

This is the accepted manuscript made available via CHORUS. The article has been published as:

$B \rightarrow \pi l l$ Form Factors for New Physics Searches from Lattice QCD

Jon A. Bailey, A. Bazavov, C. Bernard, C. M. Bouchard, C. DeTar, Daping Du, A. X. El-Khadra, E. D. Freeland, E. Gámiz, Steven Gottlieb, U. M. Heller, A. S. Kronfeld, J. Laiho, L. Levkova, Yuzhi Liu, E. Lunghi, P. B. Mackenzie, Y. Meurice, E. Neil, Si-Wei Qiu, J. N. Simone, R. Sugar, D. Toussaint, R. S. Van de Water, and Ran Zhou (Fermilab Lattice and MILC Collaborations)

Phys. Rev. Lett. **115**, 152002 — Published 7 October 2015

DOI: [10.1103/PhysRevLett.115.152002](https://doi.org/10.1103/PhysRevLett.115.152002)

$B \rightarrow \pi \ell \ell$ form factors for new-physics searches from lattice QCD

Jon A. Bailey,¹ A. Bazavov,² C. Bernard,³ C.M. Bouchard,⁴ C. DeTar,⁵ Daping Du,^{6,*}
A.X. El-Khadra,⁷ E.D. Freeland,⁸ E. Gámiz,⁹ Steven Gottlieb,¹⁰ U.M. Heller,¹¹ A.S. Kronfeld,^{12,13}
J. Laiho,⁶ L. Levkova,⁵ Yuzhi Liu,¹⁴ E. Lunghi,^{10,†} P.B. Mackenzie,¹² Y. Meurice,² E. Neil,^{14,15}
Si-Wei Qiu,⁵ J.N. Simone,¹² R. Sugar,¹⁶ D. Toussaint,¹⁷ R.S. Van de Water,^{12,‡} and Ran Zhou¹²

(Fermilab Lattice and MILC Collaborations)

¹*Department of Physics and Astronomy, Seoul National University, Seoul, South Korea*

²*Department of Physics and Astronomy, University of Iowa, Iowa City, IA, USA*

³*Department of Physics, Washington University, St. Louis, MO, USA*

⁴*Department of Physics, The Ohio State University, Columbus, OH, USA*

⁵*Department of Physics and Astronomy, University of Utah, Salt Lake City, UT, USA*

⁶*Department of Physics, Syracuse University, Syracuse, NY, USA*

⁷*Department of Physics, University of Illinois, Urbana, IL, USA*

⁸*Liberal Arts Department, School of the Art Institute of Chicago, Chicago, IL, USA*

⁹*CAFPE and Departamento de Física Teórica y del Cosmos, Universidad de Granada, Granada, Spain*

¹⁰*Department of Physics, Indiana University, Bloomington, IN, USA*

¹¹*American Physical Society, Ridge, NY, USA*

¹²*Fermi National Accelerator Laboratory, Batavia, IL, USA*

¹³*Institute for Advanced Study, Technische Universität München, Garching, Germany*

¹⁴*Department of Physics, University of Colorado, Boulder, CO, USA*

¹⁵*RIKEN-BNL Research Center, Brookhaven National Laboratory, Upton, NY, USA*

¹⁶*Department of Physics, University of California, Santa Barbara, CA, USA*

¹⁷*Physics Department, University of Arizona, Tucson, AZ, USA*

(Dated: August 17, 2015)

The rare decay $B \rightarrow \pi \ell^+ \ell^-$ arises from $b \rightarrow d$ flavor-changing neutral currents and could be sensitive to physics beyond the Standard Model. Here, we present the first *ab-initio* QCD calculation of the $B \rightarrow \pi$ tensor form factor f_T . Together with the vector and scalar form factors f_+ and f_0 from our companion work [J. A. Bailey *et al.*, Phys. Rev. D **92**, 014024 (2015)], these parameterize the hadronic contribution to $B \rightarrow \pi$ semileptonic decays in any extension of the Standard Model. We obtain the total branching ratio $\text{BR}(B^+ \rightarrow \pi^+ \mu^+ \mu^-) = 20.4(2.1) \times 10^{-9}$ in the Standard Model, which is the most precise theoretical determination to date, and agrees with the recent measurement from the LHCb experiment [R. Aaij *et al.*, JHEP **1212**, 125 (2012)]. Note added: after this paper was submitted for publication, LHCb announced a new measurement of the differential decay rate for this process [T. Tekampe, talk at DPF 2015], which we now compare to the shape and normalization of the Standard-Model prediction.

PACS numbers: 13.20.He, 12.38.Gc, 12.15.Mm

Motivation — Hadron decays that proceed through flavor-changing neutral currents may be sensitive to new physics, because their leading Standard-Model contributions are loop suppressed. Here we study the semileptonic decay $B \rightarrow \pi \ell^+ \ell^-$, which proceeds through a $b \rightarrow d$ transition. Hadronic effects in this decay are parametrized by three form factors. In this Letter, we present the first *ab-initio* QCD calculation of the tensor form factor f_T , based on lattice-QCD work that also yielded the vector and scalar form factors, f_+ and f_0 [1]. Lattice QCD has several advantages over other approaches to the form factors [2–9], particularly in providing a path to controlled uncertainties that can be systematically reduced [10].

The LHCb experiment recently made the first observation of $B^+ \rightarrow \pi^+ \mu^+ \mu^-$ [11], while the B -factories have set limits on the $e^+ e^-$ and $\tau^+ \tau^-$ channels [12–14]. Below we present the first calculations of $B \rightarrow \pi \ell^+ \ell^-$ ($\ell = e, \mu, \tau$) observables in the Standard Model using

form factors with fully controlled uncertainties.

