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Generalized Efimov effect in one dimension

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We study a one-dimensional quantum problem of two particles interacting with a third one via a scale-invariant subcritically attractive inverse square potential, which can be realized, for example, in a mixture of dipoles and charges confined to one dimension. We find that above a critical mass ratio, this version of the Calogero problem exhibits the generalized Efimov effect, the emergence of discrete scale invariance manifested by a geometric series of three-body bound states with an accumulation point at zero energy.

In quantum mechanics three identical bosons in three dimensions interacting resonantly via a short-range two-body potential have an infinite tower of bound states, whose energy spectrum forms a geometric series near the accumulation point at zero energy. This was discovered theoretically by Vitaly Efimov in 1970 [1] and is known today as the Efimov effect. This effect is a beautiful example of *few-body universality* since it is independent of the detailed form of the interaction potential provided it is tuned to the resonance (i.e., whenever a zero-energy s-wave two-body bound state if formed). The Efimov effect has been extended to systems of distinguishable particles [2–6], liberated from three dimensions [7] and found in other systems [8, 9]. During the last decade a number of experiments [10–14] with cold atoms near Feshbach resonances [15] verified various universal aspects related to Efimov physics— the Efimov ^{4}He trimer has also been recently observed in [16]— and demonstrated the experimental capability to explore fundamental aspects of few-body systems in exotic regimes.

From a more general perspective, the most startling feature of the Efimov effect is *discrete scale invariance* of the three-body problem, manifested in both bound and scattering three-body observables, that originates from *continuous scale invariance* of the two-body interaction. It thus appears natural to us to generalize the Efimov effect to systems whose two-body interaction is not necessarily short-range and define it as *the emergence of discrete scaling symmetry in a three-body problem if the particles attract each other via a two-body scale invariant potential* [17].

Motivated by this broader perspective on the Efimov effect, we study a three-body problem with a two-body *long-range* attractive potential of the form

$$V(r) = -\frac{\alpha}{2\mu r^2} \tag{1}$$

with μ being the reduced mass and α the dimensionless coupling constant. The potential (1) is scale invariant and, at zero energy or for sufficiently small r, where the energy term can be neglected, in one dimension the two independent solutions of the two-body Schrödinger equation are the powers $r^{1/2\pm\sqrt{1/4-\alpha}}$. However, the (inevitable) breakdown of the $1/r^2$ law at small distances introduces a length scale b, made explicit by writing the linear combination of the two asymptotic solutions in the form

$$\psi \sim \left(\frac{r}{b}\right)^{1/2+\sqrt{1/4-\alpha}} - \left(\frac{r}{b}\right)^{1/2-\sqrt{1/4-\alpha}}. \tag{2}$$

For the further discussion it is crucial whether α is larger or smaller than $1/4$ [4, 18, 19]. The case $\alpha > 1/4$ corresponds to the fall of a particle to the center and the discrete scaling is manifest already in the two-body problem. Here, the exponents $1/2 \pm \sqrt{1/4-\alpha}$ are complex conjugate, the two terms in Eq. (2) should be treated on equal footing, and b becomes an essential parameter, which can never be neglected. By contrast, the case $\alpha < 1/4$ has two scale-invariant limits $b = 0$ and $b^{-1} = 0$ where, respectively, only the plus-branch $r^{1/2+\sqrt{1/4-\alpha}}$ or only the minus-branch $r^{1/2-\sqrt{1/4-\alpha}}$ survives in Eq. (2). In practice, these two limits require, respectively, $|b| \ll \xi$ or $|b| \gg \xi$, where ξ is a typical length-scale in the problem such as the system size, de Broglie wave length, etc. For instance, in Eq. (2) the minus-branch solution can be neglected if $(b/\xi)^{\sqrt{1-4\alpha}} \ll 1$. Thus, the plus-branch scaling is realized "automatically" by increasing the typical size of the system, whereas the minus-branch requires a fine tuning of the short-range part of the potential [20–22]. Physically, this fine tuning signals the appearance of an additional two-body bound state emerging from the zero-energy threshold which can be realized using, for example, the Feshbach resonance technique [15].

