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Large deviation analysis of rapid onset of rain showers

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Rainfall from ice-free cumulus clouds requires collisions of large numbers of microscopic droplets to create every raindrop. The onset of rain showers can be surprisingly rapid, much faster than the mean time required for a single collision. Large-deviation theory is used to explain this observation.

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The dynamics of the onset of rainfall from ice-free ('warm') cumulus clouds is poorly understood [1–3]. A rain drop grows by collisions of microscopic water droplets. A large number of microscopic droplets must combine to make one rain drop: the volume increase is a factor of approximately one million. The collision rates in the early stages of the growth process are low (typically of order one collision per hour). Given the large number of collisions required, it is very hard to understand why rain showers can be initiated over short periods, of perhaps twenty minutes.

One possible resolution is a consequence of the large number of microscopic droplets which must combine to make a raindrop. This implies that only very few drops are required to undergo rapid growth, and perhaps there are sufficient rare combinations of rapid multiple collisions to explain rainfall: this has previously been emphasised by various authors [1, 4, 5]. Kostinski and Shaw [6] introduced an elegant model for this runaway growth process. They presented numerical evidence that the model can lead to a rapid development of showers, but a transparent theoretical approach is required. Because this problem involves the analysis of rare events, methods based upon large deviation theory [7, 8] are appropriate. In this Letter these methods are used to investigate the hypothesis that rare combinations of rapid collisions trigger showers. It is shown that a rain shower can develop over a timescale which is a small fraction of the mean timescale for one collision.

This Letter will start by discussing some observations and estimates [1–3] which illustrate the difficulties in making a quantitative description of rainfall. These will be followed by describing a model for runaway droplet growth, equations (4) to (6), introduced in [6]. This model will then be analysed using a large deviation theoretic approach.

A convecting cumulus cloud may have droplets of mean radius $a_0 = 10 \mu\text{m}$, which result from condensation onto aerosol nuclei. Raindrops have a much larger size, typically 1 mm. The volume of a droplet which becomes a raindrop therefore increases by a very large factor, denoted by \mathcal{N} , which is typically $\mathcal{N} \approx 10^6$. The number density of microscopic droplets is typically of order $N_0 = 2.5 \times 10^8 \text{ m}^{-3}$, which gives a liquid water content,

expressed as a volume fraction, $\Phi_1 \approx 4\pi N_0 a_0^3/3 \approx 10^{-6}$. The cloud depth may be $h = 2 \times 10^3 \text{ m}$ and the typical vertical velocity of air inside the cloud has magnitude $U \approx 2 \text{ m s}^{-1}$, so that the turnover time for convection is approximately $\tau_h = 10^3 \text{ s}$. Rainfall from this type of cloud can develop over a timescale of approximately $20 \text{ min} \approx 10^3 \text{ s}$.

Collisions between droplets arise principally from different terminal velocities. The Stokes law for the drag on a sphere at low Reynolds number indicates that the terminal velocity is

$$v = \tau_p g = \alpha a^2, \quad \alpha = \frac{2}{9} \frac{\rho_l}{\rho_g} \frac{g}{\nu} \quad (1)$$

where τ_p is the response time characterising the Stokes drag on a droplet, ρ_l is the density of liquid water, and ρ_g and ν are, respectively, the density and kinematic viscosity of air. Inserting values for air and water at 5°C gives $\alpha \approx 1.4 \times 10^8 \text{ m}^{-1} \text{ s}^{-1}$, so that when $a_0 = 10 \mu\text{m}$ the terminal velocity is $v \approx 1.4 \times 10^{-2} \text{ m s}^{-1}$ and the response time is $1.4 \times 10^{-3} \text{ s}$. The collision rate of a drop of radius a_1 with a gas of droplets of radius a_0 is

