Robust Characterization of Loss Rates
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Randomized Loss Estimation

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Many physical implementations of qubits—including ion traps, optical lattices and linear optics—suffer from loss. A nonzero probability of irretrievably losing a qubit can be a substantial obstacle to fault-tolerant methods of processing quantum information, requiring new techniques to safeguard against loss that introduce an additional overhead that depends upon the loss rate. Here we present a scalable and platform-independent protocol for estimating the average loss rate (averaged over all input states) resulting from an arbitrary Markovian noise process, as well as an independent estimate of detector efficiency. Moreover, we show that our protocol gives an additional constraint on estimated parameters from randomized benchmarking that improves the reliability of the estimated error rate and provides a new indicator for non-Markovian signatures in the experimental data. We also derive a bound for the state-dependent loss rate in terms of the average loss rate.

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In order to build practical devices for processing and transmitting quantum information, the rate of decoherence and other errors must be below certain fault-tolerant thresholds. In particular, many experimental implementations of qubits—such as ion traps [1, 2], optical lattices [3] and linear optics [4]—suffer from irretrievable loss, that is, there is a nonzero probability of the qubit vanishing (as opposed to leaking to other energy levels). Such loss of normalization can be a substantial obstacle to many quantum information protocols, requiring different error-correction techniques to achieve fault-tolerance [5, 6]. For example, the surface code may not be used directly if there is any probability of losing a qubit, while for the topological cluster states, loss rates of less than 1% are required to avoid impractical overheads [6].

However, there are two substantial challenges in characterizing loss. Firstly, the loss rate may depend on the state of the qubit, such as when a qubit is encoded in a superposition of vacuum and single-photon states. Secondly, the loss due to imperfect operations has to be distinguished from the inefficiency of the detector [7]. Quantum process tomography [8, 9] could be used to characterize loss, however, it is inefficient in the number of qubits and is sensitive to state preparation and measurement (SPAM) errors [10] and so cannot distinguish between loss due to imperfect operations and inefficient detectors.

In this Letter, we present a robust and efficient protocol that characterizes the loss rate due to imperfect operations averaged over input states. Our protocol is platform-independent, simple to implement and analyze and only assumes that the noise is Markovian. We begin by defining survival rates and then present our protocol and derive the associated analytical decay curve under the assumption of Markovian noise. We then prove that the average loss rate estimated via our protocol provides a practical bound on the loss rate for any state. Since our protocol is robust to SPAM, the choice of state and measurement in our protocol is unconstrained. However, we discuss two particularly suitable choices. The first of these maximizes the signal and the second allows one of the parameters in randomized benchmarking [11] to be independently estimated and leads to a new test for non-Markovian effects. The second choice also allows for an estimate of the unitarity metric introduced in Ref. [12] with no additional experimental overhead. In addition, we demonstrate that our protocol produces reliable estimates of loss rates through a numerical simulations under an error model that has the greatest variation in loss over states. Finally we illustrate how the analytical model breaks down when applied to systems that have reversible (coherent) leakage to an ancillary level.

Average survival rates—In order to distinguish between inefficient detectors and lossy processes, we now define survival rates. Many methods for characterizing a process $E$ (including randomized benchmarking [11, 13, 14]) assume it is trace-preserving. However, many experimental processes are not trace-preserving, but instead a state $\rho$ has a survival rate under $E$

\[
S(\rho|E) = \frac{\text{Tr}[E(\rho)]}{\text{Tr}\rho}
\] (1)

that is less than 1, or, equivalently, a nonzero loss rate $L(\rho|E) = 1 - S(\rho|E)$. Since the trace is linear and any unnormalized density matrix is proportional to a unit-trace density matrix, the survival rate averaged over all states (hereafter the average survival rate) is simply the survival rate of the maximally mixed state, that is, $S(E) := S(\frac{1}{d}I|E)$. Correspondingly, the average loss rate is $L(E) = 1 - S(E)$.