As far as the three-body problem with the two-body interaction (1) is concerned, Calogero solved it in one dimension analytically for three *identical* particles [23] and found continuous scale invariance for all $\alpha < 1/4$, which implies the absence of the Efimov effect [24]. In this paper we show that this conclusion does not hold in general for the modified Calogero problem – two identical spinless bosons or fermions interacting with a third particle via the potential (1). In addition to the quantum statistics and the choice $b = 0$ or $b^{-1} = 0$ the modified problem is parametrized by the two continuous dimensionless quantities: α and the mass ratio. Accordingly, we calculate the critical line separating the Efimov and scale-invariant regions and describe the nature of the three-body bound state spectrum in this parameter space.

The three-body Hamiltonian relevant for our problem reads

$$H = -\frac{\partial_{R_1}^2 + \partial_{R_2}^2}{2M} - \frac{\partial_r^2}{2m} + V(r - R_1) + V(r - R_2), \quad (3)$$

where R_1 and R_2 are the coordinates of two identical particles of mass M and r is the coordinate of the third particle of mass m. The potential V is given by Eq. (1), where $\mu = mM/(M+m)$ and α denotes the interspecies dimensionless coupling.

A convenient way to solve this problem is obtained using hyperspherical coordinates. First, we introduce the center-of-mass and mass-scaled Jacobi coordinates $R_{CM} = \big[mr + M(R_1 + R_2)\big]/(2M + m)$, $x = \sqrt{\tilde{\mu}/2}(2r - R_1 - R_2)$, $y = \sqrt{2\mu_M}(R_2 - R_1)$, where $\tilde{\mu} = 2mM/(m + 2M)$ and $\mu_M = M/2$. It is then convenient to define polar (hyperspherical) coordinates $x = \mathcal{R}\cos\theta$, $y = \mathcal{R}\sin\theta$ with the mass-scaled hyperradius $\mathcal{R} = \sqrt{2\sum_i m_i(r_i - R_{CM})^2}$. The interparticle distances in the new coordinates become $r - R_1 = \mathcal{R}\sin(\Delta + \theta)/\sqrt{2\mu}$, $r - R_2 = \mathcal{R}\sin(\Delta - \theta)/\sqrt{2\mu}$ and $R_2 - R_1 = \mathcal{R}\sin\theta/\sqrt{M}$, where $\Delta = \arctan\sqrt{1 + 2M/m}$. Accordingly, after separating the center-of-mass motion, the relative part of the Hamiltonian (3) is written as a two-dimensional radial problem

$$H = -\partial_{\mathcal{R}}^2 - \frac{1}{\mathcal{R}}\partial_{\mathcal{R}} + \frac{1}{\mathcal{R}^2}\mathcal{M}_\theta^2 \quad (4)$$

with the hyperangular Schödinger operator

$$\mathcal{M}_\theta^2 = -\partial_\theta^2 - \frac{\alpha}{\sin^2(\Delta + \theta)} - \frac{\alpha}{\sin^2(\Delta - \theta)}. \quad (5)$$

Two-body scale invariance leads to *separability* of the three-body problem in hyperspherical coordinates. The relative part of the three-body wave function $\Psi(\mathcal{R}, \theta)$ can thus be written in the factorized form $\Psi(\mathcal{R}, \theta) = \Phi(\mathcal{R})\psi(\theta)$ and the problem separates into two tasks. First, one finds ψ by diagonalizing the operator $\mathcal{M}_\theta^2$

$$\mathcal{M}_\theta^2\psi = -s^2\psi. \quad (6)$$

Then, $\Phi(\mathcal{R})$ is the solution corresponding to the Hamiltonian (4) with $\mathcal{M}_\theta^2$ substituted by $-s^2$. This second task is trivially solved in terms of the Bessel functions $J_{\pm is}$, and the onset of the generalized Efimov effect coincides with the point $s^2 = 0$: for positive s^2 the system is Efimovian and for negative s^2 it is scale invariant. Thus, the problem of determining the critical mass ratio is equivalent to solving the hyperangular problem (6) and identifying the zero crossing of s^2 as a function of Δ. We will now discuss this procedure.

