Geometric Adiabatic Transport in Quantum Hall States

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Geometric adiabatic transport in quantum Hall states

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We argue that in addition to the Hall conductance and the non-dissipative component of the viscous tensor there exists a third independent transport coefficient, which is precisely quantized, taking on constant values along quantum Hall plateaus. Relying on the holomorphic properties of the quantum Hall states, we show that the new coefficient is the Chern number of a vector bundle over moduli space of surfaces of genus two or higher and therefore can not change continuously along the plateau. As such it does not transpire on a sphere or a torus. In the linear response theory this coefficient determines intensive forces exerted on electronic fluid by adiabatic deformations of geometry and represents the effect of the gravitational anomaly. We also present the method of computing the transport coefficients for QH-states.

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1. Introduction Quantum Hall states are distinguished by a precise quantization of the Hall conductance in materials with imprecisely known characteristics. A natural question is whether the Hall conductance is a unique quantized characteristic of the quantum Hall state. Are there any other independent transport coefficients precisely quantized on the QH-plateaus?

Precise quantization in materials occurs when the transport is a non-dissipative adiabatic process. QHE is an example of a system where adiabatic conditions are in place. Namely, the low energy states are separated from the rest of the spectrum by a gap and adiabatic changes of parameters produce states with the flux. However, only adiabatic processes with the nontrivial first Chern class yield quantized transport coefficients.

In this paper we show that apart from the Hall conductance there exist two more quantized transport coefficients, although at present only the former is experimentally accessible. One of these coefficients is the non-dissipative component of viscous tensor introduced in [1–3]. Indications for the existence of another precise transport coefficient appeared recently in connection with the gravitational anomaly found in the context of QHE in Refs. [4–12], see also [13].

Precise quantization on QH plateaus of the non-dissipative transport coefficients can be explained from two points of view. The first, topological explanation is through their relation to topological invariants, such as Chern numbers of vector bundles over the appropriate parameter space [14–16]. For example, in the case of the Hall conductance, the parameter space is spanned by Aharonov-Bohm fluxes piercing through the handles of the Riemann surface. For the non-dissipative viscosity the relevant parameter space is the moduli space of complex structures on the torus [1–3]. In this paper we show that the third coefficient shows up, when the parameter space is the moduli space of complex structures for the surfaces of genus 2 and higher.

For this reason, we discuss the precise transport for QH-states on compact Riemann surfaces. We develop a general method to compute all three transport coefficients at once, with the emphasis on the third coefficient, which is the most subtle. Then we construct the topological invariants which are responsible for the precise quantization. Our method also sheds new light on the relation between the adiabatic transport in QHE and conformal field theory. The method relies on holomorphic properties of the QH-states.

The second view on the transport coefficients is via the local linear response theory. Although it does not establish quantization [14, 15], the linear response theory often provides a clearer physical interpretation. The third coefficient we consider describes an intensive part of non-dissipative viscosity, which does not depend on the fluid density, and is an analog of Casimir forces.

2. Electromotive adiabatic transport We begin with an example illustrating the quantization of non-dissipative adiabatic transport and its relation to the linear response theory, which goes back to [14–17]. We adopt the units in which adiabatic parameters, transport coefficients and adiabatic curvature are dimensionless.

We consider the charge transport in QHE on a torus with Aharonov-Bohm (AB) fluxes φ_a and φ_b along the a and b cycles. In the absence of dissipative diagonal components of the conductance matrix, the electromotive force (emf) $\dot\varphi_b$ produces a current along the a cycle $I_a = \frac{1}{2\pi}\sigma_{ab}\,\dot\varphi_b$. An adiabatic increase of the AB-flux by the flux unit h/e transports the charge $Q_a = \frac{1}{2\pi}\int_0^{2\pi}\sigma_{ab}\,d\varphi_b$. The transported charge defines the adiabatic transport coefficient $\sigma_H = Q_a$ as an average of the Hall conductance over the flux period. A more general definition [15] involves a non-dissipative conductance 2-form

$$\Omega = \frac{1}{2\pi}\sigma_{ab}\,\delta\varphi_b\wedge\delta\varphi_a. \tag{1}$$

