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# Complexified path integrals, exact saddles and supersymmetry

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In the context of two illustrative examples from supersymmetric quantum mechanics we show that the semi-classical analysis of the path integral requires complexification of the configuration space and action, and the inclusion of complex saddle points, even when the parameters in the action are real. We find new exact complex saddles, and show that without their contribution the semi-classical expansion is in conflict with basic properties such as positive-semidefiniteness of the spectrum, and constraints of supersymmetry. Generic saddles are not only complex, but also possibly multi-valued, and even singular. This is in contrast to instanton solutions, which are real, smooth, and single-valued. The multi-valuedness of the action can be interpreted as a hidden topological angle, quantized in units of  $\pi$  in supersymmetric theories. The general ideas also apply to non-supersymmetric theories.

**Introduction:** We address the question of how to properly define the semi-classical expansion of the path integral in quantum mechanics and quantum field theory. This question goes beyond the problem of studying the semi-classical approximation, because the theory of resurgence shows that the semi-classical expansion encodes perturbative as well as non-perturbative effects, and may provide a complete definition of the path integral [1, 2]. We consider a set of examples for which we show that the path integral measure and action must be complexified, and that novel complex saddle points appear. The usefulness of complexification is not surprising from the point of view of the steepest descent method for ordinary integration, but important new effects appear in functional integrals. We show that in generic cases complexification is indeed essential. Our results go beyond proposals in the literature to complexify the path integral in cases where coupling constants are analytically continued away from their physical values, as described in the work of Witten on Chern-Simons theory [3], and Harlow, Maltz and Witten on Liouville theory [4], and is potentially related to the complexification of the phase space formulation of path integral [5]. Complex saddles were previously studied as a computational tool in quantum mechanics, see e.g. [6–9]. Complex path integrals were also studied in connection with the sign problem in the Euclidean path integral of QCD and related model systems at finite chemical potential [10–14]. Here, we demonstrate the *necessity* of complexification even for the physical theory with real couplings. In [15] we show that these complex saddles have a natural interpretation in terms of thimbles in Picard-Lefschetz theory.

There are several calculations in field theory that suggest the importance of complex saddle points. As an example consider  $\mathcal{N} = 1$  supersymmetric gluodynamics on  $\mathbb{R}^3 \times S_1$  with SUSY preserving boundary conditions. This theory is confining, and has a non-perturbatively generated bosonic potential for the Polyakov line. The potential for the Polyakov line can be computed using bions, molecules of monopole-instantons [16, 17]. Bions also determine the vacuum energy, with the conclusion that supersymmetry is unbroken, e.g. for  $SU(2)$  theory,  $E_{\text{gr}} \propto -e^{-2S_m} - e^{-2S_m \pm i\pi} = 0$  where the first is from

magnetic bion and the latter from neutral bion. This calculation agrees with a calculation based on supersymmetry and the monopole instanton induced superpotential [18]. A puzzle concerning this result is that the sum over different bion types give zero vacuum energy, despite the fact that contribution of real saddles is universally negative-semidefinite [33].

The calculations in [17, 19] are based on analytic continuation in the coupling constant. Ref. [20] reinterprets the relative sign between the two different bion types as a hidden topological angle (HTA), a factor  $\exp(i\pi)$  associated with the relative phase in the quasi-zero mode Lefschetz thimble, which is nothing but a direction in field space. This result suggests that the calculation can be done directly for real values of  $g$ , and that bions arise as exact (non-BPS) saddle point solutions of the complexified path integral, and furthermore that the HTA is related to the imaginary part of the complexified action.

SUSY gluodynamics on  $\mathbb{R}^3 \times S_1$  is not an isolated case. Similar phenomena occur in  $\mathcal{N} = 1$   $SU(2)$  SUSY QCD [21], in three-dimensional SUSY gauge theory [22], and in  $\mathcal{N} = 2$  SUSY QM [23]. In this paper we make the basic idea precise in the context of SUSY quantum mechanics.

