Testing Nonassociative Quantum Mechanics

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Testing non-associative quantum mechanics

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The familiar concepts of state vectors and operators in quantum mechanics rely on associative products of observables. However, these notions do not apply to some exotic systems such as magnetic monopoles, which have long been known to lead to non-associative algebras. Their quantum physics has remained obscure. This letter presents the first derivation of potentially testable physical results in non-associative quantum mechanics, based on effective potentials. They imply new effects which cannot be mimicked in usual quantum mechanics with standard magnetic fields.

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Quantum mechanics is being tested ever more precisely by experiments, even while conceptual questions remain. We suggest a new kind of test in an extended setting in which the usual concepts of wave functions or state vectors and operators do not exist. Therefore, the standard axioms about outcomes of individual measurements are unavailable, or at least not known yet, and even at a practical level, no computational methods have been available so far. Here we show how one can derive semiclassical corrections to the motion of a particle and associated new phenomena.

The formalism of state vectors and operators implies that the action of the latter on the former is associative:

$$(\hat{A}\hat{B})\hat{C}\psi = \hat{A}\hat{B}\psi' = \hat{A}(\hat{B}\psi') = \hat{A}(\hat{B}\hat{C})\psi \tag{1}$$

if $\psi' = \hat{C}\psi$, for an arbitrary ψ. However, some exotic ingredients, such as magnetic monopoles, require an underlying non-associative algebra in order to quantize such systems. Quantum observables then no longer obey $(\hat{A}\hat{B})\hat{C} = \hat{A}(\hat{B}\hat{C})$. Not much has been known about physical effects when this basic identity is not available. In this letter, we develop and utilize a novel method in order to reveal testable quantum effects in such a system.

A non-associative algebra cannot be represented by operators on a Hilbert space. Instead, one has to work with an abstract non-associative algebra, which can be constructed by methods of deformation quantization as applied in [1–5]. States in this setting are not defined as normalized vectors in a Hilbert space, but as suitable linear functionals $\hat{A} \mapsto \langle\hat{A}\rangle$ from the algebra of observables to the complex numbers. (For associative observable algebras, this definition of a state is equivalent to the Hilbert-space picture thanks to the Gelfand–Naimark–Segal theorem; see for instance [6].) The primary object is therefore not a wave function but the set of expectation values assigned by a single state to all possible observables. We will demonstrate, for the first time in the context of non-associative quantum mechanics, that algebraic properties of such expectation-value functionals can be used to derive new semiclassical effects.

The best-known example of a non-associative quantum system is a charged particle in a magnetic monopole density [7]. Even if fundamental magnetic monopoles may not exist, such systems are gaining interest from a physical perspective, following recent constructions of analog systems of magnetic monopoles in condensed-matter physics [8–13]. (Related models also play a role in string theory [1, 14–17].) As is well known, the canonical momentum of a charged particle in a magnetic field $\vec{B}$ without monopoles, $\mathrm{div}\vec{B} = 0$, is a combination of the particle velocity and the vector potential. For the kinematical momentum $\vec{p} = m\dot{\vec{q}}$, one has non-canonical commutators of momentum components,

$$[\hat{p}_j, \hat{p}_k] = ie\hbar \sum_{l=1}^{3} \epsilon_{jkl} \hat{B}^l \tag{2}$$

where e is the particle's electric charge.

This relation depends only on the magnetic field and does not require a vector potential. Therefore, it can be used to define the basic commutators of a charged particle moving in a magnetic field with $\mathrm{div}\vec{B} \neq 0$. Regarding quantization, such fields can be of two types. If $\mathrm{div}\vec{B} \neq 0$ at isolated points of Dirac monopoles, these points can be excised and standard quantum mechanics applies. We are interested in the second type of charges not subject to Dirac quantization, allowing us to include fields with continuous magnetic charge densities for which no vector potential exists. The resulting algebra then does not fulfill the Jacobi identity of commutators:

$$\begin{aligned} &[[\hat{p}_x, \hat{p}_y], \hat{p}_z] + [[\hat{p}_y, \hat{p}_z], \hat{p}_x] + [[\hat{p}_z, \hat{p}_x], \hat{p}_y] \\ &= ie\hbar \sum_{j=1}^{3} [\hat{B}^j, \hat{p}_j] = -e\hbar^2 \widehat{\mathrm{div}\vec{B}}\,. \end{aligned} \tag{3}$$

