# Improving the Precision of Weak Measurements by Postselection Measurement

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# Improving the precision of weak measurements by postselection measurement

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Postselected weak measurement is a useful protocol to amplify weak physical effects. However, there has recently been controversy over whether it gives any advantage in precision. While it is now clear that retaining failed postselections can yield more Fisher information than discarding them, the advantage of postselection measurement itself still remains to be clarified. In this Letter, we address this problem by studying two widely used estimation strategies: averaging measurement results, and maximum likelihood estimation, respectively. For the first strategy, we find a surprising result that squeezed coherent states of the pointer can give postselected weak measurements a higher signal-to-noise ratio than standard ones while all standard coherent states cannot, which suggests that raising the precision of weak measurements by postselection calls for the presence of "nonclassicality" in the pointer states. For the second strategy, we show that the quantum Fisher information of postselected weak measurements is generally larger than that of standard weak measurements, even without using the failed postselection events, but the gap can be closed with a proper choice of system state.

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*Introduction.*— Postselected weak measurement is a quantum measurement protocol first invented by Aharonov, Albert, and Vaidman in 1988 [1]. It involves weak coupling between the system and the pointer, but the postselection on the system leads to a surprisingly counterintuitive effect: the average shift of the final pointer state can go far beyond the eigenvalue spectrum of the system observable (multiplied by the coupling constant), in sharp contrast to the projective quantum measurement. The mechanism behind this effect is the coherence between the pointer states translated by different eigenvalues of the system observable, which has an enlightening interpretation based on superoscillation [2].

Postselected weak measurement has aroused enormous research interest in different fields, due to its ability to amplify tiny physical effects. Thanks to technical progress in recent years, the weak value has been measured in experiments [3–6], and postselected weak measurements have been applied to measuring small parameters in various systems, including optical systems [7–23], atomic systems [24] and NMR [25]. More experimental protocols have also been proposed [26–38]. A general framework for postselected weak measurement is given in [39], and reviews of the field can be found in [40–42]. Of course, weak value amplification cannot be arbitrarily large in practice. The condition for the validity of the weak value formalism was discussed in [43], and the limit of amplification has been studied in [44–47].

One of the major goals in postselected weak measurement is to enhance the sensitivity of estimating small parameters. Starling *et al.*'s experiment [9] and Feizpour *et al.*'s proposal [27] showed that postselection can significantly raise the signal-to-noise ratio (SNR) of weak measurement. Nevertheless, some other work has led to a negative conclusion [48]. In recent researches, it was shown that the failed postselections contain Fisher information [49–51], and even the distribution probabilities of postselection results can carry Fisher information [51]; thus, discarding postselection results will generally lead to a loss of precision [52].

To address the issue of low postselection efficiency, Dressel *et al.* [33] and Lyons *et al.* [38] proposed recycling the unpostselected photons to improve the precision. And it was later found [53–55] that the successful postselections can concentrate most of the Fisher information in the pointer, and the Fisher information of postselected weak measurement can approximately reach the Heisenberg limit [51, 53–56]. More surprising, weak value amplification can improve the precision in the presence of technical noise [27, 53, 55], and technical noise may increase the SNR of postselected weak measurement [57]. A review of the controversy over the advantage of weak value amplification is given in [58].

The postselection in a weak measurement includes two steps: first, measure the system, second, postselect the measurement results. Most previous researches focused on whether failed postselections should be retained or not, provided the system is measured. However, a more fundamental problem is whether the system should be measured at all in order to enhance the precision of weak measurement. If the measurement on the system could not give any advantage, then it would become meaningless to study whether the failed postselections should be used or not. So, this question lies at the heart of postselected weak measurement: what is the significance of measuring the system in a weak measurement, compared to the standard weak measurement (i.e. without measuring the system)? Since postselecting and non-postselecting the results of measuring the system only lead to a negligible difference in the Fisher information [54, 55], we will focus only on comparing postselected weak measurement to the standard weak measurement.

