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Synchronization as Aggregation: Cluster Kinetics of Pulse-Coupled Oscillators

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We consider models of identical pulse-coupled oscillators with global interactions. Previous work showed that under certain conditions such systems always end up in sync, but did not quantify how small clusters of synchronized oscillators progressively coalesce into larger ones. Using tools from the study of aggregation phenomena, we obtain exact results for the time-dependent distribution of cluster sizes as the system evolves from disorder to synchrony.

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In one of the first experiments on firefly synchronization, the biologists John and Elisabeth Buck captured hundreds of male fireflies along a tidal river near Bangkok and then released them at night, fifty at a time, in their darkened hotel room [1]. They observed that "centers of synchrony began to build up slowly among the fireflies on the wall. In one area we would notice that a pair had begun to pulse in unison; in another part of the room a group of three would be flashing together, and so on." Synchronized groups continued to emerge and grow, until as many as a dozen fireflies were blinking on and off in concert. The Bucks realized that the fireflies were phase shifting each other with their flashes, driving themselves into sync.

Here we study stylized models of oscillators akin to the fireflies, in which synchrony builds up stepwise, in expanding clusters. By borrowing techniques used to analyze aggregation phenomena in polymer physics, materials science, and related subjects [2, 3], we give the first analytical description of how these synchronized clusters emerge, coalesce, and grow. We hasten to add, however, that the models we discuss are not even remotely realistic descriptions of fireflies; they are merely intended as tractable first steps toward understanding how clusters evolve en route to synchrony.

Our work is part of a broader interdisciplinary effort [4, 5]. Oscillators coupled by sudden pulses have been used to model sensor networks [6–10], earthquakes [11, 12], economic booms and busts [13], firing neurons [14, 15], and cardiac pacemaker cells [16]. Diverse forms of collective behavior can occur in these pulse-coupled systems, depending on how the oscillators are connected in space. Systems with local coupling often display waves [17, 18] or self-organized criticality [11, 19, 20], with possible relevance to neural computation [15] and epilepsy [21]. In contrast, systems with global coupling, where every oscillator interacts equally with every other, tend to fall into perfect synchrony. Rigorous convergence results have been proven for this case [20, 22–25]. But the techniques used previously have not revealed much about the transient dynamics leading up to synchrony—the opening and middle game, as opposed to the end game. Aggregation theory offers a new set of tools to explore this prelude to synchrony.

Exact results for the transient dynamics can be obtained in at least two cases. In the Supplemental Material [26], we apply aggregation theory to the deterministic Peskin model [16], assuming the oscillators rise linearly to threshold and fire pulses of size $1/N$, where $N \gg 1$ is the number of oscillators. A simplified stochastic version of this model yields similar results, but because it illustrates the main ideas more clearly we present it here in the main text. This toy model, which we call *scrambler oscillators*, consists of N identical integrate-and-fire oscillators coupled all to all. Each oscillator has a voltage-like state variable x that increases linearly according to $\dot{x} = 1$, rising from a baseline value of 0 to a threshold value of 1. Whenever any oscillator reaches threshold, it fires and does three things. (i) It kicks every oscillator (and every synchronous cluster of oscillators) to a new random voltage, independently and uniformly—in this sense, it scrambles the other oscillators. However, no scrambling occurs within a cluster; all oscillators at the same voltage get kicked to the same new voltage. Thus, clusters never desynchronize; once formed they are preserved by the scrambling procedure. (ii) The firing oscillator then "absorbs" any scrambled oscillators that lie within a distance $1/N$ of threshold, by bringing them to threshold and thereby synchronizing with them. To avoid the complications that would be caused by chain reactions of firings, we assume that the oscillators being brought to threshold do not get to fire until the *next* time they reach threshold. (iii) The oscillator that fired resets to $x = 0$ along with the oscillators it absorbed.