The form factors f_+ , f_0 , and f_T suffice to parameterize $B \rightarrow \pi$ decays in all extensions of the Standard Model. New physics from heavy particles—such as those appearing in models with supersymmetry [3, 15–17], a fourth generation [18], or extended [15, 19–23] or composite [24] Higgs sectors—alter Wilson coefficients in the effective Hamiltonian pertaining to particle physics below the electroweak scale [25–28]. Whatever these unknown particles may be, the hadronic physics remains the same.

Lattice-QCD calculation — Our work on $f_T(q^2)$ was carried out in parallel with $f_+(q^2)$ and $f_0(q^2)$. Our aim in Ref. [1] was a precise determination of the Cabibbo-Kobayashi-Maskawa (CKM) element $|V_{ub}|$, and every step of the analysis was subjected to many tests. Further, two of the authors applied a multiplicative offset to the numerical data at an early stage. This “blinding” factor was disclosed to the others only after finalizing the error analysis. Full details of the simulation parameters,

analysis, and cross-checks are given in Ref. [1].

Our calculation uses ensembles of lattice gauge-field configurations [29] from the MILC Collaboration [30–32], which are generated with a sea of up, down, and strange quarks. In practice, the up and down sea quarks have the same mass, and the strange-quark mass is tuned close to its physical value. The statistics are high, with 600–2200 gauge-field configurations per ensemble. The physical volume is large enough that we can repeat the calculation in different parts of the lattice, thereby quadrupling the statistics. We use four lattice spacings ranging from 0.12 fm to 0.045 fm to control the extrapolation to zero lattice spacing.

The tensor form factor is defined via the matrix element of the $b \rightarrow d$ tensor current $i\bar{d}\sigma^{\mu\nu}b$:

$$\langle \pi(p_\pi) | i\bar{d}\sigma^{\mu\nu}b | B(p_B) \rangle = 2 \frac{p_B^\mu p_\pi^\nu - p_B^\nu p_\pi^\mu}{M_B + M_\pi} f_T(q^2), \quad (1)$$

where p_B and p_π are the particles' momenta and $q = p_B - p_\pi$ is the momentum carried off by the leptons. The Lorentz invariant q^2 is related to the pion energy in the B -meson rest frame via $E_\pi = (M_B^2 + M_\pi^2 - q^2)/2M_B$. In the finite volume that can be simulated on a computer, E_π takes discrete values, dictated by the spatial momenta \mathbf{p}_π compatible with periodic boundary conditions. Because statistical and discretization errors increase with pion momentum, we restrict $|\mathbf{p}_\pi| \leq |2\pi(1,1,1)/L|$. The resulting simulation range of $E_\pi \lesssim 1$ GeV is significantly smaller than the kinematically allowed range of $E_\pi \leq 2.5$ GeV. Extending this discrete set of calculations into the full q^2 dependence is the central challenge of this work, and is met in two steps.

The two light quarks (up and down) have a mass larger than it should be, but the range simulated is wide and the smallest pion mass is 175 MeV, close to Nature's 140 MeV. Therefore, we can apply an effective field theory of pions—chiral perturbation theory—to extrapolate the simulation data to the physical point. We use a form of chiral perturbation theory adapted to lattice QCD, with additional terms describing the lattice-spacing dependence [33, 34] and with modifications needed for energetic final-state pions [35]. As discussed in Ref. [1], we try several fit variations. For example, we replace the loop integrals with momentum sums appropriate to the finite volume, finding negligible changes in the results. Our final fit includes next-to-next-to-next-to-leading order analytic terms and terms to model the discretization errors of the heavy quark. The latter come from an effective field theory for heavy b quarks [36–38].

Figure 1 shows the q^2 dependence of the errors after the chiral-continuum extrapolation just described. Table I gives a numerical error budget for $f_T(q^2 = 20 \text{ GeV}^2)$. The largest uncertainty comes from the statistical errors, as increased during the chiral-continuum extrapolation. This error is under good control for q^2 corresponding to the spatial momenta that we simulate, but grows large

TABLE I. Error budget in per cent for $f_T(q^2 = 20 \text{ GeV}^2)$. The first error incorporates statistical errors from the simulation and systematics associated with the chiral-continuum fit. The last column emphasizes how the error varies with q^2 .

Source of error	δf_T	q^2 dependence
Statistics \oplus χ PT \oplus HQ \oplus g_π	3.8	important
Scale r_1	0.5	negligible
Nonperturbative matching $Z_{V_{bb}^4}, Z_{V_{ll}^4}$	0.7	negligible
Perturbative matching ρ_T	2.0	none
Heavy-quark mass tuning κ_b	0.4	none
Light-quark mass tuning m_l, m_s	0.5	negligible
Total (<i>Quadrature sum of above</i>)	4.4	important

elsewhere.

The subdominant errors are as follows. To convert from lattice units to physical units, we introduce a physical distance r_1 , which is defined via the force between static quarks [39, 40]. We use it to form physical, dimensionless quantities, which are the input data for the chiral-continuum fit. At the end, we set $r_1 = 0.3117 \pm 0.0022 \text{ fm}$ [41] based on a related lattice-QCD calculation of $r_1 f_\pi$ [42] and the pion decay constant $f_\pi = 130.41 \text{ MeV}$ [43]. To propagate the parametric uncertainty in r_1 to f_T , we repeat the fit shifting r_1 by $\pm 1\sigma_{r_1}$, leading to the second line in Table I.