$$R_1 = \pi \varepsilon N_0 (a_0 + a_1)^2 \alpha (a_1^2 - a_0^2) \quad (2)$$

where ε is the coalescence efficiency [1, 2]. The coalescence efficiencies ε of small droplets are somewhat uncertain, but it is widely accepted that they are low for typical cloud droplets [1, 2]. If the larger droplet has radius below $20 \mu\text{m}$, it is believed that $\varepsilon \leq 0.1$, and that for radius $10 \mu\text{m}$, $\varepsilon \leq 0.03$ [2]. For droplets of size $a = 50 \mu\text{m}$ colliding with droplets of size $a = 10 \mu\text{m}$, however, the efficiencies are expected to be close to unity [1, 2]. Setting $a_1 - a_0 = 2.5 \mu\text{m}$ and $\varepsilon = 0.03$ in addition to the parameters defined above gives $R_1 \approx 2 \times 10^{-5} \text{ s}^{-1}$. The rate of coalescence of typical sized water droplets due to collisions is therefore very small.

Cumulus clouds are turbulent because of convective instability. Saffman and Turner [9] investigated the role of turbulence in facilitating collisions between water droplets. In the case of very small droplets, the collision rate due to turbulence is a consequence of shearing motion. The shear rate of small-scale motions in turbulence is the inverse of the Kolmogorov timescale, $\tau_K = \sqrt{\nu/\epsilon}$,

where ϵ is the rate of dissipation per unit mass. According to the Saffman-Turner model, shear induces a collision speed of order a_0/τ_K . They argue that the corresponding collision rate is

$$R_{\text{turb}} = \sqrt{\frac{8\pi}{15}} \frac{N_0 \epsilon (2a)^3}{\tau_K}. \quad (3)$$

For the parameters of the cloud model, the rate of dissipation is $\epsilon \sim U^2/\tau_h \approx 2 \times 10^{-3} \text{ m}^2 \text{ s}^{-3}$, giving $\tau_K \approx 70 \text{ ms}$, which gives $R_{\text{turb}} \approx 10^{-6} \text{ s}^{-1}$, which is negligible. The effects of turbulence are dramatically increased when the effects of droplet inertia are significant: this was noticed in numerical experiments by Sundaram and Collins [10], who ascribed the effect to a clustering effect termed ‘preferential concentration’ [11]. More recent work has proposed an alternative mechanism, which has been termed the ‘sling effect’ [12], and which has been explained in terms of the existence of caustics in the velocity field of the droplets [13]. Inertial effects are measured by the Stokes number, $\text{St} \equiv \tau_p/\tau_K$. Recent numerical studies [14] (see also [15]) show that while the collision rate is greatly enhanced by effects due to caustics for $\text{St} > 0.3$, equation (3) is a good estimate when $\text{St} \ll 1$. Although it is in principle possible for turbulence to increase the collision rate of water droplets due to inertial effects, the parameters of the cloud model discussed above yield $\text{St} \approx 2 \times 10^{-2}$, where there is no significant effect. While there is a consensus that turbulence is important for the formation of rain showers [16], turbulent enhancement of collision rates does not appear to be sufficient.

Now consider how to model the onset of a shower, developing the approach discussed in [6]. It has already been remarked that showers occur on a timescale which may be smaller than the typical timescale for one collision. It is, therefore, reasonable to assume that the runaway droplets are falling through a background of droplets which have not yet coalesced, and which are all of similar size. As a runaway droplet falls it collides with a large number \mathcal{N} of small droplets of size a_0 . The time between successive collisions may be assumed to be independent Poisson processes. If the time between the collision with index n and the previous collision is t_n , the time for a droplet to experience runaway growth is

$$T = \sum_{n=1}^{\mathcal{N}} t_n \quad (4)$$

where the t_n are independent random variables with an exponential distribution

$$P_n(t_n) = R_n \exp(-R_n t_n). \quad (5)$$

The problem is to determine the statistics of T in the limit as $\mathcal{N} \rightarrow \infty$. Note that, because the droplets grow by collisions, the droplet volume increases by a factor of approximately \mathcal{N} as a result of these collisions, so that