As the name suggests, the Jacobi *identity* normally follows without further assumptions, provided the algebra is associative. The non-zero result just obtained can therefore be consistent only if the multiplication of momentum components is not associative. (Finite translations generated by momentum operators are not associative [18]. The classical analog is a twisted Poisson bracket [19–21].)

A physical version of the property of non-associativity is a "triple" uncertainty relation, just as the usual uncertainty relation is a consequence of non-commuting operators. As usual, (2) implies that $\Delta p_x \Delta p_y \geq \frac{1}{2} e\hbar \langle \hat{B}^z \rangle$: a large magnetic field in the z-direction deflects the particle from a straight line, making it harder to measure momentum components. A characteristic of monopole fields is that they change along the direction in which they point, for instance if $B^z = \mu z$ with a constant μ. The commutator of $\hat{p}_x$ and $\hat{p}_y$ then depends on the measurement of $\hat{z}$, which itself is subject to the standard uncertainty relation $\Delta z \Delta p_z \geq \frac{1}{2}\hbar$. Therefore, all three fluctuations, Δp_x, Δp_y and Δp_z, together determine how small the momentum fluctuations can be.

As already mentioned, states then cannot be defined as vectors in a Hilbert space, but their physical properties can be analyzed by treating them as linear expectation-value functionals on the algebra. Any such functional must be normalized, $\langle \hat{\mathbb{I}} \rangle = 1$ for the identity $\hat{\mathbb{I}}$ in the algebra, and obey a positivity condition which implies uncertainty relations. Having identified basic operators as the components of position and kinematical momentum, we can parameterize a state by the basic expectation values $\langle \hat{q}_i \rangle$ and $\langle \hat{p}_j \rangle$ as well as fluctuations, correlations and higher moments. The latter are defined to be of the form

$$\Delta(p_i p_j) := \frac{1}{2} \langle \hat{p}_i \hat{p}_j + \hat{p}_j \hat{p}_i \rangle - \langle \hat{p}_i \rangle \langle \hat{p}_j \rangle \tag{4}$$

for the example of two momentum components. (With this notation, we slightly modify the usual denotation of quantum fluctuations, identifying $\Delta(p_i^2) = (\Delta p_i)^2$.) The symmetrization in (4) takes into account the non-commuting nature of kinematical momentum components in a magnetic field. For higher moments with more than two factors of momentum or position components, different symmetrizations are possible, of which we choose, following [22], totally symmetric (or Weyl) ordering by summing with equal weights over all permutations of factors. Moreover, non-associativity requires a fixed choice for the bracketing of products of observables, which we choose to be done from the left as in [23].

A Hamiltonian leads to equations of motion for the basic expectation values coupled to moments, giving an infinite-dimensional dynamical system. In a semiclassical approximation, only a finite number of moments need be considered, corresponding to a specific order in $\hbar$. The Hamiltonian we use in this letter is of the standard form,

$$\hat{H} = \frac{1}{2m} \sum_{j=1}^{3} \hat{p}_j^2 + V(\hat{x}, \hat{y}, \hat{z}) \,, \tag{5}$$

where interactions of a charged particle with the magnetic field are implemented not by a term in the potential but by the non-trivial commutators of momentum components. (The potential $V(x, y, z)$ for an additional, non-magnetic force will be specified below.) Given a Hamiltonian, equations of motion for expectation values and moments follow by using

$$\frac{\mathrm{d}\langle \hat{O} \rangle}{\mathrm{d}t} = \frac{\langle [\hat{O}, \hat{H}] \rangle}{i\hbar} \tag{6}$$

which is still available in the non-associative case. However, the non-associative nature requires great care when evaluating commutators of products of the basic observables, for which we refer to [23].