If a cluster of j oscillators does the firing, the same rules apply, except that now any oscillators within a distance j/N of threshold get absorbed. The assumed proportionality to j is natural, if each member of the cluster contributes to the pulse strength. We study other plausible coupling rules in [26].

The motivation for this scrambler model is that it leads to the simplest possible mean-field approximation. In the infinite-N limit, we would like clusters of every size to be uniformly distributed in voltage at all times. This convenient property would greatly ease the derivation of the rate equations for the cluster kinetics. As we will see below, the predictions that follow from this approximation agree reasonably well with simulations. (For finite-N, these assumptions break down at large times and for large clusters, limitations that we analyze in [26].)

Assume the initial voltages x_i, for $i = 1, \ldots, N$, are independent and uniformly distributed. At first, nothing interesting happens. The oscillators increase their voltages without interacting. But then one oscillator reaches threshold and fires. The remaining oscillators get scrambled, and perhaps some get absorbed. Then another oscillator fires, and so on. After a while, the system has formed clusters of various sizes.

Let $N_j(t)$ denote the number of clusters of size j at time t. Thus there are $N_1(t)$ singleton oscillators, $N_2(t)$ pairs of synchronized oscillators, $N_3(t)$ triplets, and so on. The N_j are correlated random quantities. They are correlated because oscillators belonging to clusters of one size are unavailable to clusters of another size, and they are random because of the randomness in the initial conditions and the scrambling procedure. It does not seem feasible to understand the time-evolution of the N_j unless they are so large that their fluctuations from one random realization to another are negligible.

So assume from now on that $N_j \gg 1$ for all j and replace these random quantities by their ensemble averages. Let $c_j = N^{-1}\langle N_j \rangle$ denote the average cluster densities. One hopes that relative fluctuations are small; more precisely, $N^{-1}N_j = c_j + O(N^{-1/2})$. An even stronger assumption is that the densities of different sub-populations are asymptotically uncorrelated: $N^{-2}N_iN_j = c_ic_j + O(N^{-1/2})$.

These c_j allow us to define a natural disorder parameter, given by the total density $c(t) = \sum_j c_j(t)$. It measures the extent of the system's fragmentation. To see this, note that at $t = 0$ each oscillator is alone; only clusters of size 1 exist. Accordingly $c_1(0) = 1$ and all other $c_j(0) = 0$ for $j > 1$. Hence $c(0) = 1$, correctly indicating that the system starts out maximally fragmented. At the opposite extreme, as $t \to \infty$ only one giant cluster of synchronized oscillators exists. The system is then minimally fragmented: $c(t) = 1/N \to 0$ as $N \to \infty$.

To derive a rate equation for the decline of $c(t)$, let R_i be the rate at which clusters of size i fire, for $i = 1, \ldots, N$, and let L_i be the number of clusters lost to absorption in each such firing. Then $\dot{c} = -\sum_i R_iL_i$.

To find L_i, recall that when a cluster of size i fires, all the other clusters get assigned a new voltage uniformly at random. Moreover, any clusters assigned to the interval $[1-i/N, 1)$ get brought to threshold and absorbed. Since the voltages of these other clusters are uniformly distributed on $[0, 1]$, a fraction i/N of them will be absorbed. There are $\sum_j N_j$ clusters in total. Hence the number absorbed is $L_i = (i/N)\sum_j N_j = i\sum c_j = ic$.

The rate R_i takes more work to calculate. Since some clusters get absorbed, not every cluster gets the chance to fire. We must account for this depletion when calculating R_i. First consider the background rate of firing of clusters of size i in the absence of absorptions. In other words, pretend for a moment that when an i-cluster fires, it simply scrambles every other cluster and restarts its own cycle without absorbing anyone. Call this background rate R_i^0. Since all oscillators move with velocity $v_i = \dot{x}_i = 1$, and since the cluster density is c_i, the corresponding background rate of firing is $R_i^0 = c_iv_i = c_i$. Next, to find the actual R_i, we must subtract from R_i^0 the rate at which clusters of size i are being absorbed and hence deprived of their chance at firing. Call this absorption rate R_i^a. Clusters of size i are absorbed when clusters of size j fire, for $j = 1, \ldots, N$, taking a fraction j/N of the uniformly distributed i-clusters along with them. Since there are N_i clusters of size i and the j-clusters fire at rate R_j, the total rate at which i-clusters are being absorbed is given by $R_i^a = \sum_j (j/N)N_iR_j = \sum_j jc_iR_j = c_i\sum jR_j$.