In lattice gauge theory, the tensor current does not have the normalization used in QCD phenomenology. We obtain most of the normalization nonperturbatively [44] from $b \rightarrow b$ and $d \rightarrow d$ transitions with the vector current, with statistical errors below 1%. Another matching factor ρ_T remains, but, by design and in practice, it is close to unity. We calculate ρ_T at the renormalization scale $\mu = m_{b,\text{pole}}$ through first order in the QCD coupling α_s .

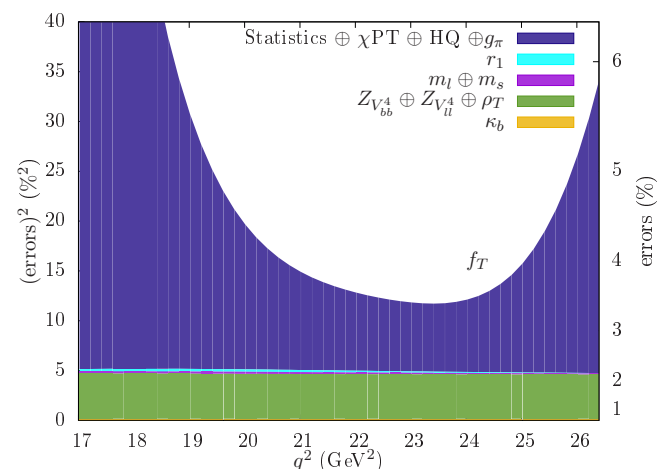


FIG. 1. (color online) Error budget for f_T as a function of q^2 for the range of simulated lattice momenta. The filled bands show the relative size of each error contribution to the total. The quadrature sum is shown on the left y axis and the error itself, in per cent, on the right.

We estimate the resulting error of order α_s^2 after removing a logarithmic dependence on the matching scale μ , which is present in continuum QCD too. We then examine how the one-loop coefficient depends on heavy-quark mass, identifying the largest value, $\rho_{T,\max}^{[1]}$. Finally, we estimate the error in ρ_T to be $2\alpha_s^2|\rho_{T,\max}^{[1]}|$, evaluating α_s on the second-finest lattice with $a \approx 0.06$ fm. This yields the 2% perturbative-matching uncertainty in Table I.

The last two uncertainties arise as follows. When generating data, we choose the simulation quark masses based on short runs and previous experience. The full analysis yields better estimates. To correct the simulation b -quark mass *a posteriori*, we recompute f_T on one ensemble with two additional values of the bare b -quark mass. Using the slope from all three mass values, we interpolate the data for f_T slightly from the production b -quark mass to the physical value. This leaves an error due to the uncertainty in the size of the b -quark mass correction. The details for f_T are nearly identical to those for f_+ [1], leading to the same estimate, 0.4%, for this error. The light-quark mass dependence is embedded in the chiral-continuum extrapolation, described above. The parametric uncertainty from the input light-quark mass [32] is propagated to f_T by repeating the fit with $\pm 1\sigma_{m_q}$ shifts to these parameters, and is given in the penultimate line of Table I.

The final line in Table I and the upper edge of the stack in Fig. 1 represent the quadrature sum of the systematic uncertainties with the chiral-continuum fit error.

Extension to all q^2 — To extend f_T in the chiral-continuum limit from the range of simulated lattice momenta to the full kinematic range, $0 < q^2 \leq (M_B - M_\pi)^2$, with controlled errors, we use a method based on the analytic structure of the form factor.

In the complex q^2 plane, $f_T(q^2)$ has a cut for timelike $q^2 \geq t_+ \equiv (M_B + M_\pi)^2$ and a pole at $q^2 = M_{B^*}^2$ but is analytic elsewhere. The variable

$$z(q^2, t_0) = \frac{\sqrt{t_+ - q^2} - \sqrt{t_+ - t_0}}{\sqrt{t_+ - q^2} + \sqrt{t_+ - t_0}} \quad (2)$$

maps the whole q^2 plane into the unit disk, with the cut mapped to the boundary and the semileptonic region mapped to an interval on the real axis. Unitarity bounds then guarantee that an expansion of f_T in z (with the B^* pole removed) converges for $|z| < 1$ [45–48]. Following Bourrely, Caprini, and Lellouch (BCL) [49], we factor out the B^* pole and expand in z :

$$(1 - q^2/M_{B^*}^2)f_T(z) = \sum_{n=0}^{N_z-1} b_n^T \left[z^n - (-1)^{n-N_z} \frac{n}{N_z} z^{N_z} \right], \quad (3)$$

choosing $t_0 = (M_B + M_\pi)(\sqrt{M_B} - \sqrt{M_\pi})^2$ to minimize $|z|$ in the semileptonic region. Although Eq. (3) was derived for the vector form factor f_+ , we use it for the tensor form

TABLE II. Best-fit values b_n with total errors and correlation matrix ρ_{nm} of the $N_z = 4$ BCL z expansion of f_T . The lower two panels show correlations with the f_+ and f_0 coefficients in Table XIV of Ref. [1] obtained from *ab-initio* QCD.

Fit: 0.393(17) -0.65(23) -0.6(1.5) 0.1(2.8)				
ρ	b_0^T	b_1^T	b_2^T	b_3^T
b_0^T	1.000	0.400	0.204	0.166
b_1^T		1.000	0.862	0.806
b_2^T			1.000	0.989
b_3^T				1.000
b_0^+	0.638	0.321	0.123	0.084
b_1^+	0.321	0.397	0.162	0.109
b_2^+	0.114	0.202	0.198	0.179
b_3^+	0.070	0.152	0.192	0.180
b_0^0	0.331	0.136	0.089	0.073
b_1^0	0.203	0.313	0.198	0.162
b_2^0	0.204	0.268	0.186	0.155
b_3^0	0.151	0.203	0.169	0.149

factor f_T because the two form factors are proportional to each other at leading order in the $1/m_b$ expansion.