Then the adiabatic transport coefficient is the average of this 2-form over a closed 2-cycle in the parameter space (in this case, a torus $T_\varphi : 0\le\varphi_a,\varphi_b<2\pi$)

$$\sigma_H = \frac{1}{2\pi}\int_{T_\varphi}\Omega. \tag{2}$$

Following the arguments of [14, 15, 18], the conductance 2-form (1) is proportional to the adiabatic curvature

$$\hbar\Omega = -\mathrm{i}\,\langle\delta\psi|\delta\psi\rangle. \tag{3}$$

In this formula ψ is a normalized ground state and $\delta\psi$ its external derivative over the parameter space. In the fractional QH case, when the ground state on a closed surface is degenerate, the symbol $\langle\delta\psi|\delta\psi\rangle$ includes the trace over all degenerate states divided by their total number. For example, on the torus there are $m = 1/\nu$ Laughlin states $\psi_1, \dots, \psi_m$, where ν is the filling fraction. Then (3) reads $\hbar\Omega = -\mathrm{i}\,\nu\sum_{r=1}^{m}\langle\delta\psi_r|\delta\psi_r\rangle$. Mathematically the vector of the ground states is a section of the rank m hermitian vector bundle over the parameter space, whose first Chern number $\frac{\mathrm{i}}{2\pi}\sum_{r=1}^{m}\int\langle\delta\psi_r|\delta\psi_r\rangle$ is an integer. Thus the conductance (2) is quantized in units of the filling fraction ν.

A subtle difference between adiabatic transport and the conductance matrix was emphasized in [14, 16]: while the conductance 2-form (1) may fluctuate in mesoscopic systems, the adiabatic transport coefficient (2) does not. The conductance 2-form consists of a precisely quantized part that saturates the adiabatic transport (2), and a non-universal exact 2-form which does not affect it. We emphasize the difference by labeling the precise adiabatic transport by a subscript H, like σ_H in (2) to distinguish it from a non-precise linear response coefficient σ in (1).

This point reflects the difference of approaches of "effective action" [9–12] and the generating functional [4–7] with the adiabatic transport (2). The effective action is given by the integral of the adiabatic curvature (3) over a surface in the parameter space enclosed by the adiabatic process. Hence the entire conductance form (1), including the part, which is an exact 2-form, is relevant. In contrast, the adiabatic transport coefficient is given by the integration of the adiabatic curvature over a closed 2-cycle, as in (2). For this integral only the universal part of the conductance form (2) is relevant, while the exact part of the 2-form does not contribute.

3. Geometric adiabatic transport In addition to the charge transport, there is another set of adiabatic parameters related to the deformations of geometry. In the seminal papers Avron, Seiler, Zograf [1] and Lévay [2] computed the adiabatic transport associated with deformations of the complex modulus $\tau = \tau_1 + \mathrm{i}\tau_2$ of the torus. The modulus defines a complex structure via complex coordinate $z = x + \tau y$. In these coordinates the metric has the form $ds^2 = g_{z\bar z}|dz|^2$, with the diagonal components vanishing $g_{zz} = g_{\bar z\bar z} = 0$, and $g_{z\bar z} = V/\tau_2$, where V is the area of the surface.

An infinitesimal change of the modulus $\tau \to \tau + \delta\tau$ preserves the area but transforms the metric

$$\delta(ds^2) = \delta g_{z\bar z}|dz|^2 + \delta g_{zz}(dz)^2 + \delta g_{\bar z\bar z}(d\bar z)^2,\ \text{where}$$
$$g_{z\bar z}^{-1}\delta g_{z\bar z} = 2|\delta\mu|^2, \quad g_{z\bar z}^{-1}\delta g_{zz} = \delta\bar\mu, \quad g_{z\bar z}^{-1}\delta g_{\bar z\bar z} = \delta\mu. \tag{4}$$

and $\delta\mu$ is called Beltrami differential. In the case of the torus $\delta\mu = \frac{\mathrm{i}\delta\tau}{2\tau_2}$ does not depend on the coordinates.

