Four-Dimensional Entropy from Three-Dimensional Gravity

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Abstract

At the horizon of a black hole, the action of (3+1)-dimensional loop quantum gravity acquires a boundary term that is formally identical to an action for three-dimensional gravity. I show how to use this correspondence to obtain the entropy of the (3+1)-dimensional black hole from well-understood conformal field theory computations of the entropy in (2+1)-dimensional de Sitter space.

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The ability to explain black hole thermodynamics is a key test of any quantum theory of gravity. In this regard, loop quantum gravity has a mixed record. The correct area dependence of black hole entropy appears quite naturally [1, 2]. But to obtain quantitative agreement with the semiclassical results of Bekenstein and Hawking, it seems necessary to tune a rather mysterious parameter, the Barbero-Immirzi parameter γ, to a peculiar value determined by a complex combinatorial computation [3, 4].

In the past few years, there have been intriguing hints that the entropy can also be obtained by setting $\gamma = i$ [5–8]. This is the natural value: it makes the theory self-dual [9], and is the only choice for which the Ashtekar-Barbero-Sen connection (1.1) is a fully diffeomorphism-invariant spacetime connection [10, 11]. Unfortunately, with this choice one must impose a reality conditions, a procedure that remains poorly defined. As a consequence, the theory with $\gamma = i$ is not nearly as mathematically sophisticated as the version with real γ, and far fewer results have been established.

In this paper, I will describe a simple new method for computing black hole entropy in loop quantum gravity with $\gamma = i$. The key observation is that loop quantum gravity requires a boundary term at a black hole horizon that is formally identical to an action for three-dimensional gravity with a positive cosmological constant. The identification is not an obvious geometric one, but the four-dimensional horizon maps to a well understood three-dimensional spacetime, and one can exploit this association to use standard techniques of conformal field theory to count the states.

1. Two $\mathrm{SL}(2, \mathbb{C})$ actions

We start with (3+1)-dimensional gravity in first-order form, treating the tetrad one-form $e^I = e_\mu{}^I dx^\mu$ and the spin connection one-form $\omega^{IJ} = \omega_\mu{}^{IJ} dx^\mu$ as independent variables. The Ashtekar-Sen self-dual connection [9, 12] is $A^{IJ} = \frac{1}{2} \left(\omega^{IJ} + \frac{i}{2} \epsilon^{IJ}{}_{KL} \omega^{KL} \right)$, but to avoid double-counting components, it is sufficient to consider the complexified SU(2)—or equivalently, $\mathrm{SL}(2, \mathbb{C})$—connection

$$A^i = i\omega^{0i} + \frac{1}{2}\epsilon^{ijk}\omega_{jk}, \tag{1.1}$$

where lower case Roman indices run from 1 to 3 (see, for instance, section 4.3 of [13]). The gravitational action can then be written in the form [14, 15]

$$I_4 = -\frac{i}{16\pi G_4} \int d^4x \, \Sigma_i \wedge F^i, \tag{1.2}$$

where $F^i = dA^i + \epsilon^{ijk} A_j \wedge A_k$ is the curvature of the connection and $\Sigma^i = ie^0 \wedge e^i + \frac{1}{2}\epsilon^{ijk} e_j \wedge e_k$ is the self-dual projection of $e^I \wedge e^J$. The real part of (1.2) is equal to the standard Einstein-Hilbert action, while the imaginary part is essentially irrelevant: it is extremal whenever the real part is, so it does not change the equations of motion, and it vanishes on shell. In loop quantum gravity, the factor of i in (1.1) is often replaced by an arbitrary parameter γ, the Barbero-Immirzi parameter. The quantization becomes much simpler when γ is chosen to be real, but as noted above, hints are now appearing that the self-dual choice $\gamma = \pm i$ simplifies and clarifies the description of black hole entropy.