We determine the b_n^T with a functional method connecting the independent functions of the chiral-continuum fit with the first several powers of z [1]. Our preferred fit uses $N_z = 4$; adding higher-order terms in z does not significantly change the central value. Table II presents our final result for f_T as coefficients of the $N_z = 4$ BCL z fit and the correlation matrix between them, where the errors include statistical and all systematic uncertainties. This information can be used to reconstruct $f_T(q^2)$ over the full kinematic range. Table II also provides the (mostly statistical) correlations between f_T , f_+ , and f_0 . Figure 2 shows the extrapolation of f_T to $q^2 = 0$. Table II and Fig. 2 represent the first main result of this Letter.

Implications — The largest contribution in the Standard Model to the amplitude for $B \rightarrow \pi \ell^+ \ell^-$ is proportional to the vector form factor. Assuming that new physics does not contribute significantly to the tree-level

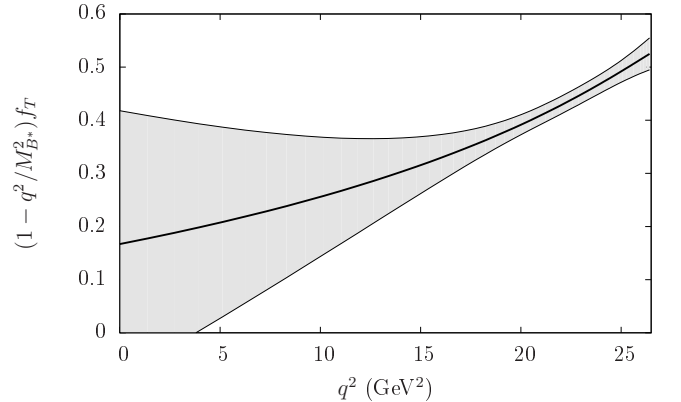


FIG. 2. *Ab-initio* result for $f_T(q^2)$ from lattice QCD.

TABLE III. Correlations between BCL coefficients for f_T with those for f_+ and f_0 from Table XIX of Ref. [1], which include experimental shape information from $B \rightarrow \pi \ell \nu$ decay.

ρ	b_0^T	b_1^T	b_2^T	b_3^T
b_0^+	0.514	0.140	0.078	0.065
b_1^+	0.111	0.221	-0.010	-0.049
b_2^+	-0.271	-0.232	-0.012	0.029
b_3^+	-0.204	-0.215	-0.013	0.023
b_0^0	0.243	-0.015	-0.025	-0.024
b_1^0	0.005	0.134	0.070	0.057
b_2^0	-0.002	-0.034	-0.032	-0.030
b_3^0	-0.044	-0.061	0.005	0.017

decay $B \rightarrow \pi \ell \nu$, one can use experimental measurements of this process to constrain the shape of $f_+(q^2)$, especially at low q^2 . In Ref. [1], we obtain the CKM element $|V_{ub}|$ from a combined z fit to our lattice-QCD results for f_+ and f_0 and measurements of $\tau_B d\Gamma(B \rightarrow \pi \ell \nu)/dq^2$ from BaBar [50, 51] and Belle [52, 53]. This joint fit also yields the most precise current determinations of f_+ and f_0 . To enable them to be combined with the results for f_T from Table II, Table III provides the correlations between the z -expansion coefficients for all three form factors. The correlations are small, because f_+ contains independent experimental information.

Using f_T from this work and f_+ and f_0 just described, we show the Standard-Model partial branching fractions for $B \rightarrow \pi \ell^+ \ell^-$ in Fig. 3. Other ingredients are needed besides the form factors. They appear many places throughout the literature, and, for convenience, some of us have collected them into an appendix of Ref. [55]. In brief, we calculate contributions that cannot be parameterized by the form factors with standard methods, employing QCD factorization at low q^2 [56–64] and an operator product expansion (OPE) in powers of $E_\pi/\sqrt{q^2}$ at large q^2 [65–72]. We take the Wilson coefficients from Ref. [27], the CKM elements from Ref. [73], the meson masses and lifetimes from Ref. [43], and the b - and c -quark masses from Ref. [7]; the numerical values for all parametric inputs used are also tabulated in Ref. [55].

Table IV presents numerical predictions for selected q^2 bins. The last error in parenthesis contains effects of parametric uncertainties in α_s , m_t , m_b , m_c ; of missing power corrections, taking 10% of contributions not directly proportional to the form factors; and of violations of quark-hadron duality, estimated to be 2% at high- q^2 [70]. At low q^2 , the uncertainty predominantly stems from the form factors; at high q^2 , the CKM elements $|V_{td}^* V_{tb}|$ and form factors each contribute similar errors. Figure 3 and Table IV represent the second main result of this Letter.