According to [1, 2] the adiabatic curvature is proportional to invariant area form on the moduli space

$$\Omega = -2\mathrm{i}\eta_H(\delta\mu\wedge\delta\bar\mu), \quad \delta\mu = \frac{\mathrm{i}\delta\tau}{2\tau_2}, \tag{5}$$

where η_H is a universal transport coefficient. The authors of [1] interpreted the $(\hbar/V)\eta_H$ as a non-dissipative component of the viscosity.

The computations of [1, 2] have been carried out for the integer QHE and on the torus. They have been extended in [19, 20] to the fractional QH-states, also on the torus. It was shown that on the torus the coefficient η_H is extensive, i. e. proportional to the number of flux quanta

$$\text{torus}: \quad \eta_H = \varsigma_H N_\Phi, \quad N_\Phi = \frac{1}{2\pi}\int BdV \tag{6}$$

For the usual Laughlin states $\varsigma_H = 1/4$. The parameter space $\mathcal{M}$ in the case of the torus is the fundamental domain of a certain subgroup of the modular group, with the volume equal to $\mathrm{vol}\,\mathcal{M} = \mathrm{i}\int\delta\mu\wedge\delta\bar\mu = \pi$. The integral of the adiabatic curvature (5) over this space

$$\frac{1}{2\pi}\int_{\mathcal{M}}\Omega = -\frac{\eta_H}{\pi}\mathrm{vol}\,\mathcal{M} \tag{7}$$

is the Chern number. Albeit non-integer it is a topological invariant which ensures precise quantization of η_H.

Now we turn to another universal coefficient. We will show that the relation (6) acquires an intensive quantum correction (9), which becomes visible only on surfaces with genus two and higher. Relation (7) establishes its preciseness. We comment that the integer QHE on compact surfaces with a constant negative curvature was first studied in the important paper of Lévay [3].

4. Geometric adiabatic transport - the main result We state the main result first and then sketch its derivation.

We briefly recall the basic notions of the moduli space of complex structures [21]. We would like to consider deformations of the metric (4), which exclude unphysical coordinate reparameterizations, or diffeomorphisms, $z \to z + \epsilon(z, \bar z)$. These correspond to Beltrami differentials of the form $\partial_{\bar z}\epsilon$, as follows from (4). Physical deformations $\delta\bar\mu$ are orthogonal to diffeomorphisms with respect to the standard inner product: $\int_\Sigma(\partial_{\bar z}\epsilon)\delta\bar\mu\, g_{z\bar z}dzd\bar z = 0$. Thus they are given by holomorphic differentials

$$\partial_{\bar z}(g_{z\bar z}\delta\bar\mu) = 0. \tag{8}$$

For surfaces of a genus $\mathrm{g} \geq 2$ there are $3\mathrm{g}-3$ independent holomorphic differentials η_l. The corresponding Beltrami differential $\delta\mu = g_{z\bar z}^{-1}\sum_{l=1}^{3\mathrm{g}-3}\bar\eta_l\delta y_l$ is characterized by $3\mathrm{g}-3$ complex coordinates $\delta y_1, \dots, \delta y_{3\mathrm{g}-3}$ on the tangent space to the moduli space. On the torus the moduli space has complex dimension one. On the sphere the moduli space is just a point.

We recall the notion of the Weil-Petersson form on the moduli space. It is the form invariant with respect to a coordinate choice of the moduli space

$$\Omega_{WP} = \mathrm{i}\int_\Sigma (\delta\mu \wedge \delta\bar{\mu})dV.$$

Here $dV = g_{z\bar{z}}dzd\bar{z}$ is the volume element of the surface Σ. We will show that the *universal* part of the adiabatic curvature of QH-states on the moduli space is

$$\Omega = -2\eta_H\Omega_{WP}, \quad \eta_H = \varsigma_H N_\Phi - \frac{c_H}{12}\chi(\Sigma), \qquad (9)$$

where c_H is a new precise transport coefficient, and $\chi(\Sigma) = 2-2\mathrm{g}$ is the Euler characteristic of the surface.