Putting all this together gives $R_i = R_i^0 - R_i^a = c_i - c_i\sum_j jR_j = c_i(1 - \sum_j jR_j)$. Let $\beta = 1 - \sum_j jR_j$. Note that β is the same for all i, which enables it to be determined self-consistently, as follows. From $R_i = \beta c_i$ we obtain $\beta = 1 - \sum_j jR_j = 1 - \sum j(\beta c_j)$. Now invoke the identity $\sum_j jc_j = j(N_j/N) = 1$, which expresses conservation of oscillators. Solving for β then gives $\beta = 1/2$ and therefore $R_i = c_i/2$.

Next, plug the expressions derived for R_i and L_i into the rate equation $\dot{c} = -\sum_i R_iL_i$. The result is $\dot{c} = -\sum_i (c_i/2)(ic) = -(c/2)\sum_i ic_i = -c/2$. Recalling that $c(0) = 1$, we conclude that

$$c(t) = \exp(-t/2). \tag{1}$$

Figure 1 shows this result matches simulations.

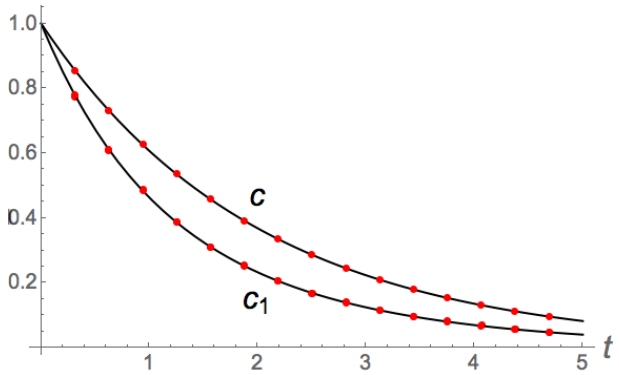

FIG. 1: Theoretical and simulated $c(t)$ and $c_1(t)$. Solid lines show theoretical curves obtained analytically (see text). Data points show simulation results for $N = 10^4$ oscillators.

How do the individual cluster densities c_i behave? To derive their rate equations, note that since the voltage space is the interval $[0,1]$, a segment of length N^{-1} contains on average $Nc\times N^{-1}=c$ clusters. In fact, the probability that it contains n clusters (of any sizes) is given by the Poisson distribution: $\Pi_n = c^n e^{-c}/n!$. This is the mathematical expression of the assumption that clusters are distributed randomly without correlations.

With this in mind, let us solve for $c_1(t)$, the density of singletons. It is the easiest $c_j(t)$ to analyze, since it can only decrease. Two mechanisms decrease $c_1(t)$: (i) The loss of a *firing* singleton when it absorbs a cluster of any size, and (ii) the loss of $p>1$ *absorbed* singletons, when a cluster of any size fires.

Consider mechanism (i). Since $R_i = c_i/2$ as shown above, singletons fire at a rate $R_1 = c_1/2$. When they fire, they absorb any cluster lying in the voltage segment $[1-1/N,1)$. The probability that this segment contains one or more clusters is, according to the Poisson distribution, $1-e^{-c}$. In this case, the firing singleton is lost by its absorption of a cluster, thus decreasing N_1 by 1. Otherwise N_1 is unchanged. Hence singletons are lost by mechanism (i) at an expected rate $(c_1/2)\left[1\times(1-e^{-c(t)})+0\times e^{-c(t)}\right] = (c_1/2)\left[1-e^{-c(t)}\right]$. Note: we only account for the loss of the *firing* singleton here; any singletons it *absorbs* are accounted for in the following mechanism (ii).