Now suppose that a black hole is present, with a horizon Δ of area A_Δ. For the surface Δ to be an isolated horizon [16], it must obey a geometric restriction, which translates to the condition [1, 17]

$$F^i = -\frac{2\pi}{A_\Delta}\Sigma^i \quad \text{on } \Delta. \tag{1.3}$$

Although the horizon is not a physical boundary, the imposition of (1.3) forces us to add a "boundary" term to the action. As first noted by Smolin in a slightly different context [18], the required term is

a Chern-Simons action. The specific form depends on the Barbero-Immirzi parameter; for our choice $\gamma = i$, it is a chiral $SL(2,\mathbb{C})$ Chern-Simons action

$$I_\Delta = \frac{k}{4\pi} \int_\Delta \mathrm{Tr} \left\{ A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right\}, \tag{1.4}$$

where $A = A^i T_i$ is the sl$(2,\mathbb{C})$-valued connection with generators normalized so $\mathrm{Tr}(T_i T_j) = \frac{1}{2}\eta_{ij}$, and the coupling constant k is expressed in terms of (3+1)-dimensional gravitational quantities as

$$k_{4D} = \frac{iA_\Delta}{8\pi G_4}. \tag{1.5}$$

Moreover, the symplectic form—that is, the set of Poisson brackets—also acquires a boundary term for the connection at the horizon, which is identical to the symplectic form of Chern-Simons theory (see, e.g., [19]). Thus components of the connection, which commute in the bulk, become canonically conjugate at Δ, and by the usual rules of quantization we expect a Hilbert space $\mathcal{H}_{bulk} \otimes \mathcal{H}_\Delta$, with the bulk and horizon states related by the operator version of the boundary conditions (1.3) [2].

So far, I have not used loop quantum gravity. I now exploit one general feature of that quantization. Classically, the boundary conditions (1.3) imply that the boundary $SL(2,\mathbb{C})$ connection is not flat, and is thus not an extremum of the Chern-Simons action. In loop quantum gravity, though, quantum states are described by spin networks, and the area element on the right-hand side of (1.3) is distributional, differing from zero only at the "punctures" where spin network edges intersect the horizon. The boundary conditions *are* then equivalent to the equations of motion for a Chern-Simons theory, but now on a sphere with punctures (or, technically, a manifold $\mathbb{R} \times S^2$ with Wilson lines) [20]. Hence the boundary Hilbert space $\mathcal{H}_\Delta$ is that of a Chern-Simons theory on a sphere with punctures. In standard loop quantum gravity, one can say much more—holonomies around punctures give calculable elements of area—but we shall not need any of those details; it is enough that the boundary theory acts as an independent Chern-Simons theory coupled to the bulk through a set of punctures.

The action (1.4) also appears in a very different context, though: it is the first-order action for (2+1)-dimensional gravity with a positive cosmological constant $\Lambda = 1/\ell^2$ [21]. The connection is now

$$\tilde{A}^a = \frac{1}{2}\epsilon^{abc}\tilde{\omega}_{bc} + \frac{i}{\ell}\tilde{e}^a, \tag{1.6}$$

where $\tilde{e}^a$ and $\tilde{\omega}^{bc}$ are the three-dimensional triad and spin connection, and the coupling constant k is

$$k_{3D} = \frac{i\ell}{2G_3}, \tag{1.7}$$

now expressed in terms of (2+1)-dimensional quantities. Much as in the four-dimensional case, the real part of (1.4) gives the usual Einstein-Hilbert action, while the imaginary part is an "exotic" term that is extremal when the real part is extremal and vanishes on shell.* The $SL(2,\mathbb{C})$ Chern-Simons action is also related to "Euclidean anti-de Sitter space"; I will return to this point in the conclusion.

Although the two appearances of the Chern-Simons action both involve gravity, their mathematical equivalence is, as far as I know, purely accidental. One can try to construct a geometrical relationship, but if one exists, it is subtle. Indeed, comparing (1.1) and (1.6), we see that while the connections can be made to match, the triad $\tilde{e}^a$ in three dimensions corresponds to the extrinsic curvature in four dimensions. Hence we might not expect the three-dimensional theory to give a simple geometrical picture of the states (although see [8,22]). Still, the formal equivalence of the actions will be enough to determine the (3+1)-dimensional Bekenstein-Hawking entropy.