**Formalism and holomorphic Newton’s equation:** Consider the Euclidean quantum mechanical path integral as a sum over real paths,  $Z = \int Dx(t) \exp(-\frac{1}{\hbar} S_E)$ , with  $S_E = \int dt (\frac{1}{2} \dot{x}^2 + V(x))$ . The critical points solve Newton’s equation in the inverted potential,  $\frac{d^2x}{dt^2} = +\frac{\partial V}{\partial x}$ . This leads to the standard multi-instanton calculus in quantum mechanics. More general saddle points appear in the complexified path integral

$$Z = \int_{\Gamma} Dz e^{-\frac{1}{\hbar} S[z(t)]}, \quad S[z(t)] = \int dt (\frac{1}{2} \dot{z}^2 + V(z)), \quad (1)$$

where  $\Gamma$  is an integration cycle that has the same dimensionality as the original real path integral. The critical points of the complexified path integral solve the *holomorphic Newton’s equation* in the inverted potential  $-V(z)$ :  $\frac{\delta S}{\delta z} = 0 \Rightarrow \frac{d^2z}{dt^2} = +\frac{\partial V}{\partial z}$ . In terms of real and imaginary parts of the potential,  $V(z) = V_r(x, y) + iV_i(x, y)$ , we get

$$\frac{d^2x}{dt^2} = +\frac{\partial V_r}{\partial x}, \quad \frac{d^2y}{dt^2} = -\frac{\partial V_r}{\partial y}, \quad (2)$$

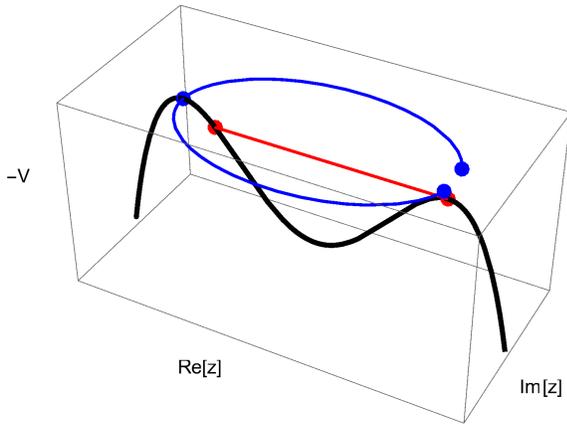


FIG. 1: Real and complex solutions in the inverted tilted double well potential. The inverted potential (on the real axis) is shown in black, the real bounce and associated critical and turning points are shown in red, and the pair of complex bions and turning and critical points are blue. The blue points correspond to  $z_1^{\text{cr}}$  and  $z_T, z_T^*$  in (6). Note that the motion takes place in the real and imaginary parts of the complex potential, as explained in the text.

where we have used the Cauchy-Riemann equations  $\partial_x V_r = \partial_y V_i$ , and  $\partial_y V_r = -\partial_x V_i$ . An important aspect of (2) is that it does **not** describe an ordinary two-dimensional classical mechanical system: the holomorphic classical mechanics is not the same as the motion of a particle in the two-dimensional inverted potential  $-V_r(x, y)$ . Instead of the usual Newton equations with force  $-\nabla_r V_r(x, y)$ , the force in the  $x$ -direction is due to  $\nabla_x V_r(x, y)$  while the force in the  $y$ -direction is due to  $-\nabla_y V_r(x, y)$ . This has interesting consequences.

**Supersymmetric quantum mechanics:** Consider supersymmetric quantum mechanics with the superpotential  $\mathcal{W}(\mathfrak{r})$

$$S = \int dt \left( \frac{1}{2} \dot{\mathfrak{r}}^2 + \frac{1}{2} (\mathcal{W}')^2 + [\bar{\psi} \dot{\psi} + p \mathcal{W}'' \bar{\psi} \psi] \right), \quad (3)$$

corresponding to  $p = 1$ . The parameter  $p$  will be used to deform the theory away from the supersymmetric point [9]. We choose  $\mathcal{W}(\mathfrak{r})$  with more than one critical point, so that there will be real instantons. By projecting to fermion number eigenstates one obtains a pair of Hamiltonians  $H_{\pm}$  [24]:

$$H_{\pm} = \frac{1}{2} \hat{p}^2 + V_{\pm}(\mathfrak{r}), \quad V_{\pm}(\mathfrak{r}) = \frac{1}{2} (\mathcal{W}'(\mathfrak{r}))^2 \pm \frac{p}{2} \mathcal{W}''(\mathfrak{r}). \quad (4)$$