Suppose p singletons lie in the interval $[1-j/N,1)$ when a cluster of size j fires, for $j=1,\ldots,N$. This event happens with probability $e^{-jc_1}(jc_1)^p/p!$, and when it does, it consumes p singletons. (If a singleton did the firing, the loss would be $p+1$. However the loss of the firing singleton was already counted in mechanism (i). So the consumption factor of p for each firing j-cluster is valid even for $j=1$.) As before, j-clusters fire at a rate $R_j = c_j/2$. Hence singletons are lost by mechanism (ii) at a rate

$$\sum_{j\geq 1}\frac{c_j}{2}\times\sum_{p\geq 1}p\frac{(jc_1)^p e^{-jc_1}}{p!} = c_1/2. \tag{2}$$

Summing the loss rates from (i) and (ii) gives

$$\frac{dc_1}{dt} = -\frac{c_1}{2}(2-e^{-c(t)}). \tag{3}$$

This equation has a closed-form solution in terms of exponential integrals:

$$c_1(t) = \exp(-t+\mathrm{Ei}(-1)-\mathrm{Ei}(-e^{-t/2})), \tag{4}$$

where we have used the initial condition $c_1(0)=1$. Figure 1 shows good agreement between the theoretical and numerical $c_1(t)$.

For $i>1$, the rate equation for c_i includes gain terms as well as loss terms. Clusters of size $i>1$ can be created when two or more smaller clusters coalesce, or destroyed when they themselves coalesce with at least one other cluster. The loss term is a straightforward generalization of that for c_1, and is given by $(c_i/2)\left[2-e^{-ic(t)}\right]$.

To find the gain term, imagine that a cluster of size k fires. The segment $[1-k/N,1)$ may contain a_1 clusters of size 1, a_2 clusters of size 2, etc. This event happens with probability $\frac{(kc_1)^{a_1}}{a_1!}e^{-kc_1}\times\frac{(kc_2)^{a_2}}{a_2!}e^{-kc_2}\times\ldots$ (where we are using the assumption that clusters of different sizes are independent as well as Poisson distributed). If the segment contains a combination of clusters such that $k+a_1+2a_2+3a_3+\cdots=i$, then a cluster of size i will form. We sum over all such combinations for a cluster of size k firing, and then sum over all k, to get the rate at which clusters of size i are created:

$$\sum_{k=1}^{i-1}\frac{c_k}{2}e^{-kc}\sum_{a_1+2a_2+\cdots=i-k}\left(\prod_{p\geq 1}\frac{(kc_p)^{a_p}}{a_p!}\right). \tag{5}$$

Combining the loss and gain terms, and transferring $c_i e^{-ic}$ into the gain term, we finally obtain

$$\dot{c_i} = -c_i+\sum_{k=1}^{i}\frac{c_k}{2}e^{-kc}\sum_{\sum pa_p=i-k}\left(\prod_{p\geq 1}\frac{(kc_p)^{a_p}}{a_p!}\right). \tag{6}$$

We see from the sum that the equations (6) are recursive. They can be solved one by one, though not analytically, so we resort to numerical integration. Figure 2 shows that the theoretical and simulated c_i agree.

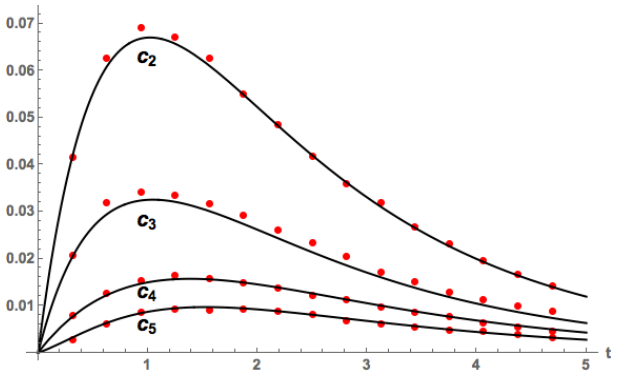

FIG. 2: Theoretical and simulated cluster densities $c_2(t)$ through $c_5(t)$. Solid lines show theoretical predictions computed from numerical integration of Eq. (6). Data points show simulation results for $N=5\times 10^4$ oscillators.