At first glance, this question seems easy to answer: since measuring the system with proper postselection can amplify the signal, the SNR can then be also increased. However, the efficiency of postselection is rather low, which may cancel the benefits of the amplification effect in the SNR, so the problem becomes subtle. In fact, the numerical results in [59] showed that postselecting the system with Gaussian pointer states cannot improve the SNR compared with standard weak measurements, and [60] found similar results for the Fisher information of measuring the position or momentum of the pointer, with the pointer states being real or Gaussian and the weak values being real or imaginary respectively.

However, it is important to note that those studies did not optimize over the choice of the system and pointer states, so they do not rule out the existence of other choices that may allow postselected weak measurements to have higher precision. In particular, the Gaussian states considered heretofore are quite "classical", so it is of great interest whether using more "quantum" states can bring any advantage for precision. In fact, it has been shown that nonclassical quantum states can be favorable to some other weak measurement protocols, e.g. consecutive violations of CHSH inequalities [61]. Moreover, the measurement basis of the pointer was not optimized either, hence it is also possible to have the Fisher information increased by measurements other than those along the basis of position or momentum of the pointer.

Answering these questions will clarify the advantage of postselection in weak measurements, and it is exactly the aim of this paper. We study the optimal precision of both postselected and standard weak measurements for general system and pointer states, and investigate when or whether postselected weak measurements can have higher precision than standard weak measurements. Moreover, different estimation strategies may also influence the precision, so we consider two principal estimation strategies: averaging the measurement results of the pointer (AMR), and maximum likelihood estimation (MLE), both of which have been widely used in practice.

For the strategy of AMR, an interesting result we find is that all standard (i.e., unsqueezed) coherent states do not give weak measurements an improvement in SNR with postselection, but properly squeezed coherent states do. This suggests that for weak value amplification to enhance the precision, a necessary ingredient is some *nonclassicality* in the initial pointer states, which was missing from previous studies. This result extends the understanding and feasibility of postselected weak measurement in parameter estimation.

For the strategy of MLE, we obtain the optimum quantum Fisher information, and show that even without using the failed postselections, the quantum Fisher information of postselected weak measurements is generally higher than that of standard weak measurements.

*Weak value formalism.*— We first review the weak value formalism for postselected weak measurement. Suppose the initial state of the system is $|\Psi_i\rangle$ and the initial state of the pointer is $|D\rangle$. The interaction Hamiltonian between the system and the pointer is

$$H_{\rm int} = gA \otimes \Omega\delta(t-t_0), \tag{1}$$

where the $\delta$ function indicates that the interaction is instantaneous at time $t_0$. Let $\hbar = 1$. After the interaction (1), the system is postselected to $|\Phi_f\rangle$, then the state of the pointer collapses to $|D_f\rangle = \langle\Phi_f|\exp(-{\rm i}gA\otimes\Omega)|\Phi_i\rangle|D\rangle$ (unnormalized). It can be derived that $|D_f\rangle \approx \langle\Phi_f|\Phi_i\rangle(1-{\rm i}gA_w\Omega)|D\rangle$ when $gA_w \ll 1$, where $A_w$ is the *weak value*, defined as

$$A_w = \frac{\langle\Phi_f|A|\Phi_i\rangle}{\langle\Phi_f|\Phi_i\rangle}, \tag{2}$$

If one measures an observable $M$ on the pointer state $|D_f\rangle$, it can be obtained [62] that the average shift is

$$\begin{aligned}\langle\Delta M\rangle_f \approx\, & g{\rm Im}A_w(\langle\{\Omega, M\}\rangle_D - 2\langle\Omega\rangle_D\langle M\rangle_D) \\ & + {\rm i}g{\rm Re}A_w\langle[\Omega, M]\rangle_D,\end{aligned} \tag{3}$$

where $\langle D|\cdot|D\rangle$ is denoted as $\langle\cdot\rangle_D$ for short. And the success probability of postselection is $P_s \approx |\langle\Phi_f|\Phi_i\rangle|^2$.

The weak value (2) can be very large when $\langle\Phi_f|\Phi_i\rangle \ll 1$, and the dependence of $\langle\Delta M\rangle_f$ on $A_w$ in Eq. (3) indicates that the average shift can go beyond any eigenvalue of $A$ in this case. This is the origin of the weak value amplification.