The coincidence angles $\theta = 0, \pi$ ($M - M$ coincidence) and $\theta = \pm\Delta, \pi \pm \Delta$ ($M - m$ coincidences) partition the hyperangular circle into six regions (see Fig. 1). Since two particles of mass M are identical, the wave function satisfies $\psi(\theta) = \psi(-\theta)$ or $\psi(\theta) = -\psi(-\theta)$, respectively, for bosons or fermions. In addition, the hyperspherical Hamiltonian is symmetric under $\theta \to \pi - \theta$ and the wave function ψ is either even or odd under this transformation. It is thus sufficient to solve the angular problem only in the domain $\theta \in (0, \pi/2)$. Moreover, we will assume that the distinguishable particles are *impenetrable*. Physically, this is realized by regularizing the inverse square potential (1) with a short-range potential that has a strong repulsive core. Due to the interspecies impenetrability, sectors I and II in Fig. 1 decouple and can be addressed separately. In sector I the hyperangular wave function, $\psi(\theta)$, should satisfy the following boundary conditions for $\theta = 0$ [25]

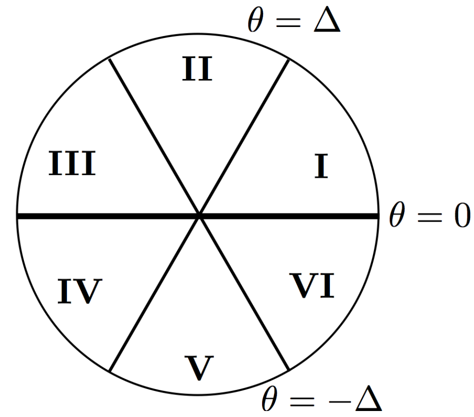

FIG. 1. Hyperangular domain partitioning.

$$\begin{aligned} \psi &= 0 \quad \text{fermions,} \\ \psi' &= 0 \quad \text{bosons} \end{aligned} \quad (7)$$

and for $\theta \to \Delta^-$

$$\begin{aligned} \psi &\sim (\Delta - \theta)^{1/2+\sqrt{1/4-\alpha}} \quad \text{plus-branch,} \\ \psi &\sim (\Delta - \theta)^{1/2-\sqrt{1/4-\alpha}} \quad \text{minus-branch.} \end{aligned} \quad (8)$$

The critical mass ratio is determined by solving Eq. (6) in sector I and is plotted in Fig. 2 (a) and (b) for bosons and fermions, respectively. We found that s^2 is an increasing function of the mass ratio M/m for any choice of α and boundary conditions. In addition, we find no zero-energy ($s = 0$) solution in sector II that satisfies the proper scale-invariant boundary conditions at the interspecies coincidence point. The wave function is thus zero in sector II, i.e., the probability to find the particle of mass m in between the two identical particles of mass M vanishes.

It should be noted that the modified Calogero problem of the type (3) is exactly solvable and scale invariant for the plus-branch under the condition $M/m = 1/(1/2 + \sqrt{1/4 - \alpha})$ [26–28]. The Efimov region corresponds to higher values of M/m and, since the problem is not solvable, we solve it numerically. We use the Numerov method [29] on a logarithmic grid (see Ref. [22]). Nevertheless, we also find approximate analytic solutions for this problem in limiting cases discussed below.

For the plus-branch the critical mass ratio diverges at $\alpha = 1/8$. In fact, for $\alpha > 1/8$ both branches give rise to the Efimov effect for sufficiently large M/m[30]. Indeed, for