We list the value of all three precise coefficients for the spin-j Laughlin states which we defined in [6, 7]

$$\sigma_H = \nu,\ \varsigma_H = \frac{1}{4}(1-2j\nu),\ c_H = 1 - \frac{3}{\nu}(1-2j\nu)^2 \quad (10)$$

and compute them below at once. Notice that the value of c_H for $\nu = 1/3$ Laughlin state is $c_H = -8$, and that ς_H may have any sign and even vanish for spin-j states. In sec. 8 we identify the coefficient ς_H and c_H with the background charge and the central charge of the relevant conformal field theory.

The formula (10) generalizes the result of [1–3], and also [19, 20] to Laughlin states on an arbitrary surface. We emphasize that as an adiabatic transport coefficient, c_H cannot be seen on the torus, since Eq.(9) then reduces to (5) (cf.,[3]).

In [20] it was argued that the extensive part $\varsigma_H N_\Phi$ of η_H in (9) is linked to the difference between the admissible number of electrons and the magnetic flux. The relation between these two quantities has been suggested in [22],

$$N = \sigma_H N_\Phi + 2\varsigma_H\chi(\Sigma). \qquad (11)$$

With the help of (11) we can write the non-dissipative viscosity coefficient in (9) as

$$\eta_H = \frac{1}{4\nu}(1-2j\nu)N - \frac{\chi(\Sigma)}{12}. \qquad (12)$$

We observe that the kinematic viscosity $\hbar\eta_H/N$ receives a universal finite size correction $-\frac{\chi(\Sigma)}{12}$. This is analogous to the Casimir effect, where forces receive a volume independent contribution. The origin of this correction is the gravitational anomaly, as we demonstrate below.

The same arguments as in Sec.3 establish precise quantization of the coefficient c_H. Since the integral of the left hand side of Eq. (9) over any closed 2-cycle in the moduli space is a topological invariant and the volume of these cycles in Weil-Petersson metric is a rational number [23], the coefficients ς_H and c_H are precisely quantized. It is more difficult to establish the units in which these coefficients are quantized, since the fundamental domain is an orbifold.

We emphasize that deformations of the metric which do not change the moduli, such as variations of the conformal factor $g_{z\bar{z}}$ (Weyl transformations) or diffeomorphisms do not lead to new precise transport coefficients.

5. Defining relation for holomorphic states The fundamental principle behind the precise quantization of the adiabatic transport coefficients is the holomorphic properties of states on the lowest Landau level. These are many-particle states built from one-particle states annihilated by the operator

$$D^\dagger = g_{z\bar{z}}^{-1/2}(-\mathrm{i}\partial_{\bar{z}} - A_{\bar{z}} + j\omega_{\bar{z}}).$$

Here $A_{\bar{z}}$ and $\omega_{\bar{z}}$ are complex components of the (non-uniform) gauge field and the spin connection, and the spin j is a parameter. We recall that the spin connection is defined such that its exterior derivative is the (scalar) curvature $\mathrm{d}\omega = \frac{1}{2}RdV$. Similarly the exterior derivative of the gauge field $\mathrm{d}A = BdV$ is the magnetic field.

Thus the states are holomorphic functions of the coordinates if the gauge field and the spin connection are treated as adiabatic parameters. But there is more to it. Unnormalized states are also holomorphic functions on the space of adiabatic parameters, in our case the space of complex structure moduli. Under a deformation of the metric (4) the operator $D^\dagger$ deforms holomorphically with μ as $\delta D^\dagger = \delta\mu D$ and so do unnormalized wave-functions

$$\psi_r(z_1,\ldots,z_N|\mu,\bar{\mu}) = \frac{1}{\sqrt{\mathcal{Z}[\mu,\bar{\mu}]}}F_r(z_1,\ldots,z_N|\mu), \quad (13)$$