Although we cannot find all the $c_i(t)$ explicitly, we can get their moments $M_n(t)=\sum_j j^n c_j(t)$ through the use of a generating function. We already know two moments: $M_0(t)=c(t)$, given by Eq. (1), and $M_1=1$. A few of the higher moments are

$$\begin{aligned} M_2(t) &= e^{3t/2} \\ M_3(t) &= 7e^{7t/2}-6e^{3t} \\ M_4(t) &= \frac{448}{5}e^{5t}-128e^{9t/2}+\frac{217}{5}e^{15t/4}-4e^{27t/8}. \end{aligned} \tag{7}$$

These and further results are discussed in [26].

We also studied two modifications of the scrambler model. For example, suppose that when a cluster of size j fires, it absorbs all oscillators within a distance kj/N of threshold, where $k > 0$ is a tunable coupling strength. Or suppose that the pulse strength is k/N, independent of the size j of the firing cluster. We discuss both cases in [26]. In the latter case the disorder parameter $c(t)$ decays algebraically rather than exponentially. This makes sense physically: by assuming that larger clusters no longer fire larger pulses proportional to their size, we cut the positive feedback loop underlying the exponential growth of synchrony in the original scrambler model.

The stochastic scrambler model approximates the deterministic models studied by Peskin [16] and Mirollo and Strogatz [22]. In those models, when a cluster of size j fires it adds a voltage pulse $j\epsilon$ to every other oscillator, or pulls it up to threshold, whichever is less. For the case where $\epsilon = 1/N$ and the charging curve is linear, we show in [26] that these deterministic systems can also be analyzed by the methods above. The main new feature is that $c(t)$ and the cluster densities $c_i(t)$ become piecewise linear. But their overall shapes still resemble those seen in the scrambler model.

Intuitively, the piecewise linearity in the deterministic case arises because the speed of each oscillator, and the effect of a pulse on each oscillator, is the same. Hence the oscillators, or clusters of oscillators, maintain their initial ordering; they all march forward through $[0, 1]$ in a line with no passing. This property then implies, in a mean-field sense discussed in [26], that the oscillators condense into clusters whose size doubles periodically. At the end of the first period, all oscillators synchronize into pairs spaced equally apart. At the end of the second period, those pairs merge into clusters of size 4, and so on. Moreover, the clusters begin each period evenly spaced from each other (again, in a mean-field sense where fluctuations are neglected), which yields the piecewise constant firing rate mentioned above.

One limitation of our analysis, for both the scrambler and deterministic models, is that each oscillator obeys $\dot{x_i} = 1$ between firings. Such linear sawtooth waveforms are reasonable for the oscillators used in sensor networks [8], but not for neurons or cardiac pacemaker cells. In [26] we show that our results for the deterministic model are robust to the addition of small concavity in the charging curve. But large concavity introduces new effects, not yet understood theoretically. The analysis becomes more difficult because clusters are no longer uniformly distributed as we have assumed throughout.

There are many avenues to explore in future work. It would be interesting to study cluster kinetics in oscillator systems with local coupling, network structure, heterogeneity, delays, and other realistic features. Several of these features would break clusters apart, and so would require including fragmentation processes in the analysis. By incorporating suitable new loss and gain terms in the rate equations, one could perhaps derive useful estimates for synchronization speeds in more complex but random networks where synchronization is guaranteed but speed estimates are lacking [27].