In the following we consider superpotentials of the form  $\mathcal{W}(\mathfrak{r}) = \frac{1}{g} W(\sqrt{g}\mathfrak{r})$ , and rescale  $x = \sqrt{g}\mathfrak{r}$ . Then the Euclidean action takes the form  $S_E = \frac{1}{g} \int dt \left( \frac{1}{2} \dot{x}^2 + V_{\pm}(x) \right)$ . We work with the bosonized description (4). Note that compared to the original bosonic potential  $\frac{1}{2} (W')^2$  the bosonized theory contains an  $O(g)$  term that arises from integrating out the fermions. The quantum modified holomorphic equations of motion in the inverted potential  $-V_+(z)$  is

$$\frac{d^2 z}{dt^2} = W'(z) W''(z) + \frac{pg}{2} W'''(z). \quad (5)$$

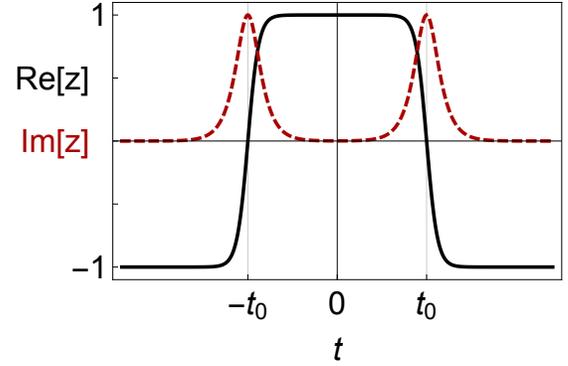


FIG. 2: Complex bion solution in supersymmetric quantum mechanics with a double well potential. The black and red lines show the real and imaginary part of the solution for  $pg = 1 \cdot 10^{-6}$ . The characteristic size of the solution is  $\text{Re}[2t_0] \simeq \frac{1}{2} \log \frac{16}{pg}$ . For larger values of  $pg$  the two tunneling event merge.

**Double well potential:** Consider  $W(x) = x^3/3 - x$ , so that  $V(x)$  is an asymmetric double well potential with an  $O(g)$  “tilt”. The ground state energy of the system is zero to all orders in perturbation theory, but non-perturbatively supersymmetry is spontaneously broken and the ground state energy is non-zero and positive [24]. Note that the positivity of the ground state energy is a consequence of the SUSY algebra,  $H = \frac{1}{2} \{Q, \bar{Q}\}$ , where  $Q$  and  $\bar{Q}$  are the SUSY generators.

In the original formulation (3) this can be understood as the contribution from *approximate* instanton-anti-instanton solutions of the bosonic potential  $\frac{1}{2} (W')^2$  [9]. In the bosonized version we seek classical solutions in the inverted potential  $-V_+$ . However, the real equations of motion in the inverted potential have no finite action configurations except for the trivial perturbative saddle, and an exact (real) bounce solution. But this bounce is related to the false vacuum and is not directly relevant for ground state properties, which are determined by saddles starting at the *global* maximum of the inverted potential. But the real motion of a classical particle starting at such a global maximum is unbounded, and has infinite action.

On the other hand, the holomorphic Newton’s equation (5) does support finite action solutions starting from the global maximum. There are *exact* finite action complex solutions that start at the global maximum of the inverted potential and bounce back from one of the two complex turning points, whose real part is located near the top of the local maximum, see Fig. 1. We refer to this as the “complex bion” solution:

$$z_{\text{cb}}(t) = z_1^{\text{cr}} - \frac{z_1^{\text{cr}} - z_T}{2} \coth \left( \frac{\omega_{\text{cb}} t_0}{2} \right) \left[ \tanh \left( \frac{\omega_{\text{cb}} (t + t_0)}{2} \right) - \tanh \left( \frac{\omega_{\text{cb}} (t - t_0)}{2} \right) \right], \quad (6)$$

where  $z_{\text{cb}}(\pm\infty) = z_1^{\text{cr}}$  is the global maximum of the inverted potential, and  $z_T = -z_1^{\text{cr}} \pm i\sqrt{pg/(-z_1^{\text{cr}})}$  are the complex turn-

ing points.  $\omega_{cb} = \sqrt{V''(z_1^{cr})}$  is the natural frequency at  $z_1^{cr}$ , and the complex parameter  $t_0$  is

$$t_0 = \frac{2}{\omega_{cb}} \operatorname{arccosh} \left[ \frac{3\omega_{cb}^2}{\omega_{cb}^2 - V''(z_1^{cr})} \right]^{1/2} \approx \frac{1}{2\omega_{cb}} \ln \left( -\frac{16}{pg} \right) \quad (7)$$

where  $\operatorname{Re}[2t_0]$  is the complex bion size.