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*The (2+1)-dimensional action is usually written as a sum of (1.6) and its complex conjugate, in which case the coupling constant for each term is half of (1.7). Here, though, we wish to match the (3+1)-dimensional action, which is chiral.

2. Entropy

Let us focus for now on (1.4) as an action for (2+1)-dimensional gravity. For the case of a *negative* cosmological constant, the counting of states in such a theory is well understood [23, 24], although the exact nature of those states is not [25]. As Brown and Henneaux showed, the asymptotic symmetry in such a theory is a two-dimensional conformal symmetry [26], which is powerful enough that the Cardy formula determines the asymptotic density of states without requiring any further details [27–29].

For the case of a positive cosmological constant, there is no asymptotic spatial boundary, and the picture is not as clean. One can, however, look at the asymptotic symmetries at timelike infinity [30–32]; or impose boundary conditions on a tube, which can be viewed as the world line of an observer [33]; or continue to negative Λ [34]; or perhaps obtain a central charge directly from the symmetries of the phase space [35]. One obtains a consistent answer: a "puncture" with $SL(2, \mathbb{C})$ holonomy conjugate to

$$H = \begin{pmatrix} e^{\pi i r_+/\ell} & a \\ 0 & e^{-\pi i r_+/\ell} \end{pmatrix} \tag{2.1}$$

gives a local geometry of a cone, and contributes an entropy

$$S = \frac{2\pi r_+}{4G_3} = -ik_{3D}\frac{\pi r_+}{\ell}. \tag{2.2}$$

(For subtleties coming from the fact that we are considering a purely chiral action, see [36].)

While (2.2) was derived in 2+1 dimensions, it is ultimately a statement about the quantum mechanics of the action (1.4), which is also the boundary action in $3 + 1$ dimensions. We now use a single fact from the four-dimensional picture: a cross-section of the horizon Δ at a fixed time is a two-sphere S^2. Consider a loop on this two-sphere surrounding all of the punctures. On the one hand, the $SL(2, \mathbb{C})$ holonomy of this loop is the product of the holonomies around each puncture. On the other hand, the loop also surrounds a region with no punctures, for which the holonomy must be the identity. Assuming that all of the holonomies are in the same conjugacy class (2.1)—I will return to this below—it is easy to see that this requires that

$$\sum_{punctures} \frac{\pi r_+}{\ell} = 2\pi. \tag{2.3}$$

Thus from (2.2) and (1.5),

$$S = -2\pi i k_{3D} = -2\pi i k_{4D} = \frac{A_\Delta}{4G_4}, \tag{2.4}$$

reproducing the correct Bekenstein-Hawking entropy for the *four*-dimensional black hole.

3. The Schwarzschild black hole

To make the discussion more concrete, let us specialize to the Schwarzschild black hole. Following Kaul and Majumdar [37], we write the metric in Kruskal-Szekeres form as

$$ds^2 = -2B(r)dvdw + r^2(v, w)(d\theta^2 + \sin^2\theta d\varphi^2) \quad \text{with } B = \frac{4r_+^3}{r}e^{-r/r_+}, \ -2vw = \left(\frac{r}{r_+} - 1\right)e^{r/r_+} \tag{3.1}$$

and choose a tetrad

$$e^0 = \sqrt{\frac{B}{2}}\left(\frac{w}{\alpha}dv + \frac{\alpha}{w}dw\right), \qquad e^1 = \sqrt{\frac{B}{2}}\left(\frac{w}{\alpha}dv - \frac{\alpha}{w}dw\right), \qquad e^2 = rd\theta, \qquad e^3 = r\sin\theta d\varphi, \tag{3.2}$$