The specific effect we will derive, related to stable motion in an effective potential, requires the particle to be completely confined. A magnetic field in the z-direction, $B^x = 0 = B^y$, confines the particle motion to a plane normal to the magnetic field. For complete confinement, we combine the magnetic force with a harmonic force in the same direction, choosing the potential to be $V(x, y, z) = \frac{1}{2} m\omega^2 z^2$. This force could be generated by an electric field. For simplicity, we consider a linear z-component $B^z = \mu z$, so that μ is the magnetic charge density. The resulting Hamiltonian is

$$\hat{H} = \frac{1}{2m} \sum_{j=1}^{3} \hat{p}_j^2 + \frac{1}{2} m\omega^2 \hat{z}^2 \,, \tag{7}$$

and the magnetic field enters via $[\hat{p}_x, \hat{p}_y] = ie\mu\hbar\hat{z}$ while the other pairs of momentum components commute.

We are interested in deriving an effective potential for the motion of such a particle. If one knows a suitable state of the system, the effective potential can be obtained from the expectation value of the Hamiltonian in which one sets all momentum expectation values to zero in order to remove the kinetic term, $V_{\rm eff}=\langle\hat{H}\rangle_{\langle\hat{p}_i\rangle=0}$. (We do not require solutions for momentum expectation values to be zero at all times.) In the given case with a quadratic Hamiltonian, the effective potential is the classical potential plus a sum of fluctuations:

$$\begin{aligned} V_{\rm eff}(\langle\hat{z}\rangle) &= \frac{1}{2}m\omega^2\langle\hat{z}\rangle^2 \\ &+\frac{1}{2m}\left(\Delta(p_x^2)+\Delta(p_y^2)+\Delta(p_z^2)\right)+\frac{1}{2}m\omega^2\Delta(z^2)\,. \end{aligned} \tag{8}$$

In order to express this potential as a function of the coordinates, we have to compute the values of quantum fluctuations. Following the methods of [24], we can compute the relevant state properties without using a wave function. Instead, we solve equations of motion for fluctuations in an adiabatic approximation (giving stationary moments in a near-coherent state) and saturating uncertainty relations (minimizing fluctuations). For well-understood (associative) systems such as anharmonic oscillators [25] or Coleman–Weinberg potentials in self-interacting scalar field theories [26], the correct results are obtained in this way [22, 24]. In our new situation, we minimize fluctuations by saturating the uncertainty relations, in the standard form for q_i and p_i and for non-commuting momentum components.

We can derive Ehrenfest-type equations of motion by using (6) for the moments. (See also [23].) Expanded up to first order in $\hbar$ for semiclassical states, thus including no moments of higher than second order, they are

$$\begin{aligned} m\dot{\Delta}(q_iq_j) &= \Delta(p_xq_i)\delta_{jx}+\Delta(p_xq_j)\delta_{ix}+\Delta(p_yq_i)\delta_{jx} \\ &+\Delta(p_yq_j)\delta_{ix}+\Delta(p_zq_i)\delta_{jx}+\Delta(p_zq_j)\delta_{ix} \end{aligned} \tag{9}$$

for all position moments,

$$\begin{aligned} m\dot{\Delta}(p_xq_i) &= \Delta(p_x^2)\delta_{ix}+\Delta(p_xp_y)\delta_{iy}+\Delta(p_xp_z)\delta_{iz} \\ &+e\mu\left(\langle\hat{z}\rangle\Delta(p_yq_i)-\langle\hat{q}_i\rangle\Delta(p_yz)\right)\,, \end{aligned} \tag{10}$$