It is straightforward to verify that (6) is a solution to the holomorphic equation of motion. The solution is shown in Fig. 2. The real part resembles an instanton-anti-instanton pair with size  $\ln \frac{16}{pg}$ , and the action is

$$S_{cb} \simeq \left( \frac{8}{3g} + p \ln \frac{16}{pg} + \dots \right) \pm i p \pi, \quad (8)$$

whose real part is slightly larger than two-instanton action,  $2S_I$ . The sign of the imaginary part  $\operatorname{Im}S_{cb} = \pm p\pi$  corresponds to the choice between two complex conjugate saddles.

The imaginary part of the action is defined modulo  $2\pi$ , so the choice between the two complex conjugate saddles does not lead to an ambiguity in the amplitude for  $p = 1$ . However, the factor  $e^{i\pi}$  is related to a hidden topological angle which is crucial to obtain the correct sign for the ground state energy. In the semi-classical limit the ground state can be understood as a dilute gas of complex bions:

$$E_{gs} \sim -e^{\pm i\pi} e^{-2S_I} \sim +e^{-2S_I} > 0, \quad (9)$$

in agreement with known results [9, 25]. The bosonized description makes the most crucial point clear. From a semi-classical view point, the positivity of the ground state energy and hence, consistency with the supersymmetry algebra owes its existence to the complexity of the exact solution, and to the hidden topological angle associated with it.

This complex bion solution can also be constructed by analytic continuation of the real bounce solution. To this end, we consider analytic continuation in the parameter  $p \rightarrow pe^{i\theta}$ , with corresponding potential

$$V_\theta(x) = \frac{1}{2}(W'(x))^2 + \frac{pe^{i\theta}g}{2}W''(x). \quad (10)$$

Only  $\theta = 0, \pi$  are physical theories, but the continuous  $\theta$  parameter is useful in order to understand the relation between different saddle points. The regular bounce solution starts at the local (smaller) maximum of the inverted potential, and gets reflected at a turning point below the global (larger) maximum. This solution is described by an ordinary elliptic integral. The analytic continuation,  $p \rightarrow pe^{i\theta}$ , produces a complex solution with finite action solution at any  $\theta$ , and can be continued all the way to  $\theta = \pi$ , where the local and the global maxima of the inverted potential are interchanged. The solution comes back to itself after  $4\pi$  rotation, it has order 2 monodromy. At  $\theta = \pm\pi$  we obtain exactly the complex conjugate pair of ‘‘complex bion’’ solutions (6).

**Quantization of hidden topological angle in supersymmetric theory:** In supersymmetric theories, since ground state energy is zero to all orders in perturbation theory, to avoid

an ambiguity in the ground state energy, it is essential that the imaginary part of the complex action is a multiple of  $\pi$ . Here, we will give a simple proof of this fact for the double well potential, which extends easily to the periodic potential. There are two complex bion solutions with complex conjugate turning points, and complex conjugate actions. This means that the imaginary part of the action can be computed from the difference of the action of the two complex bions,  $\operatorname{Im}S_{cb} = \pm \frac{1}{2i}(S_{cb}^1 - S_{cb}^2)$ . We also note that the action is computed as a line integral of the quantity  $\sqrt{2(E+V)}$  along the branch cut that connects the turning points of the solution. Here,  $E$  is the energy of the solution in the inverted potential. This implies that the imaginary part of the action can be written as

$$\operatorname{Im}S_{cb} = \frac{1}{2g} \oint_C dz \sqrt{2E + (W')^2 + pgW''}, \quad (11)$$

where the contour  $C$ , which arises from first going around the branch cut connecting  $z_1^{cr}$  and  $z_T$ , and then around the cut connecting  $z_1^{cr}$  and  $z_T^*$ , encircles all the points  $z_1^{cr}, z_T, z_T^*$ . This implies that we can deform the contour into a large circle in the complex  $z$ -plane. If  $W$  grows as a positive power of  $z$  we have  $(W')^2 \gg W''$ , so that the integrand can be expanded in powers of  $E/(W')^2$  and  $W''/(W')^2$ . The first term is a total derivative, and the second term vanishes because its residue is zero. Terms of second order and higher in  $1/(W')$  vanish faster than  $1/z$  as  $z \rightarrow \infty$ . The only contribution comes from

$$\operatorname{Im}S_{cb} = \frac{p}{4} \oint dz \frac{W''}{W'} = \frac{p}{2} \oint \frac{dW'}{W'} = ip\pi, \quad (12)$$

where we used the fact  $W' \sim z^2$  winds twice as  $z$  encircles the critical points. This proves the quantization of the HTA in the supersymmetric  $p = 1$  limit.