Experimental protocol—We now present a protocol for characterizing the average survival rate $S(E)$ in the experimental implementations $\{E_g\}$ of a set of gates $G = \{E_g\}$. In order to build practical devices for processing and transmitting quantum information, the rate of decoherence and other errors must be below certain fault-tolerant thresholds. In particular, many experimental implementations of qubits—such as ion traps [1, 2], optical lattices [3] and linear optics [4]—suffer from irretrievable loss, that is, there is a nonzero probability of the qubit vanishing (as opposed to leaking to other energy levels). Such loss of normalization can be a substantial obstacle to many quantum information protocols, requiring new techniques to safeguard against loss that introduce an additional overhead that depends upon the loss rate. Here we present a scalable and platform-independent protocol for estimating the average loss rate (averaged over all input states) resulting from an arbitrary Markovian noise process, as well as an independent estimate of detector efficiency. Moreover, we show that our protocol gives an additional constraint on estimated parameters from randomized benchmarking that improves the reliability of the estimated error rate and provides a new indicator for non-Markovian signatures in the experimental data. We also derive a bound for the state-dependent loss rate in terms of the average loss rate.

In order to build practical devices for processing and transmitting quantum information, the rate of decoherence and other errors must be below certain fault-tolerant thresholds. In particular, many experimental implementations of qubits—such as ion traps [1, 2], optical lattices [3] and linear optics [4]—suffer from irretrievable loss, that is, there is a nonzero probability of the qubit vanishing (as opposed to leaking to other energy levels). Such loss of normalization can be a substantial obstacle to many quantum information protocols, requiring new techniques to safeguard against loss that introduce an additional overhead that depends upon the loss rate. Here we present a scalable and platform-independent protocol for estimating the average loss rate (averaged over all input states) resulting from an arbitrary Markovian noise process, as well as an independent estimate of detector efficiency. Moreover, we show that our protocol gives an additional constraint on estimated parameters from randomized benchmarking that improves the reliability of the estimated error rate and provides a new indicator for non-Markovian signatures in the experimental data. We also derive a bound for the state-dependent loss rate in terms of the average loss rate.

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\{g_1, \ldots, g_d\} that are at least a unitary 1-design (e.g., the Pauli or Clifford groups) \cite{15}. For simplicity, we assume the noise is time- and gate-independent Markovian noise, so that \(\mathcal{E}_{g} = g \circ \mathcal{E}\) for some fixed map \(\mathcal{E}\) where \(\circ\) denotes composition (i.e., apply \(\mathcal{E}\) then \(g\)). This approach can be extended to accommodate time- and gate-dependent noise and a model of non-Markovian noise by applying the approaches of Refs. \cite{16,18}.

Our protocol for estimating \(S(\mathcal{E})\) is as follows.

1. Choose a sequence length \(m \in \mathbb{N}\).
2. Choose a random sequence \(k = (k_1, \ldots, k_m)\) of \(m\) integers uniformly at random, where \(k_j \in \{1, \ldots, |\mathcal{G}|\}\).
3. Prepare a state \(\rho\).
4. Apply the sequence of gates \(g_{k_m} \circ \ldots \circ g_{k_1}\).
5. Measure some operator \(Q\) (e.g., a self-adjoint operator or POVM element).
6. Repeat steps 3–5 to estimate
   \[
   Q_k = \text{Tr} [Q g_{k_m} \circ \mathcal{E} \circ \ldots \circ g_{k_1} \circ \mathcal{E}(\rho)]
   \]
   to a desired precision.
7. Repeat steps 2–6 to estimate
   \[
   E_k Q_k = |\mathcal{G}|^{-m} \sum_k Q_k
   \]
   to a desired precision (see, e.g., Ref. \cite{17} for methods to bound the number of sequences required to obtain a given precision).
8. Repeat steps 1–7 for multiple \(m\) and fit to the decay curve
   \[
   E_k Q_k = D(Q) S(\rho|\mathcal{E}) S^{m-1}(\mathcal{E}),
   \]
   derived below, to obtain estimates of \(S(\mathcal{E})\) and \(S(\rho|\mathcal{E})D(Q)\) where \(D(Q) = \text{Tr}Q/d\).

(Note that the above protocol differs from the randomized benchmarking protocol of Ref. \cite{11} in that no inversion gate is applied prior to the measurement.)

Results of numerical simulations of our protocol for a specific loss model are illustrated in Fig. 1 demonstrating the robust performance of our protocol.