where the index r labels degenerate fractional QH-states, for surfaces of genus $\mathrm{g} \geq 1$. Laughlin states on the torus transform under a unitary representation of the appropriate subgroup of the modular group, see e. g. [24]. Hence the modular invariant normalization factor is the same for each state. We assume this property to hold on higher genus surfaces. Under this assumption the common normalization factor, also known as a generating functional, determines the adiabatic curvature

$$\Omega = \int_\Sigma (\bar{\mathbf{d}}\mathbf{d}\log\mathcal{Z})dV, \qquad (14)$$

where $\mathbf{d} = \delta\mu\frac{\delta}{\delta\mu}$ and $\bar{\mathbf{d}} = \delta\bar{\mu}\frac{\delta}{\delta\bar{\mu}}$ and similar for AB fluxes. The formula (14) follows directly from the definition (3) and the property (13).

The defining relation (14) is valid for any states with the holomorphic dependence on complex parameters. Such states occur in a broad scope of physical systems, notably in conformal field theory, see e.g. [25, 26].

6. Generating functional Thus, in order to compute the adiabatic curvature one needs to know the generating functional $\mathcal{Z}$. For the Laughlin states it has been obtained in Ref. [6] (cf.[4, 5, 7]). It consists of two parts

$$\log\mathcal{Z} = \log\mathcal{Z}_H + \mathcal{F}[B,R]. \qquad (15)$$

The first term is a bilinear combination of the gauge and spin connections A_z and ω_z. The second term is a local functional of the magnetic field, scalar curvature and their derivatives.

Assuming the transversal gauge $\partial_z A_{\bar z}+\partial_{\bar z}A_z = \partial_z\omega_{\bar z}+\partial_{\bar z}\omega_z = 0$, the result of [4–7] for the first term in (15) can be written in the matrix form

$$\log \mathcal{Z}_H = \frac{2}{\pi}\int (A_{\bar z}\ \omega_{\bar z}) \begin{pmatrix} \sigma_H & 2\varsigma_H \\ 2\varsigma_H & -\frac{c_H}{12}\end{pmatrix}\begin{pmatrix} A_z \\ \omega_z\end{pmatrix} dz d\bar z. \quad (16)$$

Now we have all the necessary data to compute the adiabatic transport coefficients. We will focus on the geometric transport.

Enforcing the condition (8), which excludes diffeomorphisms, the deformation of the spin connection is composed of two distinct parts: the variation of the conformal factor $g_{z\bar z}$, which deforms the curvature but keeps the moduli fixed, and the deformations along the moduli space. The variation of the generating functional (15) with respect to the conformal factor is an exact one-form. Hence it does not contribute to the adiabatic transport. The only source for the adiabatic transport is the deformation of the moduli. Under these deformations the spin connection deforms as

$$\bar{\mathbf{d}}\mathbf{d}\omega_z = \frac{\mathrm{i}}{2}\partial_z(\delta\bar\mu\wedge\delta\mu). \quad (17)$$

Then formulas (17, 16) and (14) yield

$$\Omega = -\frac{\mathrm{i}}{2\pi}\int_\Sigma \left(\varsigma_H B - \frac{c_H}{24}R\right)(\delta\mu\wedge\delta\bar\mu)dV. \quad (18)$$

Restricting to constant magnetic field $2\pi N_\Phi/V$ and the constant curvature $4\pi\chi(\Sigma)/V$, the result (9) immediately follows.

The second term in (15) contributes only to the exact part of the conductance 2-form and therefore is not relevant for the precise adiabatic coefficients. Nevertheless it can be computed for a model wave function [6], [27].