$$\begin{aligned} m\dot{\Delta}(p_yq_i) &= \Delta(p_xp_y)\delta_{ix}+\Delta(p_y^2)\delta_{iy}+\Delta(p_yp_z)\delta_{iz} \\ &-e\mu\left(\langle\hat{z}\rangle\Delta(p_xq_i)-\langle\hat{q}_i\rangle\Delta(p_xz)\right)\,, \end{aligned} \tag{11}$$

$$\begin{aligned} m\dot{\Delta}(p_zq_i) &= \Delta(p_xp_z)\delta_{ix}+\Delta(p_yp_z)\delta_{iy}+\Delta(p_z^2)\delta_{iz} \\ &-m^2\omega^2\Delta(q_iz) \end{aligned} \tag{12}$$

for the position-momentum covariances,

$$\begin{aligned} m\dot{\Delta}(p_xp_y) &= -e\mu\left(\langle\hat{z}\rangle\Delta(p_x^2)-\langle\hat{z}\rangle\Delta(p_y^2)\right. \\ &\left.-\langle\hat{p}_x\rangle\Delta(p_xz)+\langle\hat{p}_y\rangle\Delta(p_yz)\right)\,, \end{aligned} \tag{13}$$

$$\begin{aligned} m\dot{\Delta}(p_yp_z) &= -e\mu\left(\langle\hat{z}\rangle\Delta(p_xp_z)+\langle\hat{p}_x\rangle\Delta(p_zz)\right) \\ &-m^2\omega^2\Delta(p_yz)\,, \end{aligned} \tag{14}$$

$$\begin{aligned} m\dot{\Delta}(p_xp_z) &= e\mu\left(\langle\hat{z}\rangle\Delta(p_yp_z)+\langle\hat{p}_y\rangle\Delta(p_zz)\right) \\ &-m^2\omega^2\Delta(p_xz) \end{aligned} \tag{15}$$

for momentum covariances, and

$$m\dot{\Delta}(p_x^2) = 2e\mu\left(\langle\hat{z}\rangle\Delta(p_xp_y)+2\langle\hat{p}_y\rangle\Delta(p_xz)+\langle\hat{p}_x\rangle\Delta(p_yz)\right) \tag{16}$$

$$m\dot{\Delta}(p_y^2) = -2e\mu\left(\langle\hat{z}\rangle\Delta(p_xp_y)+\langle\hat{p}_y\rangle\Delta(p_xz)+2\langle\hat{p}_x\rangle\Delta(p_yz)\right) \tag{17}$$

$$m\dot{\Delta}(p_z^2) = -2m^2\omega^2\Delta(zp_z) \tag{18}$$

for momentum fluctuations.

We solve these equations to zeroth adiabatic order in the moments, so that all time derivatives on the left-hand sides can be set to zero. The moments are then subject to linear algebraic equations. In order to solve the set of coupled equations, we use (9) for all possible index combinations to conclude that

$$\Delta(xp_x)=\Delta(yp_y)=\Delta(zp_z) = 0 \tag{19}$$

$$\Delta(yp_x)+\Delta(xp_y) = 0 \tag{20}$$

$$\Delta(zp_x)+\Delta(xp_z) = 0 \tag{21}$$

$$\Delta(zp_y)+\Delta(yp_z) = 0\,. \tag{22}$$

Using equations for mixed position-momentum moments, we obtain

$$\begin{aligned}
\Delta(p_x^2) &= -e\mu\left(\langle\hat{z}\rangle\Delta(xp_y)-\langle\hat{x}\rangle\Delta(zp_y)\right)\,, && (23)\\
\Delta(p_xp_y) &= -e\mu\left(\langle\hat{z}\rangle\Delta(yp_y)-\langle\hat{y}\rangle\Delta(zp_y)\right)\,, && (24)\\
\Delta(p_xp_z) &= 0 && (25)
\end{aligned}$$

from (10),

$$\begin{aligned}
\Delta(p_xp_y) &= e\mu\left(\langle\hat{z}\rangle\Delta(xp_x)-\langle\hat{x}\rangle\Delta(zp_x)\right)\,, && (26)\\
\Delta(p_y^2) &= e\mu\left(\langle\hat{z}\rangle\Delta(yp_x)-\langle\hat{y}\rangle\Delta(zp_x)\right)\,, && (27)\\
\Delta(p_yp_z) &= 0 && (28)
\end{aligned}$$