For the numerical simulation, the set of operations \(\mathcal{G}\) is the set of single-qubit Paulis, and we modeled the error as \(\mathcal{E}\) as
\[
\mathcal{E}(\rho) = (|0\rangle\langle 0| + \alpha|1\rangle\langle 1|)\rho (|0\rangle\langle 0| + \alpha|1\rangle\langle 1|),
\]
where \(\alpha = 0.99\). The channel \(\mathcal{E}\) corresponds to loss from the \(|1\rangle\) state and, as proven in Proposition 1 below, has the greatest variation of loss over states. The measurement was set to \(0.87|\phi\rangle\langle \phi| + 0.95|\phi^\perp\rangle\langle \phi^\perp|\) where \(\{|\phi\rangle, |\phi^\perp\rangle\}\) is a randomly-chosen orthonormal basis to model an imperfect detector.

Analysis.—To derive the decay curve in Eq. (4), note that averaging over all sequences corresponds to independently averaging over all \(g_{k_j}\), so that
\[
E_k Q_k = \text{Tr} \left[ Q \mathcal{G} \circ \mathcal{E} \circ \ldots \circ \mathcal{G} \circ \mathcal{E}(\rho) \right]
\]
where \(\mathcal{G}(\rho) = |\mathcal{G}|^{-1} \sum_{g \in \mathcal{G}} g \rho g\dagger\) (noting that a unitary channel corresponds to unitary conjugation). Since \(\mathcal{G}\) is a unitary 1-design (and a linear map), \(\mathcal{G}(A) = \text{Tr}(A)I/d\) for all \(d \times d\) matrices \(A\) \cite{18,19}. Therefore \(\mathcal{G} \circ \mathcal{E}(\rho) = S(\rho|\mathcal{E})I/d\) and \(\mathcal{G} \circ \mathcal{E}(I/d) = S(\mathcal{E})I/d\) and so Eq. (4) simplifies to Eq. (1).

The average survival rate obtained via our protocol is one possible figure of merit that could be used to characterize loss, with an important alternative being the worst-case loss. However, as we now prove, the average loss provides a practical bound for the worst-case loss:

**Proposition 1.** For any quantum channel \(\mathcal{E}\) and state \(\rho\) for a \(d\)-dimensional system,
\[
L(\rho|\mathcal{E}) \leq L(\mathcal{E})d.
\]

Moreover, for all \(d\) there exist channels \(\mathcal{E}\) and states \(\rho\) that saturate this bound.

**Proof.** Let \(\rho\) and \(\mathcal{E}\) be arbitrary states of and channels for a \(d\)-dimensional system. Since the trace is linear and any
valid state can be written as $\rho = \tau \text{Tr} \rho$ where $\tau$ is a unit-
trace density matrix, the survival rate is independent of
$\text{Tr} \rho$, so we assume $\text{Tr} \rho = 1$ without loss of generality.

Let $\rho' = (I - \rho)/(d - 1)$, which is a valid quantum
state since it is Hermitian and positive-semidefinite by
construction and has unit trace. Since $\rho'$ is a valid quan-
tum state, the probability of detecting a system in the
state $\rho'$ after applying $\mathcal{E}$ is a true probability and thus
\[
\frac{dS(\mathcal{E}) - S(\rho|\mathcal{E})}{d - 1} = \frac{\text{Tr}[\mathcal{E}(I) - \mathcal{E}(\rho)]}{d - 1} = \text{Tr}(\rho') \leq 1, \tag{7}
\]
where we have used the fact that quantum channels and
the trace are linear. Rearranging and substituting $L = 1 - S$ gives the desired bound.

To see that the bound is saturated, fix $d$ and consider the channel
\[
\mathcal{E}(\rho) = [I + (\alpha - 1)|0\rangle\langle 0|][\rho|I + (\alpha - 1)|0\rangle\langle 0|] \tag{8}
\]
for $\alpha \in [0, 1]$. For this channel,
\[
\mathcal{E}(|j\rangle\langle j|) = \begin{cases} 
\alpha^2|0\rangle\langle 0| & j = 0 \\
|j\rangle\langle j| & j \neq 0,
\end{cases} \tag{9}
\]
so $L(|j\rangle\langle j|\mathcal{E}) = \delta_j(1 - \alpha^2)$ and
\[
L(\mathcal{E}) = \frac{1}{d} \sum_j L(|j\rangle\langle j|\mathcal{E}) = 1 - \alpha^2/d. \tag{10}
\]
Therefore there exists a channel $\mathcal{E}$ and a state $\rho$ such that
$L(\rho|\mathcal{E}) = L(\mathcal{E})d$. \hfill \Box