7. Linear response Now we explain the physical meaning of transport coefficients in terms of the linear response theory.

The meaning of (14), is that the Hessian of $\log\mathcal{Z}$ with respect to adiabatic parameters is a conductance matrix. Consider, as an example, an adiabatic process where the spin and the gauge connections evolve along an open path $A(t)$, $\omega(t)$, while the complex structure remains constant. The matrix elements of the Hessian are the linear response functions. We denote them as follows

$$\sigma = \frac{\pi}{2}\frac{\delta^2\log\mathcal{Z}}{\delta A_z\delta A_{\bar z}},\ 2\varsigma = \frac{\pi}{2}\frac{\delta^2\log\mathcal{Z}}{\delta\omega_z\delta A_{\bar z}},\ -\frac{c}{12} = \frac{\pi}{2}\frac{\delta^2\log\mathcal{Z}}{\delta\omega_z\delta\omega_{\bar z}},$$

where we used a short cut notation $\sigma(z,z') = \frac{\pi}{2}\frac{\delta^2\log\mathcal{Z}}{\delta A_z(z)\delta A_{\bar z}(z')}$. The linear response functions include the universal contribution from $\mathcal{Z}_H$ in (15), related to the transport coefficients, and also non-precise gradients corrections from $\mathcal{F}$ in (15).

The adiabatic process $A(t)$, $\omega(t)$ gives rise to an electric field $E_z = \dot A_z$ and its gravitational counterpart $\mathcal{E}_z = \frac{1}{2}\dot\omega_z$. They in turn create an electric current I_z and a shear stress π_{zz}. In complex coordinates the stress tensor reads $\pi_{ij}dx^i dx^j = \pi_{zz}(dz)^2+\pi_{\bar z\bar z}(d\bar z)^2$. Then

$$I_z = \mathrm{i}\frac{\pi}{2}\frac{d}{dt}\frac{\delta\log\mathcal{Z}}{\delta A_{\bar z}},\quad \pi_{zz} = h\frac{\pi}{2}\partial_z\frac{d}{dt}\frac{\delta\log\mathcal{Z}}{\delta\omega_{\bar z}}. \quad (19)$$

The universal parts of currents and stress are determined by the term $\log\mathcal{Z}_H$ in (15). For the Laughlin state they follow from (19), (16) and (10)

$$I_z = \mathrm{i}(\nu E_z+\mathcal{E}_z),\quad \pi_{zz} = -\mathrm{i}\frac{h}{4\nu}\partial_z I_z - \frac{h}{12}\partial_z\mathcal{E}_z. \quad (20)$$

These formulas extend the notion of the Hall conductance: the e.m. current is a sum of Lorentz forces caused by the electric and gravitational fields. The last term in (20) appears because the conductance matrix is non-degenerate. It has the same origin as the finite size correction in (9) and is yet another manifestation of the gravitational anomaly.

Using the continuity equation $\dot\rho+\nabla\cdot I = 0$, we obtain the extension of the Strěda formula $\rho = \frac{1}{2\pi}(\sigma_H B+\varsigma_H R)$ connecting the density of electrons to the magnetic field and curvature. Integrating the density with the help of the Gauss-Bonnet formula $\int_\Sigma R = 4\pi\chi(\Sigma)$ we obtain the relation (11) connecting ς_H to the number of particles and the number of fluxes.

8. QH-state as a string of vertex operators As we have seen, the generating functional (16) is the central object of the theory of transport in QH-states. In the remaining part of the paper we outline one of available methods to obtain it. The method is based on the construction of Ref. [28] where QH-states are expressed by a string of N vertex operators in a relevant field theory, coupled to the magnetic field. This approach has been recently developed in [7] (see also [6]). We illustrate this method for the Laughlin spin j-states, defined in [6, 7], and assume no AB-fluxes.

We look for a field theory which represents the unnormalized part of the Laughlin wave function. Since this state consists of only one type of particles, it is described by one Gaussian field Φ coupled to the magnetic field and curvature

$$S[\Phi] = \frac{\sigma_H}{8\pi}\int(\nabla\Phi)^2 dV + \frac{\mathrm{i}}{2\pi}\int(\sigma_H B+\varsigma_H R)\,\Phi\,dV, \quad (21)$$

where the coupling constants σ_H, ς_H are fixed by the requirements:

(i) An electron is represented by a holomorphic primary operator $V(z)$ with electric charge 1. Identification of the vertex operator with $e^{\mathrm{i}\Phi(z,\bar z)} = V(z)V(\bar z)$ fixes the coupling to the gauge field.