For the  $p \neq 1$  non-supersymmetric deformation of the theory, the perturbative ground state energy does not vanish anymore, but the energy spectrum must still be unambiguous. In that case, we show in [15] that the ambiguity inherent to the Borel resummation of perturbation theory cancels exactly the two-fold ambiguous complex bion amplitude, as an explicit illustration of resurgence.

**Periodic potential:** Now consider the superpotential  $W(x) = 4\cos(x/2)$ . In this system supersymmetry is unbroken [24]. There are two degenerate ground states, one bosonic and one fermionic, both with vanishing ground state energy. After the fermion is integrated out we obtain the bosonic potential

$$V_\pm(x) = 2\sin^2(x/2) \pm \frac{pg}{2}\cos(x/2). \quad (13)$$

The inverted potential, Fig. 3, has global maxima at  $x = 4n\pi$ , and local maxima at  $x = (4n+2)\pi$ . (The potential has period  $4\pi$ ). There is an exact real bounce solution starting at the local maximum, and bouncing from a real turning point, but again this is not directly relevant for ground state properties. Now we find two types of exact bion solutions, shown in Fig. 3. The first is a ‘‘real bion’’, connecting neighboring

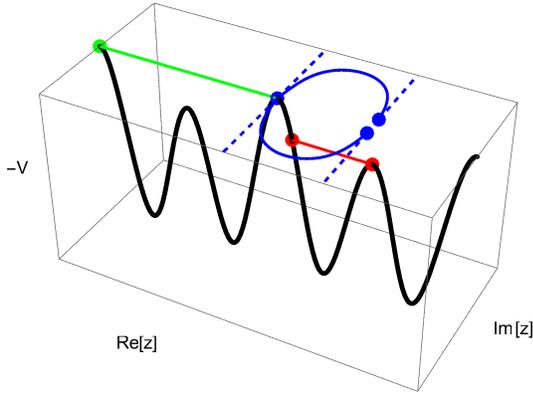


FIG. 3: Real and complex solutions in the quantum modified inverted Sine-Gordon potential. The inverted potential (on the real axis) is shown in black, the real bounce and associated critical and turning points are shown in red, the pair of complex bions and associated turning as well as critical points are blue, and the real bion is shown in green. In order to smoothen the (singular) complex bion the solution is plotted at  $\theta = 0.95\pi$ . The singular limit is shown as the dashed line. Note that the vacuum properties are governed by the real and complex bion solutions.

global maxima, say at  $x = 0$  and  $x = 4\pi$ . It has the form of an instanton-instanton solution, and as such has no analogue in the double-well case. There is also a *complex* bion solution that starts from a global maximum of the inverted potential, and is reflected from a complex turning point, with real part near the local maximum. This solution can be found directly or by analytic continuation from the real bounce,  $p \rightarrow pe^{i\theta}$ , and leads to an exact finite action complex saddle:

$$z_{cb}(t) = 2\pi \pm 4 \left( \arctan e^{-\omega_{cb}(t-t_0)} + \arctan e^{\omega_{cb}(t+t_0)} \right), \quad (14)$$

where  $\omega_{cb} = \sqrt{V''(0)} = \sqrt{1 + \frac{pg}{8}}$ . The complex parameter  $t_0 \simeq \frac{1}{2\omega_{cb}} \ln \left( -\frac{32}{pg} \right)$ , where  $\text{Re}[2t_0]$  is the complex bion size. The action is

$$S_{cb} \simeq \left( \frac{16}{g} + p \ln \frac{32}{pg} + \dots \right) \pm i p \pi. \quad (15)$$

The complex bion has the form of a complex instanton/anti-instanton molecule. An interesting new feature of this solution is that it is singular at  $t = \pm t_0$ , even though the action is finite. Physically this is because the real part of the holomorphic potential has ridges along the  $y$  direction, and the holomorphic equations of motion allow the particle to *roll up* (notice relative signs in (2)) along one ridge and then jump to the next ridge at infinity before rolling back again.

