{d\pm,y}}$ then respectively create charge-$e/2$ and neutral excitations with time-reversal properties $a_{\pm,y} \to a_{\pm,y}$ and $d_{\pm,y} \to d^\dagger_{\pm,y}$. These quasiparticles have bosonic self-statistics but exhibit mutual statistics $e^{i\pi/2}$, implying that $a^4_{\pm,y}$ and $d^4_{\pm,y}$ represent local bosons. Interactions

$$(\psi_R\psi_L)^2 (a_- a_+)^4 + H.c. \sim \cos 4\theta_c \quad (2)$$

$$\left(\psi_R^\dagger\psi_L\right)^2 \left(d_-^\dagger d_+\right)^4 + H.c. \sim \cos 4\theta_s \quad (3)$$

$$\left(a_-^\dagger a_+\right)^4 + H.c. \sim \cos 4\theta_n, \quad (4)$$

are therefore physical. (We suppress y-dependence whenever unneeded.) The fields $\theta_{c,s,n}$ defined above mutually commute and can therefore be simultaneously pinned to gap the interfaces. Moreover, the interactions preserve both $\mathcal{T}$ and charge conservation. Thus uniformly condensing $\langle e^{i\theta_{c,s,n}}\rangle \neq 0$ respects all symmetries; for details see the Supplementary Material [24].

Identification of topological order. Determining the resulting surface topological order requires identifying the deconfined anyons, i.e., those that can move continuously throughout the surface. The plates each carry 16 quasiparticles built from combinations of a and d. Only electrons can move between plates, and in the absence of the gapping interactions (2)-(4) the fractional quasiparticles are therefore confined in the y direction. How then can anyons propagate along y when interactions 'stitch together' plates in the gapped topologically ordered surface?

Consider first dragging an a charge-$e/2$ anyon from one plate to the next. Since fractional excitations cannot directly cross between plates, this process leaves a dipole described by $a_{y-} a^\dagger_{y+} \sim e^{i\theta_n}$ at the interface as Figure 3(a) illustrates. However, the condensate $\langle e^{i\theta_n}\rangle$ readily absorbs the dipole—which is effectively invisible—negating any energy cost. The a quasiparticle thus propagates freely across the surface, precisely as for a Laughlin quasiparticle jumping between two strongly hybridized $\nu = 1/3$ quantum Hall strips.

TABLE I. Topological data for the fundamental anyons $\tilde{d}$ and a in the symmetrically gapped weak topological insulator surface.

Anyon	Charge	$\mathcal{T}$	T_y	Braid with $\tilde{d}$	Braid with a
$\tilde{d}$	0	$\tilde{d}^*$	$\tilde{d}a^2(\times\text{electron})$	0	i
a	$e/2$	a	a	i	0

In contrast, asking the same question about the d quasiparticle reveals physics unique to the WTI surface. Dragging a neutral d anyon between plates does not simply leave behind a $d_{y_-}d^\dagger_{y_+}$ dipole since the interactions (2-4) do not generate condensation of such a dipole. To specify its fate we define a neutral fermion

$$\tilde{\psi}_{R/L} = \psi_{R/L}a^2 \sim e^{2i\theta_c}\left(\psi_{L/R}a^2\right)^\dagger \tag{5}$$

The condensates created by the interactions (2-4) identify $\tilde{\psi}_R$ and $\tilde{\psi}^\dagger_L$; we therefore refer to both as simply $\tilde{\psi}$. When d crosses an interface it leaves the condensed combination $d_{y_-}d^\dagger_{y_+}\tilde{\psi}$ at the interface, and turns into a *different* anyon corresponding to d augmented by the neutral fermion $\tilde{\psi}$. Thus quasiparticles $\tilde{d}$ given by

$$\tilde{d} = \begin{cases} d, & \text{even plates} \\ d\tilde{\psi}^\dagger, & \text{odd plates} \end{cases} \tag{6}$$

may propagate freely across the surface. Remarkably, translations act nontrivially on these anyons:

$$T_y\tilde{d}T_y^{-1} = \tilde{d}\tilde{\psi}. \tag{7}$$

This property, which manifests an even-odd effect, crucially distinguishes the symmetrically gapped WTI surface and the topological order formed by individual plates. Table I summarizes the topological data for the surface.

Related phases in 2D. As emphasized earlier, any symmetric phase for the WTI surface cannot exist in a purely 2D system with the same symmetries. It is instructive to analyze how breaking either translation or time-reversal invariance allows the topological order to appear in strict 2D. Breaking one of these two symmetries allows for many ways to reduce the system to a 2D gapped system with topological order. Perhaps the simplest way is by decoupling the WTI from the $4\sigma_x$ plate, gapping its surface, and gluing the $4\sigma_x$ plate to one another. This forms a trivial insulator in parallel to a 2D $4\sigma_x$ state, but does not retain the unique transformation under translation embodied in (7).