For average survival rates close to 1, the estimate of
$S(\rho|\mathcal{E})D(Q)$ can be used to directly estimate $D(Q)$, since
\[
S(\rho|\mathcal{E})D(Q) \in [(1 - d[1 - S(\mathcal{E})])D(Q), D(Q)] \tag{11}
\]
by Proposition \[1\]. Consequently, $S(\rho|\mathcal{E})D(Q)/S(\mathcal{E})$ will give an estimate of $D(Q)$ that is accurate to within a
factor of $(d - 1)L(\mathcal{E})$. Estimating $D(Q)$ can be used to estimate the efficiency of the detector as
\[
\eta = \frac{D(Q)}{D(Q_{\text{ideal}})}, \tag{12}
\]
where $Q_{\text{ideal}}$ and $Q$ are the ideal and actual measurement
operators. That is, $\eta$ is the ratio of observed to expected detector “clicks”, averaged (independently) over all states.

Choosing states and measurements.—Our protocol is
robust to SPAM errors, in that the choice of the state $\rho$ and
measurement operator $Q$ only effect the value of the
constant $S(\rho|\mathcal{E})D(Q)$. However, there are two choices of
$Q$ and $\rho$ that have particular benefits.

(i) The most useful scenario corresponds to choosing $G$
and $\rho, Q \approx |0\rangle\langle 0|$ as in randomized benchmarking \[11\]. There are two ma-
jor advantages to this choice. Firstly, with this choice
the same data can also be used to estimate the unitari-
ity of $\mathcal{E}$, a quantitative measure of how the noise $\mathcal{E}$
effects the purity of input states \[12\]. Secondly, estimating
the constant prefactor in Eq. \[4\] with this choice is par-
ticularly useful because it allows an additional and vital
constraint to be imposed when fitting randomized bench-
marking data to the fidelity decay curve. In Ref. \[11\], it
was shown that the fidelity decay curve is
\[
A(\mathcal{E}') p^m + B(\mathcal{E}') \tag{13}
\]
where $p$ is related to the average gate fidelity, $\mathcal{E}'$ is the average error under the convention that the experimental
implementation of $g$ is written as $\mathcal{E}_g = \mathcal{E}' \circ g$ (in contrast to our choice of $\mathcal{E}_g = g \circ \mathcal{E}$) and
\[
A(\mathcal{E}') = \text{Tr}[Q\mathcal{E}'(\rho - \frac{1}{2})], \\
B(\mathcal{E}') = \text{Tr}[Q(\mathcal{E}'(\frac{1}{2}))]. \tag{14}
\]
If the alternative convention of writing errors as $\mathcal{E}' \circ g$
is applied to Eq. \[6\], then the constant prefactor $S(\rho|\mathcal{E})D(Q)$ in Eq. \[11\] becomes $B(\mathcal{E}')$. Since the fi-
delity decay curve is in terms of observable properties,
it is independent of the choice of convention and so
$B(\mathcal{E}') = S(\rho|\mathcal{E})D(Q)$. Obtaining a precise estimate of
the constant term for randomized benchmarking is im-
portant for two reasons. First, underestimating the con-
stant term $B(\mathcal{E}')$ [and hence overestimating the coeffi-
cient $A(\mathcal{E}')$] will lead to an overestimate of the decay
parameter $p$, or, equivalently, an underestimate of the
average gate infidelity. That is, underestimating the con-
stant term will falsely indicate that the gates are perform-
ing better than they actually are. Second, the values of
the constants $A$ and $B$ in randomized benchmarking are
not completely independent: they must satisfy particular
constraints in order to correspond to physical Markovian
noise processes. In particular, for qubits, note that
\[
B(\mathcal{E}) - A(\mathcal{E}) = \text{Tr}[Q\mathcal{E}(\rho^\perp)] \tag{15}
\]
where $\rho^\perp$ is the state whose Bloch vector is anti-parallel
to that of $\rho$. Therefore $B(\mathcal{E}) - A(\mathcal{E})$ is a probability
and so must be nonnegative if the noise is truly Markovian.
Consequently, if $B(\mathcal{E}) - A(\mathcal{E})$ is (strongly) negative, then
either the noise is non-Markovian or strongly gate de-
dependent and so the estimate of the average gate infidelity
in randomized benchmarking is not known to be accurate.
Moreover, if the prefactor $S(\rho|\mathcal{E})D(Q)$ in Eq. \[4\] is esti-
mated by setting $m = 1$, then the resulting estimate is
unaffected by the presence of non-Markovian effects
between sequential operations (since there is only one
operation applied). Therefore if the estimate obtained
by setting $m = 1$ differs from the estimate obtained from
fitting the randomized benchmarking data under the pro-
tocol of Ref. \[11\], then this disagreement indicates that
non-Markovian effects are present in the data for the lat-
ter.
(ii) Alternatively, given any allowed choice of $G$, choosing $Q \approx I$ and $\rho$ to be any unit-trace density matrix will maximize the value of the constant prefactor in Eq. (1), reducing the number of experiments required to obtain a desired precision (since $E_k Q_k$ is close to one for sufficiently small $m$). Note that this data can be collected under the same experimental configuration as case (i), where $Q = |0\rangle\langle 0|$ and $G$ is a unitary 2-design, by simply re-incorporating the outcomes associated with $I - Q$ that are discarded in case (i). This data gives independent information because by assumption the probabilities of these two outcomes are not constrained to add to 1 due to presence of loss.