Another possible application concerns the detection of network topology. Arenas et al. showed that in the Kuramoto model, the time course of cluster formation en route to synchronization can be used to shed light on a network's topology [28] and community structure [29]. While the mean-field approach used above is suitable for homogenous topologies, extensions of our approach using degree distributions might prove useful in probing a network's hidden structure.

[1] J. Buck and E. Buck, Science **159**, 1319 (1968).
[2] F. Leyvraz, Physics Reports **383**, 95 (2003).
[3] P. L. Krapivsky, S. Redner, and E. Ben-Naim, *A Kinetic View of Statistical Physics* (Cambridge University Press, Cambridge, UK, 2010).
[4] A. Pikovsky, M. Rosenblum, and J. Kurths, *Synchronization* (Cambridge University Press, Cambridge, UK, 2003).
[5] S. Strogatz, *Sync* (Hyperion, New York, 2003).
[6] Y.-W. Hong and A. Scaglione, in *2003 IEEE Conference on Ultra Wideband Systems and Technologies* (IEEE, 2003), pp. 190–194.
[7] X. Y. Wang and A. B. Apsel, in *50th Midwest Symposium on Circuits and Systems, 2007* (IEEE, 2007), pp. 1524–1527.
[8] G. Werner-Allen, G. Tewari, A. Patel, M. Welsh, and R. Nagpal, in *Proceedings of the 3rd International Conference on Embedded Networked Sensor Systems* (ACM, 2005), pp. 142–153.
[9] J. Nishimura and E. J. Friedman, Phys. Rev. Lett. **106**, 194101 (2011).
[10] J. Nishimura and E. J. Friedman, Phys. Rev. E **86**, 025201 (2012).
[11] A. V. M. Herz and J. J. Hopfield, Phys. Rev. Lett. **75**, 1222 (1995).
[12] S. Bottani and B. Delamotte, Physica D **103**, 430 (1997).
[13] S. Gualdi, J.-P. Bouchaud, G. Cencetti, M. Tarzia, and F. Zamponi, ArXiv e-prints (2014), 1409.3296.
[14] C. Kirst, T. Geisel, and M. Timme, Phys. Rev. Lett. **102**, 068101 (2009).
[15] J. J. Hopfield and A. V. M. Herz, Proceedings of the National Academy of Sciences **92**, 6655 (1995).
[16] C. S. Peskin, *Mathematical Aspects of Heart Physiology* (Courant Institute of Mathematical Sciences, New York, 1975), pp. 268–278.
[17] P. C. Bressloff, S. Coombes, and B. de Souza, Phys. Rev. Lett. **79**, 2791 (1997).
[18] P. C. Bressloff and S. Coombes, Phys. Rev. Lett. **80**, 4815 (1998).
[19] A. Corral, C. J. Pérez, A. Diaz-Guilera, and A. Arenas, Phys. Rev. Lett. **74**, 118 (1995).
[20] S. Bottani, Phys. Rev. Lett. **74**, 4189 (1995).
[21] G. A. Worrell, S. D. Cranstoun, J. Echauz, and B. Litt,

NeuroReport **13**, 2017 (2002).
[22] R. E. Mirollo and S. H. Strogatz, SIAM Journal on Applied Mathematics **50**, 1645 (1990).
[23] W. Gerstner and J. L. van Hemmen, Phys. Rev. Lett. **71**, 312 (1993).
[24] U. Ernst, K. Pawelzik, and T. Geisel, Phys. Rev. Lett. **74**, 1570 (1995).
[25] S. Bottani, Phys. Rev. E **54**, 2334 (1996).
[26] See Supplemental Material at `http://link.aps.org/supplemental/` for further results and analysis.
[27] J. Klinglmayr, C. Kirst, C. Bettstetter, and M. Timme, New Journal of Physics **14**, 073031 (2012).
[28] J. Gómez-Gardeñes, Y. Moreno, and A. Arenas, Phys. Rev. Lett. **98**, 034101 (2007).
[29] A. Arenas and A. Diaz-Guilera, The European Physical Journal-Special Topics **143**, 19 (2007).