In the regions $q^2 \lesssim 1 \text{ GeV}^2$ and $6 \text{ GeV}^2 \lesssim q^2 \lesssim 14 \text{ GeV}^2$, $u\bar{u}$ and $c\bar{c}$ resonances dominate the rate. To estimate the total BR, we simply disregard them and interpolate linearly in q^2 between the QCD-factorization

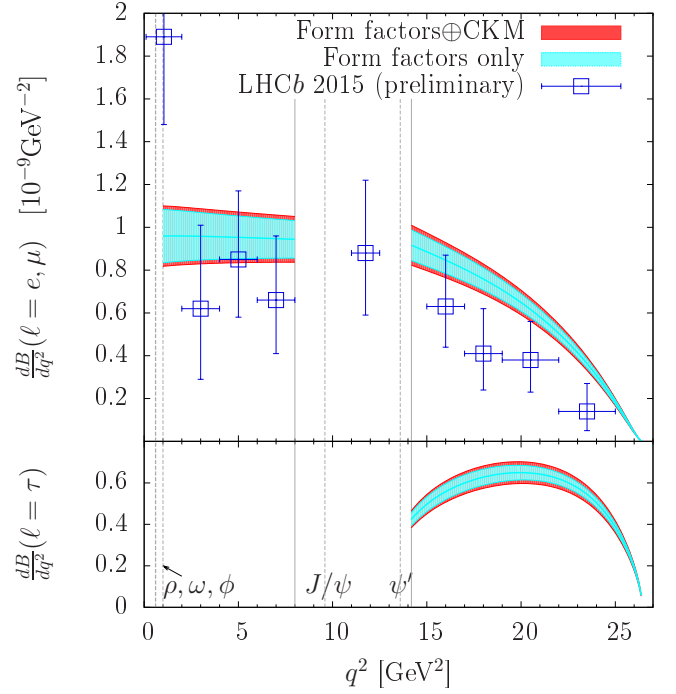


FIG. 3. (color online) Partial branching fractions for $B^+ \rightarrow \pi^+ \mu^+ \mu^-$ (upper panel) and $B^+ \rightarrow \pi^+ \tau^+ \tau^-$ (lower panel) outside the resonance regions. Different patterns (colors) show the contributions from the main sources of uncertainty; those from the remaining sources are too small to be visible. For $B^+ \rightarrow \pi^+ \mu^+ \mu^-$, new measurements from LHCb [54], which were announced after our paper appeared, are overlaid.

TABLE IV. Standard-Model predictions for $B^+ \rightarrow \pi^+ \ell^+ \ell^-$ partial branching fractions. Those for B^0 decays can be obtained by multiplying by the lifetime ratio $(\tau_{B^0}/\tau_{B^+})/2 = 0.463$. Errors shown are from the CKM elements, form factors, variation of the high and low matching scales, and the quadrature sum of all other contributions, respectively.

$[q_{\min}^2, q_{\max}^2]$ (GeV ²)	$10^9 \times \text{BR}(B^+ \rightarrow \pi^+ \ell^+ \ell^-)$	
	$\ell = e, \mu$	$\ell = \tau$
[0.1, 2.0]	1.81(11,24,6,2)	
[2.0, 4.0]	1.92(11,22,6,3)	
[4.0, 6.0]	1.91(11,20,6,3)	
[6.0, 8.0]	1.89(11,18,5,3)	
[15, 17]	1.69(10,13,3,5)	1.11(7,8,2,4)
[17, 19]	1.52(9,10,2,4)	1.25(8,8,2,3)
[19, 22]	1.84(11,11,3,5)	1.93(12,10,4,5)
[22, 25]	1.07(6,6,3,3)	1.59(10,7,4,4)
[1, 6]	4.78(29,54,15,6)	
[15, 22]	5.05(30,34,7,15)	4.29(26,25,7,12)
$[4m_\ell^2, 26.4]$	20.4(1.2,1.6,0.3,0.5)	

result at $q^2 \approx 8.5 \text{ GeV}^2$ and the OPE result at $q^2 \approx 13 \text{ GeV}^2$. While this treatment does not yield the full branching ratio, it does enable a comparison with LHCb's published result, $\text{BR}(B^+ \rightarrow \pi^+ \mu^+ \mu^-) = 23(6) \times 10^{-9}$ [11], which was obtained from a similar interpolation over these regions. Our result $\text{BR}(B^+ \rightarrow$

$\pi^+\mu^+\mu^- = 20.4(2.1) \times 10^{-9}$ agrees with LHCb, and is more precise than the best previous theoretical estimate [7] because we use f_T directly, which avoids a large uncertainty from varying the matching scale μ .

Outlook – The largest uncertainty in our determination of the $B \rightarrow \pi$ form factors is the combined error from statistics with chiral-extrapolation and discretization effects included. We will be able to reduce these with calculations on the MILC Collaboration’s recently generated four-flavor ensembles with physical light-quark masses [74]. LHCb’s measurement of $\text{BR}(B^+ \rightarrow \pi^+\mu^+\mu^-)$ will improve, and Belle II expects to observe the neutral decay mode $B^0 \rightarrow \pi^0\ell^+\ell^-$. If a deviation from the Standard Model is observed, our form factors can be used to compute other observables such as asymmetries, thereby providing information about new heavy particles, such as their masses, spin, and couplings.

Note added: after this paper was submitted for publication, the LHCb experiment announced a new measurement for the $B^+ \rightarrow \pi^+\mu^+\mu^-$ differential decay rate [54]. The new results are shown in Fig. 3. The large difference in the lowest q^2 bin is due to the presence of light (ρ, ω, ϕ) resonances, whose effects are important but cannot be estimated in a model-independent manner. Given the present experimental and theoretical uncertainties, it is too early to discern possible new-physics contributions to this process.