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where α is an arbitrary function labeling a gauge choice for local Lorentz transformations. It is then straightforward to compute the connection (1.1); at the horizon $r = r_+$, $B = B_+$, $w = 0$, one finds [37]

$$A^1 = \cos\theta d\varphi + i\frac{d\alpha}{\alpha}, \quad A^2 = -\sqrt{\frac{B_+}{2}}\frac{1}{2r_+}\alpha\left(id\theta - \sin\theta d\varphi\right), \quad A^3 = -\sqrt{\frac{B_+}{2}}\frac{1}{2r_+}\alpha\left(i\sin\theta d\varphi + d\theta\right). \quad (3.3)$$

Now, by (1.6), the imaginary part of the connection (3.3) should give the triad in the (2+1)-dimensional picture. Defining

$$\frac{1}{\beta} = \sqrt{\frac{B_+}{2}}\frac{1}{2r_+}\alpha \quad (3.4)$$

we see that the classical (2+1)-dimensional metric is

$$ds^2 = \frac{\ell^2}{\beta^2}\left(-d\beta^2 + d\theta^2 + \sin^2\theta d\varphi^2\right). \quad (3.5)$$

This is *almost* the de Sitter metric on an expanding patch. It is not quite; the curvature is not constant, but satisfies an equivalent of (1.3). But as noted above, in loop quantum gravity we should replace the continuous curvature by a collection of punctures, of the type first introduced by Deser and Jackiw [38]. That is, as in Regge calculus, we should replace (3.5) by a locally de Sitter metric

$$ds^2 = \frac{\ell^2}{\beta^2}\left(-d\beta^2 + dzd\bar{z}\right) \quad \text{with } z = x + iy \quad (3.6)$$

with a set of conical singularities that reproduce the curvature of (3.5) in the large.

Now, the isometry group of the de Sitter metric (3.6) is $SL(2, \mathbb{C})$, with an action [39]

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} : (z, \beta) \to \left(\frac{(az+b)(\bar{c}\bar{z}+\bar{d}) + a\bar{c}\beta^2}{|cz+d|^2 + |c|^2\beta^2}, \frac{\beta}{|cz+d|^2 + |c|^2\beta^2}\right). \quad (3.7)$$

To obtain the metric (3.5) from (3.6), we must add a set of conical points on surfaces of constant β. The condition for an isometry (3.7) to preserve such surfaces is that $c = 0$, $|d| = 1$, and the resulting isometries are precisely the ones given by (2.1).

In slightly more detail, an elliptic element

$$R = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} \quad (3.8)$$

of $SL(2, \mathbb{C})$ rotates z by an angle 2θ around the origin, while a parabolic element

$$T = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \quad (3.9)$$

translates z by a. An individual puncture at position a thus corresponds to a holonomy TRT^{-1}, equivalent to (2.1), and the total holonomy is

$$H = T_1 R_1 T_1^{-1} T_2 R_2 T_2^{-1} T_3 R_3 T_3^{-1} \ldots \quad (3.10)$$

in agreement with the analysis of the preceding section.

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We can now go further. The parabolic element T can be written as

$$\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} = \exp\left\{a(J_1 + K_2)\right\}, \tag{3.11}$$

where the generators of complexified SU(2) are $J_i = \frac{1}{2}\sigma_i$, $K_i = \frac{i}{2}\sigma_i$. From a (3+1)-dimensional viewpoint, this is a null rotation, a Lorentz transformation that leaves a null vector fixed. Similarly, RTR^{-1} is a null rotation fixing a different, rotated null vector. The holonomy (3.10) can be rewritten as

$$H = \left(T_1\right)\left(R_1(T_1^{-1}T_2)R_1^{-1}\right)\left((R_1R_2)(T_2^{-1}T_3)(R_1R_2)^{-1}\right)\ldots \tag{3.12}$$

that is, as a product of null rotations.