its radius increases by a factor of $\mathcal{N}^{1/3}$. Equation (2) shows that the rates for successive collisions increase as the size of the falling drop grows. Because all of the collision rates R_n scale in the same way as a function of the droplet size a_0 and the number density N_0 , write $R_n = R_1 f(n)$. Here R_1 depends upon the properties of the cloud but the function $f(n)$ is the same for all clouds. In order to identify the form of $f(n)$, consider the rate of collision of a large droplet resulting from n previous collisions with a gas of small droplets of radius a_0 . The radius of the large droplet is $a_n = n^{1/3} a_0$. When n is large it may be assumed that the collision efficiency is $\epsilon \approx 1$ and $a_n \gg a_0$ so that $R_n \sim \pi N_0 \alpha a_n^4 \propto n^{4/3}$, which suggests setting $f(n) = n^{4/3}$. However during the early stages of droplet growth, the collision efficiency for the first few collisions is small, but increases rapidly with n . In what follows $f(n)$ is assumed to be a power-law, so that

$$R_n = R_1 n^\gamma. \quad (6)$$

If the collision efficiency of droplets were unity, it would be appropriate to set $\gamma = 4/3$. Because the collision efficiency of droplets at the crucial initial stage of their growth is small, the collision rate increases more rapidly as the size of the falling droplet increases. When the droplets are between $10 \mu\text{m}$ and $50 \mu\text{m}$ it is reasonable to model the product of the collision rate and the collision efficiency as being proportional to a^6 , that is to n^2 , where n is the number of collisions [6]. In other cases, such as solid precipitation (snow), other values of γ may be appropriate. In the following γ is left as an adjustable parameter, but special consideration is given to $\gamma = 2$, because it gives a good approximation to terrestrial rainfall, and to $\gamma = 4/3$, because this may be a good approximation for atmospheres on other planets where the collision efficiency might not limit the rate of coalescence. The remainder of this Letter is concerned with using large deviation theory to analyse the consequences of the model contained in equations (3) to (6).

It is necessary to determine the probability density for the time T being a very small fraction of its mean value, $\langle T \rangle$. Inspired by large deviation theory [7, 8], the probability density of T is written in an exponential form:

$$P(T) = \frac{1}{\langle T \rangle} \exp[-J(\tau)], \quad \tau = \frac{T}{\langle T \rangle}. \quad (7)$$

When R_n is given by (6), the mean time for explosive growth converges as $\mathcal{N} \rightarrow \infty$ when $\gamma > 1$:

$$\lim_{\mathcal{N} \rightarrow \infty} \langle T \rangle = \lim_{\mathcal{N} \rightarrow \infty} \frac{1}{R_1} \sum_{n=1}^{\mathcal{N}} n^{-\gamma} = \frac{1}{R_1} \zeta(\gamma). \quad (8)$$

where ζ is the Riemann zeta function. The function $J(\tau)$ in (7) is often termed the entropy in texts on large deviation theory [7, 8]. It will be necessary to determine the entropy function $J(\tau)$ from the rates R_n .

After a drop has grown to a size where it is much larger than the typical droplets, and where the collision efficiency is approximately unity, it falls rapidly and collects other droplets in its path. Consider a drop of size a_1 falling through a ‘gas’ of much smaller droplets, with liquid volume fraction Φ_1 . The larger drop falls with velocity v and sweeps out a volume $\pi a_1^2 v$ per unit time, so that it grows in volume at a rate $\dot{V} = \pi \varepsilon a_1^2 \Phi_1 v = 4\pi a_1^2 \dot{a}_1$, where \dot{a}_1 is the rate of increase of the drop radius. The rate of increase of the radius of the ‘collector’ drop as a function of the distance x through which it has fallen is therefore

$$\frac{da}{dx} = \frac{\varepsilon \Phi_1}{4} . \quad (9)$$

Note that this expression is valid whether or not the terminal velocity is given by the small Reynolds number approximation, (1). In the case of droplets which reach a radius of approximately 1 mm, the collision efficiency ε is close to unity throughout most of the fall. The droplet radius after falling through a cloud of depth h is therefore $a(h) \sim \Phi_1 h/4$. It will be assumed that the most relevant collector drops are those that started at the top of the cloud. Because droplets grow by coalescence of microscopic droplets as they fall, the ratio of the final volume to the initial volume is equal to the number of collisions: this is

$$\mathcal{N} = \left(\frac{h}{4a_0} \right)^3 \Phi_1^3 . \quad (10)$$

Using the representative values given above gives $\mathcal{N} \approx 10^5$. Kostinski [17] has used related arguments to explain observations that the rate of production of drizzle from maritime stratus cloud is proportional to the cube of its depth.