If we break time-reversal infinitesimally and preserve translation symmetry, however, we can have a 2D system that retains Eq. (7). To that end, we look for a K-matrix and charge vector that yield $\sigma_{xy} = 0$ since infinitesimal breaking of time-reversal does not yield a Hall conductivity for a gapped system. We also look for a single-unit-cell translation matrix M_y which implements translations of quasiparticles $e^{i\vec{n}\cdot\vec{\Phi}}$ ($\vec{n}$ is an integer vector) as $T_y e^{i\vec{n}\cdot\vec{\Phi}}T_y^{-1} = e^{i(M_y\vec{n})\cdot\vec{\Phi}}$. This matrix must obey $M_y^T K^{-1} M_y = K^{-1}$ and $M_y^2 = 1$. The first condition ensures that statistics are invariant and is required for a system symmetric to translation by one unit cell. The second condition states that *all* quasiparticles transform onto themselves under translations by two unit cells. M_y must further encode Eq. (7) while acting trivially on a quasiparticles and on electrons. Trivial action means here that a translated excitation at most acquires a local boson that transforms trivially under all present symmetries. For example, the operation

$$T_y\psi_R T_y^{-1} = a^4\psi_R^\dagger \sim \psi_R\left(\psi_R^\dagger\psi_R^\dagger a^4\right) \sim \psi_R(\psi_R^\dagger\psi_L e^{2i\theta_c}) \tag{8}$$

multiplies ψ_R by the local, charge-neutral boson in parenthesis. This set of requirements is satisfied by [25]

$$K = \begin{pmatrix} 0 & 4 & 0 & 0 \\ 4 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, Q = \begin{pmatrix} 2 \\ 0 \\ 1 \\ 1 \end{pmatrix}, M_y = \begin{pmatrix} 1 & -2 & 4 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{9}$$

with translations implemented in precisely this way; the second and third columns of M_y encode (7) and (8). Note that the local boson acquired by ψ_R under translations is odd under time reversal. Consequently, Eq. (9) describes a translation-symmetric 2D state only in the absence of time-reversal. If we ignore M_y, the same K-matrix could equally well describe a time-reversal-invariant 2D phase. However, enforcing translation symmetry through M_y violates time reversal. Such 2D realizations can never simultaneously implement both time-reversal and translation symmetries as in the WTI surface, following our earlier general arguments (Fig. 1).

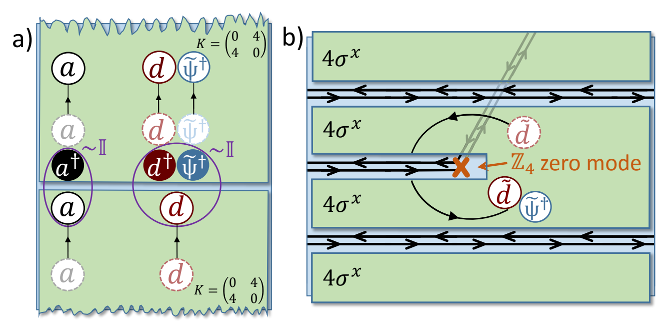

FIG. 3. (color online) (a) Dragging an anyon across plates leaves behind 'invisible' operators at the interface that get absorbed into a condensate. The condensates allow the a quasiparticle, carrying charge $e/2$, to pass freely between plates while the neutral d quasiparticle acquires a neutral fermion and thus changes anyon type. (b) Weak topological insulator surface with a screw dislocation. Upon encircling the dislocation $\tilde{d}$ anyons acquire a neutral fermion, indicating a zero mode bound to the defect.

Even-odd effect and weak symmetry breaking. It is illuminating to discuss the gapped WTI when the system consists of a finite stack of N QSH layers. The surface is then quasi-1D and hence technically cannot sustain the required topological order. Indeed, this case reveals a subtlety regarding time reversal and the possibility of weak symmetry breaking [26]. As a primer consider Fig. 2(b) where a cylindrical plate 'wraps around' a single QSH edge, leaving a gapless helical region of length Δx. The QSH/plate interface is identical to that considered above, and the same interactions (2)–(4) can open a gap—ostensibly without breaking symmetries. Furthermore the circular edge of the plate can ostensibly *also* be gapped without breaking the symmetries (either due to its finite-size or due to interactions [22]). Interestingly, symmetry-breaking must nevertheless occur [26]: A right-moving electron from the gapless QSH edge cannot penetrate into the adjacent gapped segments and must therefore reflect into an opposite-spin left-mover. This necessitates spontaneous magnetization, which we now analyze.