Coherent leakage.—A distinct, but closely related error to loss is (coherent) leakage, wherein the system is “leaked” from the qubit subspace to other energy levels. Leakage errors are non-Markovian errors on the qubit subspace, since the system can return to the qubit subspace. Coherent leakage is a known consequence of control imperfections in certain implementations of the coupling gate in ion traps [20] and the controlled-phase gate in superconducting qubits [21, 22]. Fig. 2 shows the results of our protocol given a model of coherent leakage, in particular, an error model for a random (fixed) unitary acting on a qutrit with a random relative phase between the leakage level and the qubit levels. The results initially appear to fit a single exponential decay, but then quickly converge to a constant, similar to the behavior observed in Ref. [24]. Consequently, if experimental data for our protocol does not neatly fit a single exponential, one explanation would be that there is a leakage level that has not been accounted for. A simple protocol for estimating rates of coherent leakage has been provided previously in Ref. [24].

Conclusion—In this paper, we have presented a platform-independent and robust protocol for characterizing the average loss rate due to noisy implementations of operations. Our protocol can also be used to estimate the detector efficiency, provided the loss rate due to noisy operations is sufficiently small. Since our protocol is easy to implement, it is also a promising technique for experimentally optimizing quantum control, as done, e.g. in Ref. [25] using randomized benchmarking experiments.

Experimentally implementing our protocol yields a single exponential decay curve which can be fitted to our analytical expression to obtain the average loss rate. If the experimental data deviates significantly from a single decay curve, the experimental noise is either strongly gate-dependent or non-Markovian. We have illustrated that the decay can be observed and fitted in practice through numerical simulations of loss for a specific error model and also that non-Markovian leakage to an ancillary level results in a deviation from a single exponential. However, fully characterizing how the present protocol (and other randomization-based protocols) behave in the presence of non-Markovian noise remains an open problem.

Our protocol is scalable and robust against state-preparation and measurement errors. However, particular choices of preparations and measurements give extra information. If the set of gates is chosen to be a unitary 2-design and the preparation and measurement are the same as those used in standard randomized benchmarking, then our current protocol can be applied to directly estimate one of the parameters in randomized benchmarking and thus provides a test to indicate non-Markovian noise. Furthermore, with this choice of preparation and measurement, the same data obtained via our protocol can be used to estimate the unitarity presented in Ref. [12] and thus to estimate how close the noise is to depolarizing noise.

As with standard RB, obtaining rigorous confidence intervals on the parameters obtained from our protocol is still an open problem, though techniques bounding the number of sequences to be sampled [17] and using Bayesian methods to refine prior information [26] should also be applicable to our protocol.

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