The fraction of droplets undergoing runaway growth between time t and $t + \delta t$ is $P(t)\delta t$. Because the volume of the runaway growth droplets has increased by a factor of \mathcal{N} , when these raindrops fall out of the cloud they reduce the liquid water content Φ_1 by $\mathcal{N}\Phi_1 P(t)\delta t$. The rate of change of the liquid water content of a cloud due to the runaway growth of droplets is therefore

$$\frac{d\Phi_1}{dt} = -\Phi_1 \mathcal{N} P(t) . \quad (11)$$

Note that the growth factor \mathcal{N} and the probability density for runaway growth after time t are both functions of Φ_1 , but if the objective is to understand the onset of a rain shower it suffices to evaluate these quantities with the initial value $\Phi_1(0)$. The onset of the shower is determined by the criterion that a significant fraction μ (typically a few percent) of the liquid water content of a cloud is removed by creating raindrops, resulting from \mathcal{N} collision events: from (11) this occurs when $\mathcal{N} \int_0^t dt' P(t') = \mu$. Using (7) for $P(t)$, the condition for the timescale t^* where

there is a significant reduction in $\Phi_1(t)$ is approximated by $\mathcal{N}^* \exp[-J(t^*/\langle T \rangle)] = 1$, where $\mathcal{N}^* = \mathcal{N}/\mu$ is a large number, typically $\mathcal{N}^* \approx 10^6$. The condition for the onset of a shower is, therefore,

$$t^* = \tau^* \langle T \rangle , \quad J(\tau^*) = \ln \mathcal{N}^* . \quad (12)$$

To determine the solution of (12) for t^* , it is necessary to determine the entropy function $J(\tau)$ for the random sum defined by (4) and (5).

Now consider how to compute $J(T)$. A cumulant generating function $\lambda(k)$ is defined by writing

$$\exp[-\lambda(k)] = \langle \exp(-kT) \rangle = \int_0^\infty dT P(T) \exp(-kT) . \quad (13)$$

Because the t_n are independent, with a distribution given by (5):

$$\lambda(k) = - \sum_{n=1}^{\mathcal{N}} \ln \langle \exp(-kt_n) \rangle = \sum_{n=1}^{\mathcal{N}} \ln \left(1 + \frac{k}{R_n} \right) . \quad (14)$$

Consider how to obtain $P(T)$ from $\lambda(k)$. Noting that $\exp[-\lambda(k)]$ is the Laplace transform of $P(T)$, application of the Bromwich integral formula for inversion gives

$$P(T) = \frac{1}{2\pi i} \int_{\mathcal{R}-i\infty}^{\mathcal{R}+i\infty} dz \exp[zT - \lambda(z)] \quad (15)$$

where $\mathcal{R} > -R_1$. The integral is dominated by contributions from the neighbourhood of a saddle at $z = k^*$ (on the real axis), where

$$T = \sum_{n=1}^{\mathcal{N}} \frac{1}{R_n + k^*} \quad (16)$$

which is to be solved for the saddle point k^* given a value of the time T . The probability density $P(T)$ is then approximated by

$$P(T) = \frac{1}{R_1} \frac{1}{\sqrt{2\pi J''(k^*)}} \exp[-J(\tau)] \quad (17)$$

where $\tau = T/\langle T \rangle$ and $J''(k)$ is the magnitude of the second derivative of the exponent in (15). Equations (16) and (17) cannot be solved exactly and explicitly for $J(\tau)$. Consider how to write down a parametric representation of $J(\tau)$ using a scaled variable, κ , defined by $\kappa = k^*/R_1$. Imposing the requirement that the integrand of (15) is stationary with respect to z , in the limit as $\mathcal{N} \rightarrow \infty$, the dimensionless time for raindrop formation is

$$\tau(\kappa) = \frac{1}{\zeta(\gamma)} \sum_{n=1}^{\infty} \frac{1}{\kappa + n^\gamma} \quad (18)$$