Using Eq. (3) one can express the magnetization at the gapless region's endpoints as $\langle\psi_L^\dagger\psi_R\rangle \sim e^{i2\theta_s}\langle(d_-^\dagger d_+)^2\rangle$. Three cases exist: (i) When the plate's circular edges are gapped by interactions the expectation value $\langle(d_-^\dagger d_+)^2\rangle$ must be circumference-independent. (ii) When the plate's circular edges are gapped only owing to their finite-size, $\langle d_-^\dagger d_+\rangle$ and the magnetization decay as a power-law in the cylinder circumference L. (iii) Finally, when the entire QSH edge is gapped ($\Delta x \to 0$) the circular edges are simply absent and $\langle d_-^\dagger d_+\rangle \sim e^{-L/\xi}$ with a length ξ set by the plate's bulk quasiparticle gap. This last case corresponds to the setup examined in Ref. [26].

Consider next the $N = 2$ generalization of Fig. 2(b) where plates arranged into a cylinder gap two QSH layers. The above argument for spontaneous time-reversal symmetry breaking no longer holds since a right-moving QSH electron from one layer can backscatter into the other without breaking time reversal symmetry.

These two examples signify an even-odd effect. For N layers with periodic boundary conditions between the first and N'th layers the local magnetization at an interface y analogously reads

$$\langle\psi_{L,y}^\dagger\psi_{R,y}\rangle \sim \left\langle\left(d_{-,y}^\dagger d_{+,y}\right)^2\right\rangle. \tag{10}$$

A finite expectation value generically arises if a d quasiparticle from just above the interface can propagate intact to the bottom side of the interface. The issue is subtle since d acquires a neutral fermion $\tilde\psi$ when crossing an interface; recall Fig. 3(a). Consequently, direct tunneling (which requires traversing a single interface) cannot generate a non-zero magnetization and quasiparticles must take the 'long way' around to contribute. With even N the initial d ends up dressed by $\tilde\psi$ when it reaches the bottom of the interface, and the magnetization thereby vanishes. By contrast, for odd N the d quasiparticle boldly arrives undressed yielding a non-zero value. If the entire surface is gapped this expectation value decays exponentially with N, while with adjacent gapless modes [similar to Fig. 2(b)] a power-law emerges.

Dislocation defects. The non-trivial action of translation symmetry on $\tilde d$ anyons yields interesting consequences for lattice defects. In a WTI screw dislocations terminating at position x_0 on the surface [as in Fig. 3(b)] bind a helical QSH edge state that penetrates into the bulk [27]. When interactions gap the WTI boundary, electrons from the bulk helical modes must backscatter at the surface. Such a defect thus locally violates time-reversal symmetry—yet another manifestation of weak symmetry breaking. The impact on surface anyons is even more striking: When $\tilde d$ encircles the termination point as sketched in Fig. 3 it changes anyon type and acquires a neutral fermion. This suggests that the dislocation forms an extrinsic non-Abelian defect that traps a nontrivial zero mode (similar effects arise in [28–32]).

Note that the point of the defect, x_0, may be viewed as the boundary between a region $x < x_0$ where the interface is gapped by means of the interactions (2)-(4), and a region $x > x_0$ where one QSH edge is left ungapped, and the two neighboring topologically ordered plates are healed into one. The penetration of the QSH edge into the third dimension does not affect the following considerations. To capture the spontaneous breaking of TRS we add a two-particle backscattering term $(\psi_R^\dagger\psi_L)^2 + H.c.$ to this QSH edge. The Supplementary Material [24] derives the following effective Hamiltonian density that describes the defect,

$$\mathcal{H} = \tilde\Delta\Theta(x_0 - x)\tilde\psi_R\tilde\psi_L + \tilde u\Theta(x - x_0)(\tilde\psi_R^\dagger\tilde\psi_L)^2 + H.c., \tag{11}$$

with $\tilde\psi_{R/L}$ defined in Eq. (5). (Note however that at $x > x_0$ we no longer have $\tilde\psi_R \sim \tilde\psi_L^\dagger$.) The $\tilde\Delta$ and $\tilde u$ terms respectively arise from Eq. (2) and the two-particle backscattering upon taking into account condensates involving the plates. References [33, 34] analyzed precisely Eq. (11) and showed that the defect hosts a $\mathbb{Z}_4$ parafermion zero mode. The '$\mathbb{Z}_4$-ness' reflects the two possible values for the spontaneous magnetization and the two possible values for (neutral) fermion parity.

Conclusions. We explored strongly interacting WTI surfaces using a quasi-1D formulation that permits full analytical control. We found that the surface can become gapped by entering an Abelian topologically ordered state with several unusual features. First, symmetries act on quasiparticles in a manner forbidden in purely 2D setups. Second, an interesting even-odd effect previously known for non-interacting electrons persists in the topologically ordered surface: For a WTI composed of an odd number of QSH systems, 'weak symmetry breaking' leads to a magnetization exponentially small in the number of layers. Third, lattice defects in the Abelian topologically ordered surface exhibit a non-Abelian structure, which may be viewed as a manifestation of the anomalous symmetry properties of the quasiparticles. We expect such features to persist quite generally in weak topological phases assembled from 2D symmetry-protected topological states.

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