*Optimal signal-to-noise ratio.*— We first study the precision of postselected weak measurement, then compare it with that of standard weak measurement, to determine when or whether postselection can assist weak measurement in precision.

To quantify the precision of estimating the parameter $g$, a widely used benchmark is the signal-to-noise ratio of the estimates, defined as

$${\rm SNR}_{\rm post} = \frac{\sqrt{NP_s}\langle\Delta M\rangle_f}{\sqrt{{\rm Var}(M)_f}}, \tag{4}$$

where $N$ is the total number of measurements. The factor $\sqrt{P_s}$ is due to ${\rm Var}(M)_f$ scales inversely with the number of successful postselections. In the first order approximation with respect to $g$, the spread of the pointer wave function is almost unchanged, so ${\rm Var}(M)_f \approx {\rm Var}(M)_D$.

Note that the quantity defined in Eq. (4) is the SNR of the AMR estimator, not of the measurement results, and it is directly related to the estimation precision of postselected weak measurement [62].

With different pre- and postselections of the system, the SNR is usually different, so a proper measure for the precision of postselected weak measurement is the maximum SNR over all possible pre- and postselections. Direct maximization of the SNR by usual means (such as the variation method) is rather difficult, since the variation of ${\rm SNR}_{\rm post}$ (4) produces a nonlinear equation that is not easy to deal with.

However, the results of [54] offer an alternative possible approach to this hard problem. In that paper, the largest success probability over all postselections of the system for a given weak value $A_w$ was shown to be

$$\max_{|\Phi_f\rangle} P_s \approx \frac{\mathrm{Var}(A)_i}{\langle A^2\rangle_i - 2\langle A\rangle_i \mathrm{Re} A_w + |A_w|^2}, \tag{5}$$

where $\langle\cdot\rangle_i$ is short for $\langle\Phi_i|\cdot|\Phi_i\rangle$. By exploiting this result, the task of maximizing the SNR over all pre- and post-selections can be simplified to maximizing over all weak values $A_w$.

Usually the weak value $A_w$ is complex, and can be denoted as $A_w = |A_w|\mathrm{e}^{\mathrm{i}\theta}$, so we can follow a two-step procedure to obtain the maximum of the $\mathrm{SNR}_{\mathrm{post}}$ over $A_w$: first, maximize $\mathrm{SNR}_{\mathrm{post}}$ over $|A_w|$, then maximize it over $\theta$.

The mathematical detail of this optimization is left for the Supplemental Material. The result of the maximized $\mathrm{SNR}_{\mathrm{post}}$ turns out to be

$$g\eta(\varphi)\sqrt{N\frac{(\langle\{\Omega,M\}\rangle_D - 2\langle\Omega\rangle_D\langle M\rangle_D)^2 + |\langle[\Omega,M]\rangle_D|^2}{\mathrm{Var}(M)_D}}, \tag{6}$$

where $\varphi = \arctan\frac{\mathrm{i}\langle[\Omega,M]\rangle_D}{\langle\{\Omega,M\}\rangle_D - 2\langle\Omega\rangle_D\langle M\rangle_D}$ and $\eta(\varphi) = \sqrt{\mathrm{Var}(A)_i + \langle A\rangle_i^2 \sin^2\varphi}$.

In [62], we also obtained an upper bound on the optimal $\mathrm{SNR}_{\mathrm{post}}$ based on (6).

*When can SNR be increased?*— The maximum SNR (6) quantifies the metrological performance of postselected weak measurement. To address the question of when (or whether) postselection can improve the SNR of weak measurement, we need to further compare (6) with the maximum SNR of standard weak measurement.

Before proceeding with this question, it is helpful to note that in the average shift of the pointer (3), the real part of the weak value is assigned with the commutator between $\Omega$, $M$ and the imaginary part with the covariance between $\Omega$, $M$. These coefficients can be quite large with proper pointer states, and will not be counterbalanced by the low postselection probability while the weak values may be. So it opens the possibility of increasing the SNR by postselection.