(this is a dimensionless version of (16), expressed in terms of $\tau = T/\langle T \rangle = R_1 T/\zeta(\gamma)$). The entropy function is the

value of the exponent of (15) evaluated at the stationary point κ which solves (18): in the limit as $\mathcal{N} \rightarrow \infty$ this is

$$J(\kappa) = S(\kappa) - \zeta(\gamma) \kappa \tau(\kappa) \quad (19)$$

with

$$S(\kappa) = \sum_{n=1}^{\infty} \ln(1 + \kappa n^{-\gamma}) . \quad (20)$$

In general, equation (18) must be solved numerically to determine κ for a given value of τ , but it is possible to extract asymptotic expressions which are valid for small and large τ . The small τ asymptotics are determined by the large κ asymptotics of the sums in (18) and (20). These require some delicate analysis (detailed in the Supplementary Materials), but the methods are standard. The leading order behaviour of $S(\kappa)$ is

$$S \sim \gamma A(\gamma) \kappa^{\frac{1}{\gamma}} - \frac{1}{2} \ln(\kappa) - \gamma C + O(\kappa^{-1}) \quad (21)$$

where C is a constant defined in the supplementary materials, and

$$A(\gamma) = \frac{1}{\gamma} \int_0^{\infty} dx \frac{x^{-\frac{\gamma-1}{\gamma}}}{1+x} = \frac{1}{\gamma} B\left(1 - \frac{1}{\gamma}, \frac{1}{\gamma}\right) \quad (22)$$

(here $B(v, w)$ is the Euler beta function as defined in [18]). Differentiation of $S(\kappa)$ yields an expression equivalent to (18) relating for τ to κ , which can be inverted to give an expression for the saddle point

$$\kappa^* = b\tau^{-\frac{\gamma}{\gamma-1}} \left[1 - c\tau^{\frac{1}{\gamma-1}}\right] \quad (23)$$

where b and c are functions of γ . In terms of τ , the leading order terms of the entropy are

$$J(\tau) = (\gamma - 1) A b^{\frac{1}{\gamma}} \tau^{-\frac{\gamma}{\gamma-1}} + \frac{\gamma}{2(\gamma-1)} \ln \tau + D \quad (24)$$

where D is another constant. Furthermore, in terms of τ the second derivative of the entropy, J'' , is proportional to $\tau^{\frac{2\gamma-1}{\gamma-1}}$. The probability density is therefore

$$P(\tau) = K \tau^{-\frac{3\gamma-1}{2(\gamma-1)}} \exp(-\mathcal{C}/\tau^{\frac{1}{\gamma-1}}) \quad (25)$$

where $\mathcal{C} = (\gamma - 1) A b^{\frac{1}{\gamma}}$ and K is another constant which can be explicitly constructed. For completeness, the approximation of $P(\tau)$ for $\mathcal{N} \rightarrow \infty$ and $\tau \gg 1$ is

$$P(\tau) \sim \zeta(\gamma) \exp[\mathcal{S}(\gamma) - \zeta(\gamma)\tau], \quad \mathcal{S}(\gamma) = \sum_{n=2}^{\infty} \ln(1 - n^{-\gamma}) . \quad (26)$$

Figure 1 shows the distribution of $\tau = T/\langle T \rangle$ for the case $\gamma = 2$ (the case which is most relevant to rain showers), with $\mathcal{N} = 10^4$ and $R_1 = 1$, comparing the results of simulation of (4), the Bromwich integral (15), the saddle-point approximation, equations (17), (19), (20), and the