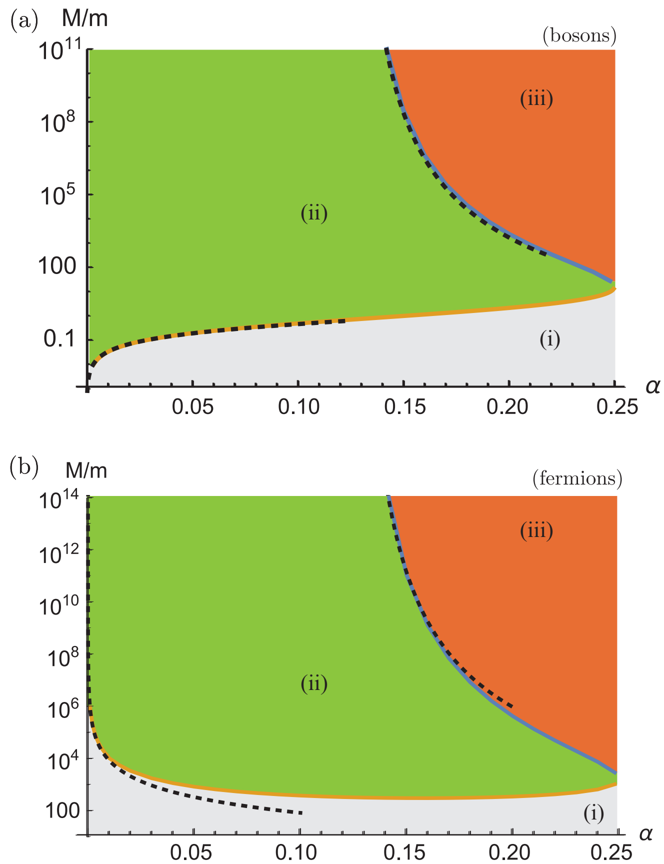

FIG. 2. (Color online) (a) Critical mass ratio as a function of α for bosons: upper blue (lower orange) line is the Numerov numerics for the plus (minus) branch, dashed lines are analytic asymptotes near $\alpha = 0$ and $\alpha = 1/8$ (see text). The shaded regions (i), (ii), and (iii), denote the regimes in which the Efimov effect does not occur, occurs for the minus-branch solution only, and occurs for both minus- and plus-branch solution. (b) same as (a) but for fermions.

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$M/m \to \infty$ the angle $\Delta = \pi/2$ and the hyperangular potential in Eq. (5) reduces to $-2\alpha/\cos^2\theta$. One can see that the hyperangular problem becomes Efimovian for $2\alpha > 1/4$ independent of the quantum statistics of the heavy particles and the branch choice. This means that the spectrum of $\mathcal{M}_\theta^2$ is unbound from below with deep bound states localized close to $\theta = \pi/2$. As a result, a finite $\pi/2 - \Delta$ is necessary to renormalize this potential and bring the ground state energy $-s^2$ to zero. Quantitatively, for the plus-branch solution in the vicinity of $\alpha = 1/8$ we obtain [22]

$$\frac{\pi}{2} - \Delta \approx \mathscr{N} e^{\pm\frac{\pi}{2} - \frac{2\pi}{\sqrt{8\alpha-1}}}, \tag{9}$$

where the upper (lower) sign corresponds to the case of bosons (fermions) and $\mathscr{N} = 16\exp[-2 - 2\sqrt{2} - H_{(-3+\sqrt{2})/2}] \approx 11.887$ with H_n being the harmonic number.

For the minus-branch the spectrum is Efimovian for any $0 < \alpha < 1/4$ for $M/m \gg 1$ [22]. The less stringent condition for the Efimov effect in this case can be explained by the fact that the minus-branch two-body interaction nearly binds two particles and is, in this sense, more attractive than the plus-branch interaction with the same α. In fact, the hyperangular problem can be solved analytically close to the non-interacting point $\alpha = 0$. In Ref. [22] we show that for the bosonic case

$$\Delta - \frac{\pi}{4} \approx \frac{\alpha\pi}{2} \tag{10}$$

and for fermions

$$\frac{\pi}{2} - \Delta \approx \frac{\alpha\pi}{4}. \tag{11}$$

The asymptotes (9), (10), and (11) are plotted in Figs. 2 as dashed lines.

The identified critical mass ratio is calculated using the wave function $\psi(\theta)$ without nodes inside sector I. As one increases the mass ratio, wave functions with increasing number of nodes will give rise to additional towers of Efimov states.

Now we describe the qualitative nature of the three-body bound state spectrum.The interaction in Eq. (1) must be regularized at short distances, see [22]. As the short-range potential depth D_0 changes one can tune between a pure plus-branch ($b = 0$) and minus-branch ($b^{-1} = 0$) solutions. The nature of the three-body spectrum will depend on which region in Fig. 2 the system falls into. There are three different regimes:

- In the region (i), below the orange curve, there is no Efimov effect for any value of b.

- In the region (ii), between the orange and blue critical curves, the spectrum behaves similar to the original Efimov problem [1]. By starting from the plus branch solution with $b = 0$ and increasing the depth D_0, three-body bound states emerge one-by-one from the three-body continuum as one approaches $b^{-1} = 0$. At the critical point $b^{-1} = 0$, where a zero-energy two-body bound state pops up, an infinite tower of Efimov states is formed with the Efimov parameter s_- (encoding the geometric factor $e^{2\pi/s_-}$ for the energy spectrum) which depends on both M/m and α. As one further increases the depth D_0, the trimers disappear one-by-one into the particle-dimer continuum.