In a standard weak measurement, the average shift in the observable $M$ on the post-interaction pointer state is $\langle\Delta M\rangle = \mathrm{i}g\langle A\rangle_i\langle[\Omega,M]\rangle_D$ [62], and $\max\langle A\rangle_i = \lambda_{\max}(A)$, so the optimal SNR is

$$\max \mathrm{SNR}_{\mathrm{std}} = g\frac{\sqrt{N}|\lambda_{\max}(A)\langle[\Omega,M]\rangle_D|}{\sqrt{\mathrm{Var}(M)_D}}. \tag{7}$$

The ratio between the optimal SNR of postselected and standard weak measurements is therefore

$$s = \frac{\sqrt{\mathrm{Var}(A)_i \csc^2\varphi + \langle A\rangle_i^2}}{|\lambda_{\max}(A)|}. \tag{8}$$

Obviously, since $\csc^2\varphi \geq 1$, when $|\Phi_i\rangle \to |\lambda_{\max}(A)\rangle$, $\sqrt{\mathrm{Var}(A)_i\csc^2\varphi + \langle A\rangle_i^2} \geq |\lambda_{\max}(A)|$, and thus $s \geq 1$, which means that postselection in weak measurement will not reduce the SNR at least. But this is still not enough. The key question is when (or whether) $\csc^2\varphi > 1$ can hold, so that postselection gives an increase of the SNR compared with standard weak measurement.

To answer this question, we move to Fock space. Suppose that $\Omega = q$, $M = p$. Then $[\Omega,M]_D = \mathrm{i}$. In Fock space, $q$ and $p$ can be represented by $q = (a+a^\dagger)/\sqrt{2}$, $p = (a-a^\dagger)/\sqrt{2}\mathrm{i}$, so $\{q,p\} = \mathrm{i}(a^{\dagger 2} - a^2)$, and $\csc^2\varphi = 1 + \left|\langle a^{\dagger 2}\rangle + \langle a\rangle_D^2 - \langle a^\dagger\rangle_D^2 - \langle a^2\rangle_D\right|^2$.

When the initial pointer state $|D\rangle$ is a standard coherent state, $\csc^2\varphi = 1$, so standard coherent states cannot give postselected weak measurements any advantage in SNR over standard weak measurements. This generalizes the results of [59, 60], and suggests that "classical" pointer states are not able to improve the SNR of postselected weak measurements.

An interesting question is whether introducing "nonclassicality" to the pointer state can "activate" the advantage of postselected weak measurement in SNR. Consider squeezed coherent states for the pointer. Suppose the initial state $|D\rangle$ of the pointer is

$$|\xi,\alpha\rangle = \exp\frac{1}{2}(\xi^* a^2 - \xi a^{\dagger 2})|\alpha\rangle, \tag{9}$$

where $\xi$ is the squeeze parameter. Let $\xi = r\mathrm{e}^{\mathrm{i}\theta}$, then one can find [62]

$$\csc^2\varphi = 1 + 4(\sin\theta\sinh r\cosh r)^2. \tag{10}$$

It is clear from (10) that when $\sin\theta \neq 0$, one can acquire $\csc^2\varphi > 1$ with a large $r$, so according to (8), if $\mathrm{Var}(A)_i \neq 0$, the SNR of postselected weak measurements exceeds that of standard weak measurements in this case. This shows that nonclassicality really can assist the postselection to improve the SNR of weak measurements! It is in a similar spirit to Ref. [27]: correlations, classical or quantum, can increase the SNR of weak measurements.

To illustrate the above result, Fig. 1 plots the contours of the ratio $s$ on the complex plane of $\xi$ for the squeezed vacuum state $|\xi,0\rangle$. Improvement of SNR can be explicitly observed in the figure.

Why are squeezed coherent states more beneficial to the SNR than standard coherent states? It can be roughly understood from the following. The SNR of postselected weak measurement can be shown to be bounded by $\sqrt{\mathrm{Var}(\Omega)_D}$ [62], and the SNR of standard weak measurement is proportional to $1/\sqrt{\mathrm{Var}(M)_D}$ (see Eq. (7)). The ratio between them is approximately $\sqrt{\mathrm{Var}(\Omega)_D\mathrm{Var}(M)_D}$. Since coherent states have minimal uncertainty, $\sqrt{\mathrm{Var}(\Omega)_D\mathrm{Var}(M)_D}$ does not change and keeps the minimum for conjugate quadratures $\Omega$ and $M$. In contrast, squeezing can increase $\mathrm{Var}(\Omega)_D$ and decrease