from (11), and

$$\begin{aligned}
\Delta(p_xp_z) &= m^2\omega^2\Delta(xz)\,, && (29)\\
\Delta(p_yp_z) &= m^2\omega^2\Delta(yz)\,, && (30)\\
\Delta(p_z^2) &= m^2\omega^2\Delta(z^2) && (31)
\end{aligned}$$

from (12). The equations of motion (13), (14) and (15) for momentum covariances provide

$$\begin{aligned}
\Delta(p_x^2)-\Delta(p_y^2) &= \frac{\langle\hat{p}_x\rangle\Delta(zp_x)-\langle\hat{p}_y\rangle\Delta(zp_y)}{\langle\hat{z}\rangle}\,, && (32)\\
\Delta(zp_z) &= -\frac{m^2\omega^2}{e\mu\langle\hat{p}_x\rangle}\Delta(zp_y)-\frac{\langle\hat{z}\rangle}{\langle\hat{p}_x\rangle}\Delta(p_xp_z)\,, && (33)\\
\Delta(zp_z) &= \frac{m^2\omega^2}{e\mu\langle\hat{p}_y\rangle}\Delta(zp_x)-\frac{\langle\hat{z}\rangle}{\langle\hat{p}_y\rangle}\Delta(p_yp_z)\,. && (34)
\end{aligned}$$

Since $\Delta(zp_z)=0$ from (19) and $\Delta(p_xp_z)=0=\Delta(p_yp_z)$ from (25) and (28), (33) and (34) imply $\Delta(zp_x)=0=\Delta(zp_y)$. From (32) and (24) or (26), we immediately conclude that

$$\Delta(p_x^2)=\Delta(p_y^2)\quad\text{and}\quad\Delta(p_xp_y)=0\,, \tag{35}$$

also using (19). These values are consistent with (23) and (27), in which the same fluctuations appear. All equations are then solved and the adiabatic approximation is self-consistent, showing that an effective potential exists.

We now consider states saturating the uncertainty relations. For the pair (z,p_z), we have the standard one,

$$\Delta(z^2)\Delta(p_z^2)-\Delta(zp_z)^2\geq\frac{\hbar^2}{4}\,, \tag{36}$$

while (2) implies an uncertainty relation

$$\Delta(p_x^2)\Delta(p_y^2)-\Delta(p_xp_y)^2\geq\frac{1}{4}e^2\hbar^2\langle\hat{B}\rangle^2\,. \tag{37}$$

If both inequalities are saturated, we obtain

$$\Delta(p_x^2)=\Delta(p_y^2)=\frac{1}{2}e\hbar\langle\hat{B}\rangle \tag{38}$$

and

$$\Delta(z^2)=\frac{\hbar}{2m\omega}\quad,\quad\Delta(p_z^2)=\frac{1}{2}\hbar\omega\,. \tag{39}$$

Finally, inserting these values in (8), we obtain

$$V_{\rm eff}(\langle\hat{z}\rangle)=\frac{1}{2}m\omega^2\langle\hat{z}\rangle^2+\frac{1}{2}\hbar\frac{eB(\langle\hat{z}\rangle)}{m}+\frac{1}{2}\hbar\omega\,. \tag{40}$$

If the magnetic field is constant, the fraction $eB/m=\omega_{\rm c}$ in (40) is the cyclotron frequency. It is well known that the Hamiltonian of a charged particle in a constant magnetic field can be transformed to one of a harmonic oscillator with the cyclotron frequency, so that our derivation provides the correct result of a constant $\hbar$-term in the effective potential given by the sum of zero-point energies $\frac{1}{2}\hbar\omega_{\rm c}$ and $\frac{1}{2}\hbar\omega$ of two uncoupled oscillators.