The analytic continuation in  $\theta$  smooths this singularity, and the solution is correspondingly multivalued as  $\theta \rightarrow \pi \pm \epsilon$ : see Fig. 4. As  $\theta \rightarrow \pi$  the real part of  $z_{cb}(t)$  has a discontinuity, and the imaginary part diverges. The action is finite, because the divergence in the action integral due to the singular behavior in  $\text{Re}z(t)$  and  $\text{Im}z(t)$  cancel. Fig. 5 shows the real and

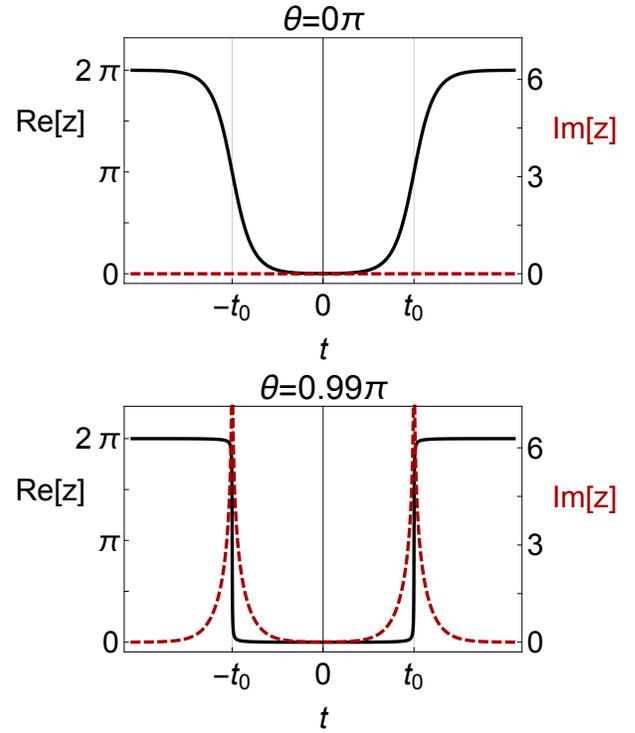


FIG. 4: Complex solutions in the quantum modified Sine-Gordon potential with  $pg = 2 \cdot 10^{-5}$ .  $\theta = 0$  correspond to the real bounce solution. At  $\theta = \pi^-$ , the real bounce turns into a complex bion. The characteristic size of the solution is  $\text{Re}[2t_0] \approx \ln \frac{32}{pg}$ .

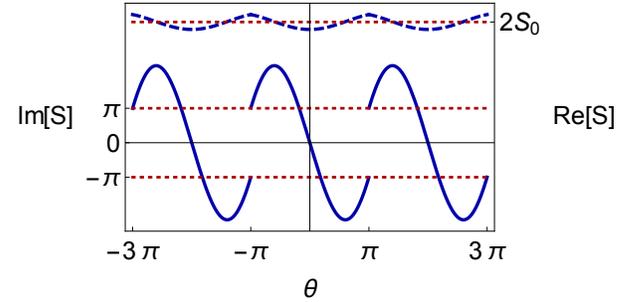


FIG. 5: Action of the complex saddle solution in the Sine-Gordon potential as a function of  $\theta$  for  $pg = 0.1$ .  $\theta = 0$  corresponds to the real bounce, and  $\theta = \pi$  is the complex bion, a multi-valued singular solution.

imaginary parts of the action as a function of the  $\theta$  parameter.

In the semi-classical limit the ground state can be described as a dilute gas of complex and real bions, with energy

$$E_{gs} \sim -e^{-S_{cb}} - e^{-S_{rb}} = -e^{\pm i\pi} e^{-2S_{rb}} - e^{-2S_{rb}} = 0, \quad (16)$$

consistent with the requirement of supersymmetry. The non-inclusion of the multi-valued saddle would result in a negative ground state energy and a conflict with the constraints of supersymmetry algebra. This proves that in order for the semi-

classical analysis to be consistent with the supersymmetry algebra, it is essential to include singular, multi-valued complex bion solution. This resolves a deep puzzle raised in [4].

**Conclusions:** We have presented two examples that demonstrate the need to include complex, and even singular and multi-valued, saddle point solutions of the path integral. We obtained exact finite action saddle points of the complexified path integral in supersymmetric quantum mechanics with a double well and Sine-Gordon potential. In both cases these new complex bion configurations are essential in order to obtain agreement with known results and the requirements of supersymmetry. This phenomenon is not restricted to quantum mechanics: analogous effects occur in several field theories, such as sigma models with fermions [2, 26–29] SUSY gluodynamics and QCD(adj) on  $R^3 \times S_1$ ,  $\mathcal{N} = 1$  [16, 30, 31],  $SU(2)$  SUSY QCD with one quark flavor [21, 32], and three dimensional SUSY  $\mathcal{N} = 2$  gauge theory [22]. Clearly, it is of interest to study these field theories, and ultimately QCD, using complexified path integrals.

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