Acknowledgements – We thank Ulrik Egede and Tobias Tekampe from LHCb for useful correspondence. Computations for this work were carried out with resources provided by the USQCD Collaboration, the Argonne Leadership Computing Facility, the National Energy Research Scientific Computing Center, and the Los Alamos National Laboratory, which are funded by the Office of Science of the United States Department of Energy; and with resources provided by the National Institute for Computational Science, the Pittsburgh Supercomputer Center, the San Diego Supercomputer Center, and the Texas Advanced Computing Center, which are funded through the National Science Foundation’s Teragrid/XSEDE Program. This work was supported in part by the U.S. Department of Energy under Grants No. DE-FG02-91ER40628 (C.B.), No. DE-FC02-06ER41446 (C.D., L.L., S.-W.Q.), No. DE-SC0010120 (S.G.), No. DE-FG02-91ER40661 (S.G.), No. DE-FG02-13ER42001 (D.D., A.X.K.), No. DE-FG02-91ER40664 (Y.M.), and No. DE-FG02-13ER41976 (D.T.); by the National Science Foundation under Grants No. PHY-1067881, No. PHY-10034278 (C.D., L.L., S.-W.Q.), No. PHY-1417805 (J.L., D.D.), and No. PHY-1316748 (R.S.); by the URA Visiting Scholars’ program (A.X.K., Y.M.); by the MINECO (Spain) under Grant FPA2013-47836-C3-1-P, and the Ramón y Cajal program (E.G.); by the Junta de Andalucía (Spain) under Grants FQM-101 and FQM-6552 (E.G.); by the European Commission under Grant No. PCIG10-GA-2011-303781 (E.G.);

by the German Excellence Initiative, the European Union Seventh Framework Programme under grant agreement No. 291763, and the European Union’s Marie Curie CO-FUND program (A.S.K.); and by the Basic Science Research Program of the National Research Foundation of Korea (NRF) funded by the Ministry of Education (No. 2014027937) and the Creative Research Initiatives Program (No. 2014001852) of the NRF grant funded by the Korean government (MEST) (J.A.B). Brookhaven National Laboratory is supported by the U. S. Department of Energy under Contract No. DE-SC0012704. Fermilab is operated by Fermi Research Alliance, LLC, under Contract No. DE-AC02-07CH11359 with the U.S. Department of Energy.

* dadu@syr.edu

† elunghi@indiana.edu

‡ ruthv@fnal.gov

- [1] J. A. Bailey *et al.* (Fermilab Lattice, MILC), *Phys. Rev.* **D92**, 014024 (2015), [arXiv:1503.07839 \[hep-lat\]](#).
- [2] P. Ball and R. Zwicky, *Phys. Rev.* **D71**, 014015 (2005), [arXiv:hep-ph/0406232](#).
- [3] J.-J. Wang, R.-M. Wang, Y.-G. Xu, and Y.-D. Yang, *Phys. Rev.* **D77**, 014017 (2008), [arXiv:0711.0321 \[hep-ph\]](#).
- [4] G. Duplancić, A. Khodjamirian, T. Mannel, B. Melić, and N. Offen, *JHEP* **0804**, 014 (2008), [arXiv:0801.1796 \[hep-ph\]](#).
- [5] X.-G. Wu and T. Huang, *Phys. Rev.* **D79**, 034013 (2009), [arXiv:0901.2636 \[hep-ph\]](#).
- [6] R. N. Faustov and V. O. Galkin, *Eur. Phys. J.* **C74**, 2911 (2014), [arXiv:1403.4466 \[hep-ph\]](#).
- [7] A. Ali, A. Y. Parkhomenko, and A. V. Rusov, *Phys. Rev.* **D89**, 094021 (2014), [arXiv:1312.2523 \[hep-ph\]](#).
- [8] Z.-H. Li, Z.-G. Si, Y. Wang, and N. Zhu, (2014), [arXiv:1411.0466 \[hep-ph\]](#).
- [9] C. Hambrook, A. Khodjamirian, and A. Rusov, (2015), [arXiv:1506.07760 \[hep-ph\]](#).
- [10] A. S. Kronfeld, *Annu. Rev. Nucl. Part. Sci.* **62**, 265 (2012), [arXiv:1203.1204 \[hep-lat\]](#).
- [11] R. Aaij *et al.* (LHCb Collaboration), *JHEP* **1212**, 125 (2012), [arXiv:1210.2645 \[hep-ex\]](#).
- [12] J.-T. Wei *et al.* (Belle Collaboration), *Phys. Rev.* **D78**, 011101 (2008), [arXiv:0804.3656 \[hep-ex\]](#).
- [13] J. P. Lees *et al.* (BaBar Collaboration), *Phys. Rev.* **D88**, 032012 (2013), [arXiv:1303.6010 \[hep-ex\]](#).
- [14] O. Lutz *et al.* (Belle Collaboration), *Phys. Rev.* **D87**, 111103 (2013), [arXiv:1303.3719 \[hep-ex\]](#).
- [15] C. Bobeth, T. Ewerth, F. Krüger, and J. Urban, *Phys. Rev.* **D64**, 074014 (2001), [arXiv:hep-ph/0104284](#).
- [16] D. A. Demir, K. A. Olive, and M. B. Voloshin, *Phys. Rev.* **D66**, 034015 (2002), [arXiv:hep-ph/0204119](#).
- [17] S. R. Choudhury and N. Gaur, *Phys. Rev.* **D66**, 094015 (2002), [arXiv:hep-ph/0206128](#).
- [18] W.-S. Hou, M. Kohda, and F. Xu, *Phys. Rev.* **D87**, 094005 (2013), [arXiv:1302.1471 \[hep-ph\]](#).
- [19] T. M. Aliev and M. Savci, *Phys. Rev.* **D60**, 014005 (1999), [arXiv:hep-ph/9812272 \[hep-ph\]](#).