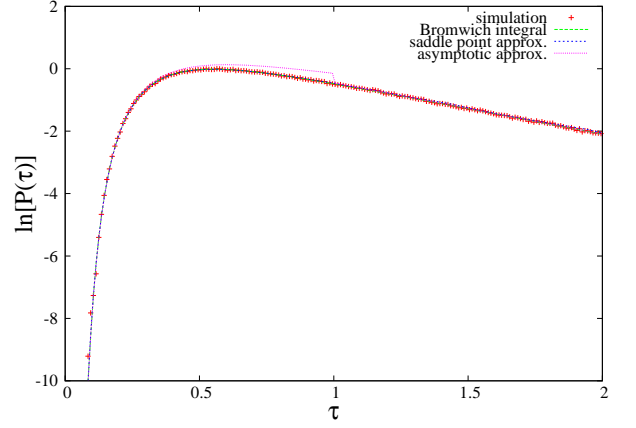


FIG. 1. (Color online). Plot of $\ln[P(\tau)]$, for $\mathcal{N} = 10^4$ and $\gamma = 2$. The results of simulation, evaluation of the Bromwich integral, and the saddle point approximation, and the explicit asymptotic formulae, equations (25) (used where $\tau \leq 1$) and (26) (used for $\tau > 1$) for all are in excellent agreement.

explicit asymptotic formulae, (25) and (26), which are all in excellent agreement. A small discontinuity in the asymptotic expression at $\tau = 1$ marks the switch between using (25) and (26). The entropy function increases very rapidly as $\tau \rightarrow 0$, indicating that the value of $\tau^* = t^*/\langle T \rangle$ is quite insensitive to the value of $\ln \mathcal{N}$. It is clear from figure 1 that the solution of equation (25) gives small values of τ^* when \mathcal{N} is large. Numerical evaluation of the solution of (12) using the Bromwich integral of $P(\tau)$ with $\gamma = 2$ gives $\tau^* \approx 0.077$ when $\mathcal{N} = 10^5$ and $\tau^* \approx 0.068$ when $\mathcal{N} = 10^6$. Alternatively, in terms of $\langle t_1 \rangle = \langle T \rangle / \zeta(\gamma)$, when $\gamma = 2$, the predicted time for onset of a shower is a small fraction of the mean time for the first collision: $t^* \approx 0.128 \langle t_1 \rangle$ when $\mathcal{N} = 10^5$ and $t^* \approx 0.112 \langle t_1 \rangle$ when $\mathcal{N} = 10^6$. These estimates for t^* are compatible with results reported by Kostinski and Shaw [6].

In the case where $\gamma = 4/3$, there is also excellent agreement between the exact evaluation of $P(\tau)$ using (15) and the saddle point approximation. While (25) is the precise asymptotic expression for $P(\tau)$ in the limit as $\tau \rightarrow 0$, for $\gamma = 4/3$ the convergence of this estimate is not good as for $\gamma = 2$. Reasons for this, and an improved asymptotic approximation which also gives excellent agreement with simulations at $\gamma = 4/3$, are discussed in the Supplementary Material.

Equations (12) and (25) imply that the timescale t^* for the initiation of a shower is smaller than the mean time for a single collision. Surprisingly, the timescale t^* decreases as the number of collisions required to make a raindrop increases: as $\mathcal{N}^* \rightarrow \infty$ the dominant term of (24) as $\tau \rightarrow 0$ gives

$$\tau^* \sim \frac{1}{R_1} [\ln \mathcal{N}^*]^{-\frac{\gamma-1}{\gamma}} . \quad (27)$$

Thus large deviation theory has resolved an apparent paradox of meteorology, that rain showers can start very quickly, on timescales which are short compared to typical mean collision times.

This calculation does not resolve all of the uncertainties about initiation of rain showers. Clouds can exist for a long period without producing a rain shower, before depositing a significant fraction of their water content over a short time. Shower activity is associated with convective motion in clouds. For typical levels of turbulence, however, turbulent enhancement of collisions does not appear to be sufficient to trigger showers. It seems as if non-collisional mechanisms involving convection must play a role in initiating the cascade [19].

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