- In the region (iii) the spectrum resembles the one appearing in the three-dimensional Efimov problem of particles with *unequal* scattering lengths [2, 31, 32]. Now both the plus- and minus-branches support Efimov states characterized by the Efimov parameter s_+ and s_-, respectively, where $0 < s_+ < s_-$. The energy spectrum contains an infinite number of three-body bound states close to the zero-energy threshold for any value of b. The interpolation from the plus- to the minus-branch can be understood as follows: Near $b = 0$ the energy spectrum close to the zero-energy threshold is controlled by the s_+ parameter. As one approaches $b^{-1} = 0$ the *virtual* dimer state of size of

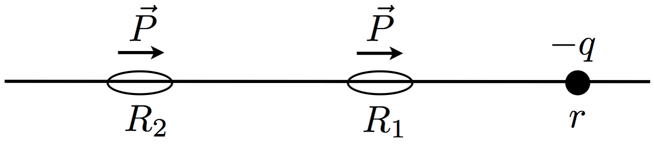

FIG. 3. Two identical dipoles and a charge confined in one dimension.

order $|b|$ is formed. As a result, the trimers with energies below (above) $\epsilon_d \sim -1/\mu b^2$ follow the geometric scaling with the Efimov parameter s_- (s_+). At resonance $b^{-1} = 0$ the geometric spectrum with s_- scaling is obtained. When the depth D_0 is increased further, a two-body *bound* state is formed and three-body states with energies below (above) ϵ_d will also follow the geometric scaling with the Efimov parameter s_- (s_+) to the point where, when away enough from $b^{-1} = 0$, the energy spectrum is again completely controlled by the s_+ parameter.

Can the Efimov effect found in this paper be discovered experimentally? A promising candidate might be a mixture of dipoles and charges that are confined to one dimension. Indeed, the dipole-charge interaction in three dimensions is given by the scale-invariant anisotropic potential $V(\mathbf{r}) \sim \cos\phi/r^2$, where ϕ is the angle between the direction of the dipole moment and the dipole-charge separation vector. Consider now a system of two identical dipoles of mass M and dipole moment P and a particle of mass m and charge $-q$ confined to a one-dimensional line. Let us also regard the dipoles as dumbbells with fixed dipole moments as illustrated in Fig. 3. This three-body problem is governed by the Hamiltonian

$$H = -\frac{1}{2M}\left(\partial^2_{R_1} + \partial^2_{R_2}\right) - \frac{1}{2m}\partial^2_r \\ \underbrace{-K_e q P\Big[\frac{1}{(r-R_1)^2} + \frac{1}{(r-R_2)^2}\Big]}_{\text{dipole-charge}} \underbrace{-\frac{2K_e P^2}{(R_1-R_2)^3}}_{\text{dipole-dipole}}, \quad (12)$$

where the Coulomb constant $K_e = 1/(4\pi\epsilon)$. If we neglect the dipole-dipole term, this Hamiltonian maps on Eq. (3) with

$$\alpha = 2\mu K_e P q \quad (13)$$

and thus gives rise to the Efimov effect provided $\alpha < 1/4$ and the mass ratio is above the critical value. The presence of the dipole-dipole term introduces a length scale $l_{dd} \sim M K_e P^2 \sim \frac{P}{q}$ and bound states of this size. In the Efimov regime the length l_{dd} provides the high-energy cutoff for the energy spectrum. This cutoff should not affect the (geometric) energy spectrum close to the zero-energy threshold. However, since our problem is one-dimensional, the dipole-dipole interaction effectively fermionizes the dipoles since their wave function is suppressed at $R_1 - R_2 \sim l_{dd}$. Thus, the critical mass ratio for the Efimov effect should be read off of Fig. 2 (b) rather than (a) even if the dipoles are identical bosons.

As an example, consider two polar molecules interacting with an electron. From Eq. (13) the dipole moment of the polar molecule should satisfy $P < P_{cr} = ea_0/8 \approx 0.318$ Debye, where e is the charge of the electron and a_0 is the Bohr radius. Such a system has a large mass ratio, and, therefore, provided it falls into the region (iii) in Fig. 2 (b), displays Efimov states (without fine-tuning to the minus branch) which can be detected spectroscopically. For a typical mass ratio $M/m = 10^5$ we find $s_+ > 1$ for $P > 0.281$ Debye. Moreover, Efimov states could also be observed if tuning of the dipole moment P is possible. In that case, near an electron-dipole resonance, dipolar losses should be enhanced every time a new Efimov state is formed.

A three-dimensional version of the problem may have better chances to be realized experimentally and should also have interesting quantum-chemistry implications. In that case $P_{cr} \approx 1.63$ Debye [33–35]. We consider it as a promising project and leave it for future studies.

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