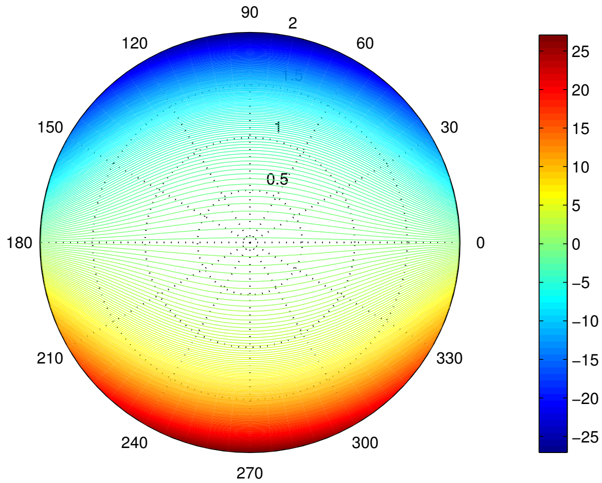


Figure 1. (Color online) The contours of $s$ are plotted for the squeezed vacuum states $|\xi,0\rangle = \exp\frac{1}{2}(\xi^* a^2 - \xi a^{\dagger 2})|0\rangle$ with $|\xi| \le 2$ and $|\arg\xi| \le \pi$. The interaction Hamiltonian is $g\sigma_z \otimes q$ with $g = 10^{-5}$. The weak value is fixed to 20i. The momentum $p$ is measured on the pointer after the postselection. Each point in the figure represents a $\xi$ on the complex plane, and the color indicates the corresponding value of $s$, which is the ratio between the SNR of postselected and standard weak measurements. It clearly shows $|s|$ can be much larger than 1 with proper $\xi$, implying an increase in the SNR by postselecting the system. The sign of $s$ denotes the relative sign between the results of postselected and standard weak measurements.

$\mathrm{Var}(M)_D$, so it can simultaneously increase the SNR of both types of weak measurements. However, squeezed coherent states no longer have the minimum uncertainty, so $\sqrt{\mathrm{Var}(\Omega)_D \mathrm{Var}(M)_D}$ can be increased. Hence, the SNR of postselected weak measurement can be raised more than that of standard weak measurement.

It is worth noting that squeezing may also simultaneously decrease $\mathrm{Var}(\Omega)_D$ and increase $\mathrm{Var}(M)_D$ instead, and the SNR of postselected weak measurement can still be higher than that of standard weak measurement. But in this case, the SNR of both types of weak measurements are decreased, so it should be avoided in practice.

*Optimal quantum Fisher information.*— Next, we turn to the precision of weak measurements using maximum likelihood estimation strategy. Once again, our goal is to determine whether postselected or standard weak measurement has greater precision, and what conditions determine the advantage.

The exact variance of the MLE estimator is usually difficult to obtain; however, Cramér and Rao [63] showed that it is inversely bounded by the Fisher information, and this bound can be saturated in the asymptotic limit. So we will use Fisher information as the measure of precision for MLE instead.

As different measurements on the pointer produce different Fisher informations, a proper benchmark for the precision of MLE is the maximum Fisher information over all possible measurements on the pointer, called the *quantum Fisher information* [64, 65], and it gives a more general bound than that can be found by working in only one specific measurement basis. For a pure $g$-dependent state $|\psi_g\rangle$, the quantum Fisher information of estimating $g$ is $F^{(Q)} = 4(\langle\partial_g\psi_g|\partial_g\psi_g\rangle - |\langle\psi_g|\partial_g\psi_g\rangle|^2)$.