This is just what one would expect in the self-dual formulation of general relativity, where the connection (1.1) involves a sum of a rotation and a boost. But we can now even identify the null vector being held fixed. The coordinate β in the (2+1)-dimensional metric (3.5) originated as a gauge-dependent parameter in the (3+1)-dimensional tetrad (3.2). But for (3.5) to be truly (2+1)-dimensional, β cannot depend on θ and φ alone, but must be a function of the null coordinate v along the horizon. Indeed, to preserve spherical symmetry, β should be a function of v alone. Hence the isometries (2.1), chosen in 2+1 dimensions to leave β invariant, fix v in 3+1 dimensions. The null vector that defines our null rotations is just the null normal to the horizon.

This choice is physically natural, and may offer insights into the underlying degrees of freedom [8]. But it is awkward to implement in a formulation with a real Barbero-Immirzi parameter, perhaps explaining why the derivation of black hole entropy is simpler with a self-dual connection.

4. Implications and open questions

I have focused on the Schwarzschild black hole, but the general arguments about the structure of holonomies hold for any black hole satisfying the isolated horizon boundary conditions (1.3). Still, it would be interesting to see an explicit extension to an arbitrary black hole. For the Kerr black hole, much of the preliminary work appears in [40], although a more general Lorentz gauge is needed.

The present derivation of black hole entropy differs from the standard loop quantum gravity approach of [1, 2] in an interesting way. The usual starting point is an ensemble of horizon configurations with arbitrarily many punctures and arbitrary holonomies, restricted only by the specified area A_Δ. Counting states is then a combinatorial problem; an entropy proportional to area appears naturally, but the Barbero-Immirzi parameter must be tuned to give the right prefactor. Here, in contrast, the entropy is derived for a *single* configuration of punctures and holonomies, now restricted only by the closure condition (2.3). This is reminiscent of the proposal that the number of punctures should be treated as a sort of "quantum hair" [41] that physically distinguishes different black holes. In essence, the question is in how fine a coarse-graining is needed to define the entropy.

The method of counting states here also differs from the standard approach. In contrast to the usual procedure, our central result (2.2) depends on no details of the Hilbert space, but only on the fact that an SL(2, $\mathbb{C}$) Chern-Simons theory implies a two-dimensional conformal symmetry, which is powerful enough to severely constrain the density of states. Similar symmetry arguments have been used in other attempts to count black hole states—see [42] for a review—and it is intriguing that the central charge $c = 6k$ here is nearly identical to the value obtained in those approaches, differing by a factor of two. Any relationship between these analyses must be a bit subtle, since the conformal methods of [42] involve symmetries in the "r–t plane" rather than symmetries of spatial sections of the horizon. But

as Pranzetti has pointed out [43], the self-dual connection (1.1) automatically links transformations in these different spaces, so a relationship might exist.

There is another direction in which this work might be extended. An $SL(2,\mathbb{C})$ Chern-Simons theory is a theory of (2+1)-dimensional de Sitter gravity, but also of "Euclidean anti-de Sitter gravity," that is, (2+1)-dimensional gravity with $\Lambda < 0$ analytically continued to Riemannian signature. Punctures then correspond to point particles in AdS, and the quantization is almost certainly related to Liouville theory [44]. An interesting new possibility now arises if we allow the elliptic holonomies (2.1) to lie in different conjugacy classes. The product of a large number of random elliptic elements of $SL(2,\mathbb{C})$ is exponentially likely to be hyperbolic [45, 46], and a hyperbolic isometry in AdS signals the appearance of a three-dimensional black hole horizon. It is not entirely clear how to count the resulting degrees of freedom—I do not know the analog of the closure condition (2.3)—but this would be interesting to pursue. One possibility is to use the canonical version of the Cardy formula; as discussed in section 5 of [8], this may again yield the correct entropy.

Finally, it is interesting to ask whether this sort of calculation can be applied in more general contexts. Boundary conditions of the form (1.3) have been suggested in broader settings [18]; it would be good to know whether the results of this paper can be extended to, for instance, general spatial boundaries, or perhaps causal horizons [47]. Note also that the derivation here made very limited use of the dynamics of general relativity [48]; while the results almost certainly depend on the use of a noncompact connection [8], they might generalize to BF theory or its deformations [49].

Acknowledgments

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