- [20] E. O. Iltan, *Int. J. Mod. Phys. A* **14**, 4365 (1999), arXiv:hep-ph/9807256 [hep-ph].
- [21] G. Erkol and G. Turan, *JHEP* **0202**, 015 (2002), arXiv:hep-ph/0201055 [hep-ph].
- [22] G. Erkol, J. W. Wagenaar, and G. Turan, *Eur. Phys. J. C* **41**, 189 (2005), arXiv:hep-ph/0408186 [hep-ph].
- [23] H.-Z. Song, L.-X. Lu, and G.-R. Lu, *Commun. Theor. Phys.* **50**, 696 (2008).
- [24] B. Gripaio, M. Nardecchia, and S. A. Renner, *JHEP* **1505**, 006 (2015), arXiv:1412.1791 [hep-ph].
- [25] B. Grinstein, M. J. Savage, and M. B. Wise, *Nucl. Phys. B* **319**, 271 (1989).
- [26] A. J. Buras, M. Misiak, M. Münz, and S. Pokorski, *Nucl. Phys. B* **424**, 374 (1994), arXiv:hep-ph/9311345.
- [27] T. Huber, E. Lunghi, M. Misiak, and D. Wyler, *Nucl. Phys. B* **740**, 105 (2006), arXiv:hep-ph/0512066.
- [28] W. Altmannshofer, P. Ball, A. Bharucha, A. J. Buras, D. M. Straub, and M. Wick, *JHEP* **0901**, 019 (2009), arXiv:0811.1214 [hep-ph].
- [29] C. Aubin *et al.* (MILC Collaboration), “asqtad ensemble 2064f21b676m010m050” (2015); “asqtad ensemble 2064f21b676m010m050b” (2015); “asqtad ensemble 2064f21b676m007m050” (2015); “asqtad ensemble 2064f21b676m007m050b” (2015); “asqtad ensemble 2464f21b676m005m050” (2015); “asqtad ensemble 2896f21b709m0062m031” (2015); “asqtad ensemble 2896f21b709m0062m031b” (2015); “asqtad ensemble 3296f21b7085m00465m031” (2015); “asqtad ensemble 4096f21b708m0031m031a” (2015); “asqtad ensemble 4096f21b708m0031m031b” (2015); “asqtad ensemble 6496f21b7075m00155m031” (2015); “asqtad ensemble 48144f21b748m0072m018a” (2015); “asqtad ensemble 48144f21b748m0072m018b” (2015); “asqtad ensemble 48144f21b747m0036m018a” (2015); “asqtad ensemble 48144f21b747m0036m018b” (2015); “asqtad ensemble 56144f21b7465m0025m018” (2015); “asqtad ensemble 64144f21b746m0018m018a” (2015); “asqtad ensemble 64144f21b746m0018m018b” (2015); “asqtad ensemble 64192f21b781m0028m014” (2015).
- [30] C. W. Bernard *et al.* (MILC Collaboration), *Phys. Rev. D* **64**, 054506 (2001), arXiv:hep-lat/0104002 [hep-lat].
- [31] C. Aubin *et al.* (MILC Collaboration), *Phys. Rev. D* **70**, 094505 (2004), arXiv:hep-lat/0402030 [hep-lat].
- [32] A. Bazavov, D. Toussaint, C. Bernard, J. Laiho, C. Detar, L. Levkova, M. B. Oktay, S. Gottlieb, U. M. Heller, J. E. Hetrick, P. B. Mackenzie, R. Sugar, and R. S. Van de Water, *Rev. Mod. Phys.* **82**, 1349 (2010), arXiv:0903.3598 [hep-lat].
- [33] C. Aubin and C. Bernard, *Phys. Rev. D* **76**, 014002 (2007), arXiv:0704.0795 [hep-lat].
- [34] J. A. Bailey *et al.* (Fermilab Lattice and MILC Collaborations), (2015), in preparation.
- [35] J. Bijnens and I. Jemos, *Nucl. Phys. B* **840**, 54 (2010), arXiv:1006.1197 [hep-ph].
- [36] A. S. Kronfeld, *Phys. Rev. D* **62**, 014505 (2000), arXiv:hep-lat/0002008 [hep-lat].
- [37] J. Harada, S. Hashimoto, K.-I. Ishikawa, A. S. Kronfeld, T. Onogi, and N. Yamada, *Phys. Rev. D* **65**, 094513 (2002), *Phys. Rev. D* **71** 019903(E) (2005), arXiv:hep-lat/0112044.
- [38] M. B. Oktay and A. S. Kronfeld, *Phys. Rev. D* **78**, 014504 (2008), arXiv:0803.0523 [hep-lat].
- [39] C. W. Bernard *et al.* (MILC Collaboration), *Phys. Rev. D* **62**, 034503 (2000), arXiv:hep-lat/0002028 [hep-lat].
- [40] R. Sommer, *Nucl. Phys. B* **411**, 839 (1994), arXiv:hep-lat/9310022.
- [41] A. Bazavov *et al.* (Fermilab Lattice and MILC Collaborations), *Phys. Rev. D* **85**, 114506 (2012), arXiv:1112.3051 [hep-lat].
- [42] A. Bazavov *et al.* (MILC Collaboration), *PoS CD09*, 007 (2009), arXiv:0910.2966 [hep-ph].
- [43] K. A. Olive *et al.* (Particle Data Group), *Chin. Phys. C* **38**, 090001 (2014).
- [44] A. X. El-Khadra, A. S. Kronfeld, P. B. Mackenzie, S. M. Ryan, and J. N. Simone, *Phys. Rev. D* **64**, 014502 (2001), arXiv:hep-ph/0101023.
- [45] C. Bourrely, B. Machet, and E. de Rafael, *Nucl. Phys. B* **189**, 157 (1981).
- [46] C. G. Boyd, B. Grinstein, and R. F. Lebed, *Phys. Rev. Lett.* **74**, 4603 (1995), arXiv:hep-ph/9412324.
- [47] L. Lellouch, *Nucl. Phys. B* **479**, 353 (1996), arXiv:hep-ph/9509358.
- [48] C. G. Boyd and M. J. Savage, *Phys. Rev. D* **56**, 303 (1997), arXiv:hep-ph/9702300.
- [49] C. Bourrely, I. Caprini, and L. Lellouch, *Phys. Rev. D* **79**, 013008 (2009), arXiv:0807.2722 [hep-ph].
- [50] P. del Amo Sanchez *et al.* (BaBar Collaboration), *Phys. Rev. D* **83**, 032007 (2011), arXiv:1005.3288 [hep-ex].
- [51] J. P. Lees *et al.* (BaBar Collaboration), *Phys. Rev. D* **86**, 092004 (2012), arXiv:1208.1253 [hep-ex].
- [52] H. Ha *et al.* (Belle Collaboration), *Phys. Rev. D* **83**, 071101 (2011), arXiv:1012.0090 [hep-ex].
- [53] A. Sibidanov *et al.* (Belle Collaboration), *Phys. Rev. D* **88**, 032005 (2013), arXiv:1306.2781 [hep-ex].
- [54] T. Tekampe (LHCb), <https://indico.cern.ch/event/361123/session/4/contribution/409> (2015), talk presented at DPF 2015.
- [55] D. Du, A. X. El Khadra, S. Gottlieb, A. S. Kronfeld, J. Laiho, E. Lunghi, R. S. Van de Water, and R. Zhou (Fermilab Lattice and MILC Collaborations), (2015), in preparation.
- [56] M. Beneke, G. Buchalla, M. Neubert, and C. T. Sachrajda, *Phys. Rev. Lett.* **83**, 1914 (1999), arXiv:hep-ph/9905312 [hep-ph].
- [57] M. Beneke, G. Buchalla, M. Neubert, and C. T. Sachrajda, *Nucl. Phys. B* **591**, 313 (2000), arXiv:hep-ph/0006124 [hep-ph].
- [58] M. Beneke and T. Feldmann, *Nucl. Phys. B* **592**, 3 (2001), arXiv:hep-ph/0008255.
- [59] H. H. Asatrian, H. M. Asatrian, C. Greub, and M. Walker, *Phys. Lett. B* **507**, 162 (2001), arXiv:hep-ph/0103087 [hep-ph].
- [60] M. Beneke, T. Feldmann, and D. Seidel, *Nucl. Phys. B* **612**, 25 (2001), arXiv:hep-ph/0106067 [hep-ph].
- [61] H. H. Asatryan, H. M. Asatrian, C. Greub, and M. Walker, *Phys. Rev. D* **65**, 074004 (2002), arXiv:hep-ph/0109140 [hep-ph].
- [62] H. M. Asatrian, K. Bieri, C. Greub, and M. Walker, *Phys. Rev. D* **69**, 074007 (2004), arXiv:hep-ph/0312063 [hep-ph].
- [63] M. Beneke, T. Feldmann, and D. Seidel, *Eur. Phys. J. C* **41**, 173 (2005), arXiv:hep-ph/0412400 [hep-ph].
- [64] C. Bobeth, G. Hiller, and G. Piranishvili, *JHEP* **0712**, 040 (2007), arXiv:0709.4174 [hep-ph].
- [65] B. Grinstein and D. Pirjol, *Phys. Lett. B* **533**, 8 (2002), arXiv:hep-ph/0201298 [hep-ph].
- [66] D. Seidel, *Phys. Rev. D* **70**, 094038 (2004), arXiv:hep-ph/0403185 [hep-ph].