In a postselected weak measurement, the pointer state after postselecting the system is $|D_f\rangle \approx \mathrm{e}^{-\mathrm{i}gA_w\Omega}|D\rangle$, so $|\partial_g D_f\rangle \approx -(\mathrm{i}A_w\Omega + \langle\Omega\rangle_D \mathrm{Im}A_w)|D\rangle$, and the quantum Fisher information is approximately [62]

$$F^{(Q)}_{\mathrm{post}} \approx 4P_s|A_w|^2 \mathrm{Var}(\Omega)_D, \tag{11}$$

where we note the dependence on the postselection probability $P_s$. The maximum $P_s$ is given by (5), therefore, the maximum quantum Fisher information over all postselections given the weak value $A_w$ is

$$F^{(Q)}_{\mathrm{post}} \approx \frac{4|A_w|^2 \mathrm{Var}(A)_i \mathrm{Var}(\Omega)_D}{\langle A^2\rangle_i - 2\langle A\rangle_i \mathrm{Re}A_w + |A_w|^2}. \tag{12}$$

Now, the task is just to maximize $F^{(Q)}_{\mathrm{post}}$ over $A_w$. This maximization can be achieved by a two-step procedure similar to maximizing $\mathrm{SNR}_{\mathrm{post}}$ [62], and the result is

$$\max F^{(Q)}_{\mathrm{post}} \approx 4\langle A^2\rangle_i \mathrm{Var}(\Omega)_D. \tag{13}$$

As a comparison, consider the standard weak measurement. In this case, the post-interaction pointer state is generally a mixed state since the pointer is entangled with the system by the weak interaction. The quantum Fisher information for mixed states is much more complex than that for pure states, and a general analytical result is unavailable.

However, with the weak coupling limit $gA_w \ll 1$, this difficulty can be significantly reduced, since the post-interaction pointer state can be approximated to a pure state $|D_f\rangle \approx \mathrm{e}^{-\mathrm{i}g\langle A\rangle_i\Omega}|D\rangle$ [62]. Then, one can immediately derive the quantum Fisher information for standard weak measurement:

$$F^{(Q)}_{\mathrm{std}} \approx 4\langle A\rangle_i^2 \mathrm{Var}(\Omega)_D. \tag{14}$$

Now, comparing $F^{(Q)}_{\mathrm{std}}$ with $F^{(Q)}_{\mathrm{post}}$, the ratio between them can be obtained:

$$\frac{F^{(Q)}_{\mathrm{post}}}{F^{(Q)}_{\mathrm{std}}} \approx \frac{\langle A^2\rangle_i}{\langle A\rangle_i^2}. \tag{15}$$

The result (15) compares the *quantum* Fisher information between postselected and standard weak measurements for *every* possible state of the system, in contrast to Ref. [58, 60] where the Fisher information of measuring the pointer along the position or momentum basis was compared between the two types of weak measurements for their respective optimal system states (with additional assumptions as reviewed in the introduction section). Eq. (15) indicates that the initial state of the system decides the ratio of quantum Fisher information, and implies the postselected weak measurement generally possesses more Fisher information than the standard weak measurement, except that the latter can catch up when the initial system is in an eigenstate of $A$.

Ref. [49–51] made the comparison between using and discarding failed postselections, given that the system is measured. Ref. [53–55] showed that the difference between the Fisher information in these two cases can be shrunk to be negligibly small. Combining (15) with those results, if we denote the quantum Fisher information retaining all failed postselections as $F_{\rm all}^{(Q)}$, then

$$F_{\rm all}^{(Q)} \gtrsim F_{\rm post}^{(Q)} \geq F_{\rm std}^{(Q)}. \qquad (16)$$

This clearly shows the relation of the quantum Fisher information between different types of weak measurements, and clarifies when the postselected weak measurement has metrological advantage. The first inequality of (16) reflects the results of [49–51, 53–55], and the equality sign of the second inequality accords with [58, 60].

*Remark.*— The results for SNR and Fisher information at first glance seem quite different: a significant advantage can be given by postselected weak measurements over standard weak measurements in SNR, while the advantage is quite limited in Fisher information. The difference is rooted in the performances of the two estimators behind them, AMR and MLE, respectively. MLE has the minimum variance over all estimators, while AMR does not, and the Fisher information is usually an upper bound on the precision of MLE (except for Gaussian distributions) which can be achieved only asymptotically. Due to these differences, the SNR has more room to be improved than the Fisher information by optimizing the measurement strategy and the initial states of the system and pointer. These results indicate that the advantage of postselected weak measurements has dependence on the choice of estimation strategy.

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