In the case of a magnetic field with constant charge density, the effective potential is linear, implying a new constant force from quantum effects. The force

$$F_{\rm eff} = -\frac{e\mu\hbar}{2m} \tag{41}$$

points in the direction of the magnetic field, or in the same direction in which the harmonic force is acting. We can combine the classical potential $\frac{1}{2}m\omega^2 z^2$ with the linear quantum potential $\frac{1}{2}e\mu\hbar z/m$ and write the effective potential as

$$V_{\rm eff}(\langle\hat{z}\rangle) = \frac{1}{2}m\omega^2\left(\langle\hat{z}\rangle + \frac{1}{2}\frac{e\mu\hbar}{m^2\omega^2}\right)^2 + \frac{1}{2}\hbar\omega\,. \tag{42}$$

(Rewriting (40) in this way generates an $\hbar^2$-term, which we do not include here because all our derivations were up to first order in $\hbar$. The $\hbar^2$-term can be absorbed in the next order corrections coming from higher moments.) The minimum of the harmonic potential is shifted by

$$\delta z = -\frac{1}{2}\frac{e\mu\hbar}{m^2\omega^2} \tag{43}$$

as the main first-order quantum effect. Interestingly, the shift is inversely proportional to ω, so that a small frequency can enlarge the quantum effect. (For $\omega \to 0$, the shift diverges. However, it is then simply ill-defined because the harmonic potential disappears in the limit and does not distinguish a center in the z-direction.) Also, as may be expected for an effect of quantum back-reaction, the shift is larger for particles with smaller mass.

In addition to harmonic oscillations around a shifted center, the charged particle would move along a circle in the $x-y$-plane as a consequence of the non-zero B^z for $z \neq 0$. Classically, without the shift in the harmonic potential, stable circular motion would not be possible because at $z = 0$, where the harmonic force vanishes, the magnetic field is zero. Given the shift of the minimum, stable circular motion is now possible with the particle confined to move in a plane at fixed $z = \delta z$.

This effect cannot be mimicked by magnetic fields without monopole densities. Such a magnetic field could produce a z-dependent potential only if there are non-vanishing components B^x or B^y, either by cancelling the z-derivative of B^z in the divergence or by having the z-dependence come only from B^x or B^y. However, the motion would then be more complicated than circular motion in the $x-y$-plane at some fixed value of z.

Our results have important conceptual and potentially observable consequences. They demonstrate that physical effects can be derived in quantum mechanics even when the usual and widely used notions of state vectors and operators are unavailable. Non-associative quantum mechanics is thereby shown to be meaningful physically, which, despite its exotic appearance, can be applied in diverse ways, including some versions of string theory and analog magnetic monopoles.

Regarding the latter, we have specialized our general methods to a system in which closed-form solutions can be obtained, providing a model system with clear new effects. Such models always play important roles in situations like the present one: not much is known about testable quantum effects of analog condensed matter monopoles, even while experimental realizations seem to be within reach [27]. Our model amounts to an idealized example which brings out new effects clearly.

In practice, although it seems hard to have a constant monopole density, for sufficiently large amplitude of the oscillating motions of a charged particle, it is conceivable that a fine lattice of magnetic monopoles could be used to test the new effect found here. Specifically, one should arrange the lattice in cylinder shape, so as to impose a preferred direction identified here with the z-direction. On scales larger than the lattice spacing (but well within the entire lattice), the complicated dynamics of electric charges moving around monopoles can be approximated by electric charges moving through a uniform monopole density to which our methods apply. Analog monopoles do have Dirac strings [27], which may still have an effect after averaging to a continuous density, making the magnetic field non-linear. For more accurate derivations of the effective potential, applying our methods to non-linear magnetic fields, the same equations for moments are available, but they are coupled in more complicated ways which are likely to require numerical input and further research. Similarly, the equations can be extended to higher orders in $\hbar$ by including higher moments, but again we are not aware of closed analytic solutions.

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