- [67] B. Grinstein and D. Pirjol, *Phys. Rev.* **D70**, 114005 (2004), [arXiv:hep-ph/0404250 \[hep-ph\]](#).
- [68] C. Greub, V. Pilipp, and C. Schupbach, *JHEP* **0812**, 040 (2008), [arXiv:0810.4077 \[hep-ph\]](#).
- [69] C. Bobeth, G. Hiller, and D. van Dyk, *JHEP* **1007**, 098 (2010), [arXiv:1006.5013 \[hep-ph\]](#).
- [70] M. Beylich, G. Buchalla, and T. Feldmann, *Eur. Phys. J.* **C71**, 1635 (2011), [arXiv:1101.5118 \[hep-ph\]](#).
- [71] C. Bobeth, G. Hiller, and D. van Dyk, *JHEP* **1107**, 067 (2011), [arXiv:1105.0376 \[hep-ph\]](#).
- [72] C. Bobeth, G. Hiller, D. van Dyk, and C. Wacker, *JHEP* **1201**, 107 (2012), [arXiv:1111.2558 \[hep-ph\]](#).
- [73] J. Charles *et al.* (CKMfitter Group), *Eur. Phys. J.* **C41**, 1 (2005), updates at <http://ckmfitter.in2p3.fr>, [arXiv:hep-ph/0406184 \[hep-ph\]](#).
- [74] A. Bazavov *et al.* (MILC Collaboration), *Phys. Rev.* **D87**, 054505 (2013), [arXiv:1212.4768 \[hep-lat\]](#).