(ii) The OPE of two operators should satisfy $V(z_1)V(z_2)\sim(z_1-z_2)^m$ as $z_1\to z_2$, where $m = 1/\nu$. This condition determines $\sigma_H = \nu$ in (21).

(iii) In the spin-j Laughlin state, a particle has the conformal spin j [6, 7]. This state is a generalization of the usual Laughlin state, for which $j = 0$. Since the state is holomorphic, its conformal dimension also equals to

j. We recall that the conformal dimension of the vertex operator $e^{\mathrm{i}\alpha\Phi}$ with respect to the action (21) is

$$h_\alpha = \frac{\alpha}{2\sigma_H}(\alpha - 4\varsigma_H).$$

Choosing $\alpha = 1$ and $h_1 = j$ we obtain $\varsigma_H = \frac{1}{4}(1 - j\nu)$ as in (10). This condition fixes the parameters of the spin-j Laughlin state. The central charge of such theory

$$c_H = 1 - 48\frac{\varsigma_H^2}{\sigma_H}$$

is given by (10).

Now let us compute the unnormalized correlation function of a string of vertex operators $e^{\mathrm{i}\Phi}$, following [7]. We reproduce the unnormalized Laughlin wave-function (13)

$$\frac{1}{\mathcal{Z}_G}\int\left[\prod_{i=1}^N e^{\mathrm{i}\Phi(z_i,\bar z_i)}\right]e^{-S[\Phi]}D\Phi = |F(z_1,\dots,z_N)|^2 \quad (22)$$

For example, on the sphere the state reads

$$F(z_1,\dots,z_N) = \prod_{i<j}^N (z_i - z_j)^m e^{\frac{1}{2}\sum_{i=1}^N Q(z_i,\bar z_i)}, \quad (23)$$

where the potential Q is such that $\partial_{\bar z}Q = 2\mathrm{i}(A_{\bar z} - j\omega_{\bar z})$.

The factor $\mathcal{Z}_G$ in (22) is

$$\mathcal{Z}_G = [\mathrm{Det}(-\Delta)]^{-\frac{1}{2}} e^{-\frac{2}{\pi\sigma_H}\int|(\sigma_H A_z + 2\varsigma_H\omega_z)|^2 dz d\bar z}, \quad (24)$$

where $\mathrm{Det}(-\Delta)$ is the spectral determinant of the Laplace operator. The next step is to integrate over positions of particles and use the relation $\int |F|^2 dV_1 \dots dV_N = \mathcal{Z}$, where $\mathcal{Z}$ is the normalization factor in (13). We denote $e^{\mathcal{F}} = \int\left[\int e^{\mathrm{i}\Phi(z,\bar z)}dV\right]^N e^{-S[\Phi]}D\Phi$. Then the integration over the coordinates yields

$$e^{\mathcal{F}} = \mathcal{Z}\cdot\mathcal{Z}_G. \quad (25)$$

To complete the argument we notice that the l.h.s. of (25) depends locally on the curvature and does not depend on moduli. Comparing to (15) we obtain the main relation

$$\mathcal{Z}_H^{-1} = \mathcal{Z}_G. \quad (26)$$

It remains to recall the value of the spectral determinant of the Laplace operator in (24). Up to metric independent terms it is given by the formula of Polyakov [29]

$$\log\mathrm{Det}(-\Delta) = -\frac{1}{3\pi}\int|\omega_z|^2 dz d\bar z. \quad (27)$$

It represents the effect of the gravitational anomaly. This term corresponds to 1 in the formula for c_H (10) and is responsible for the finite size correction to the non-dissipative viscosity (12). The result for the generating functional of QH states (16) follows from (26).

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At the time of writing this paper we became aware about the recent paper [12] on the third coefficient c, where the holomorphic properties were employed to compute the geometric part of the adiabatic curvature.

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