Thresholds for Correcting Errors, Erasures, and Faulty Syndrome Measurements in Degenerate Quantum Codes

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We suggest a technique for constructing lower (existence) bounds for the fault-tolerant threshold to scalable quantum computation applicable to degenerate quantum codes with sublinear distance scaling. We give explicit analytic expressions combining probabilities of erasures, depolarizing errors, and phenomenological syndrome measurement errors for quantum low-density parity-check (LDPC) codes with logarithmic or larger distances. These threshold estimates are parametrically better than the existing analytical bound based on percolation.

Quantum computers process coherent superpositions of exponentially many basis states instead of one binary string at a time. In theory, this parallelism makes quantum computers faster than the classical ones. However, quantum superpositions are fragile; without quantum error correction, decoherence would make computations unfeasible[1]. Further, unlike in a classical setup restricted to transmission errors, any quantum error-correcting code (QECC) requires complicated measurements prone to errors. This *syndrome* extraction from a system of qubits requires fault-tolerance (FT): all operations have to limit error propagation. Then, an arbitrarily large quantum computation is possible with a polynomial complexity if physical qubits and elementary gates exceed some accuracy threshold (*threshold theorem*) [2–7].

For years, out of many existing families of QECCs [8, 9], FT threshold was established for only two code families, concatenated[2] and surface[5] codes (also, related color codes[10]). However, both families have asymptotically zero code rates[11] and therefore require substantial hardware overhead. A new alternative is offered by quantum low-density parity-check (LDPC) codes[12], which can combine *finite* rates with a nonzero FT threshold. These are stabilizer codes[8, 13] with a limited number of qubits in each stabilizer generator (operators to be measured during QEC). Several families of such codes have finite code rates[14–18]. The threshold existence has been proven[19] by two of us using ideas from percolation theory. Subsequently, a related approach of Gottesman[20] demonstrated that such codes can achieve scalable quantum computation with a finite overhead per logical qubit.

While Ref. 19 gives a finite threshold for certain quantum LDPC codes, the actual threshold value and its dependence on the parameters are both far off. The technique[19] also fails to give a finite threshold whenever a single qubit is shared by many stabilizer generators.

Here we present an approach resulting in parametrically better lower bounds for the thresholds, for both a quantum channel and a phenomenological error model with FT setting. We consider infinite sequences of "*long*" quantum LDPC codes of increasing length n, whose distances d scale with n at least logarithmically,

$$d \geq D \ln n, \quad D > 0. \tag{1}$$

A super-logarithmic scaling of the distance (including a power law $d \geq An^{\alpha}$ with $A, \alpha > 0$) gives $D \to \infty$. At the same time, we limit all stabilizer generators to some fixed number of w or fewer qubits. For any sequence of such codes, we give an analytical lower (existence) bound combining uncorrelated qubit erasures, depolarizing errors, and syndrome measurement errors. We also give a similar bound tailored for CSS codes. These bounds no longer require that every qubit be included in a limited number of stabilizer generators. Tying our lower bound on erasure threshold with other results[21, 22], we restrict parameters of LDPC codes with certain properties.

We consider QECCs defined on the n-qubit Hilbert space $\mathcal{H}_2^{\otimes n}$. Any operator acting in $\mathcal{H}_2^{\otimes n}$ can be represented as a linear combination of *Pauli operators*, elements of the n-qubit Pauli group $\mathscr{P}_n$ of size 2^{2n+2},

$$\mathscr{P}_n = i^m \{I, X, Y, Z\}^{\otimes n}, \ m = 0, \ldots, 3, \tag{2}$$

where X, Y, and Z are the usual Pauli matrices, I is the identity matrix, and i^m a phase factor. *Weight* wgt E of a Pauli operator $E \in \mathscr{P}_n$ is the number of non-identity terms in its expansion (2). A *stabilizer code* $\mathcal{Q}$ $[[n, k, d]]$ is a 2^k-dimensional subspace of the Hilbert space $\mathcal{H}_2^{\otimes n}$, a common $+1$ eigenspace of operators in an Abelian *stabilizer* group $\mathscr{S} = \langle G_1, \ldots, G_r \rangle$ with generators G_i,

$$\mathcal{Q} = \{|\psi\rangle : S |\psi\rangle = |\psi\rangle, \forall S \in \mathscr{S}\}, \quad -\mathbb{1} \notin \mathscr{S}. \tag{3}$$

A narrower set of Calderbank-Shor-Steane (CSS) codes [23, 24] contains codes where each stabilizer generator is a product of only Pauli X or Pauli Z operators. For a stabilizer group with r independent generators, the dimension of the quantum code is $k = n - r$; for a CSS code with r_μ independent generators of type $\mu = X, Z$, respectively, $k = n - r_X - r_Z$.

Error correction is done by measuring the stabilizer generators G_i, $i = 1, \ldots, r$; the corresponding eigenvalues $(-1)^{s_i}$, $s_i \in \{0, 1\}$ form the *syndrome* $\mathbf{s} \equiv (s_1, s_2, \ldots, s_r)$ of the error. Measuring the syndrome projects any state $|\psi\rangle \in \mathcal{H}_2^{\otimes n}$ into one of the 2^r subspaces $\mathcal{Q}_\mathbf{s}$ equivalent to the code $\mathcal{Q} \equiv \mathcal{Q}_\mathbf{0}$. A *detectable* error $E \in \mathscr{P}_n$ anticommutes with some generator(s) of the stabilizer; otherwise it is called *undetectable*. Then, for any $|\psi\rangle \in \mathcal{Q}$, the syndrome measured in the state $E|\psi\rangle$ is non-zero for a detectable error, and it is zero otherwise. While operators in the stabilizer group are undetectable, they act *trivially* on the code; such errors can be ignored. Any two Pauli errors E_1, E_2 which differ by a phase and an element of the stabilizer, $E_2 = e^{i\alpha} E_1 S$, $S \in \mathscr{S}$, are called degenerate. Mutually degenerate errors act identically on the code, they cannot (and need not) be distinguished.

The *distance d* of a code $\mathcal{Q}$ is the minimum weight of an undetectable Pauli error $E \in \mathscr{P}_n$ which is not a part of the stabilizer, $E \notin \mathscr{S}$ (up to a phase). A code with distance d detects non-trivial Pauli errors of weight up to $d - 1$, and corrects such errors of weight up to $\lfloor d/2 \rfloor$.

A code is called *degenerate* if its stabilizer includes a non-trivial operator $S \in \mathscr{S}$ with weight smaller than the distance, $0 \neq \mathrm{wgt}\, S < d$. Degenerate codes are nice since generators of small weight are easier to measure; all codes with a known FT threshold are degenerate. The ultimate case of degeneracy are w-limited *quantum LDPC codes*, where every stabilizer generator has weight w or smaller.

We consider three simple error models[25]: quantum depolarizing channel, where with probability p an incoming qubit is replaced by a qubit in a random state; independent X/Z errors, where Pauli operators X and Z are applied to each qubit with probabilities p_X and p_Z, respectively, and the quantum erasure channel, where with probability y each qubit is replaced by an "erasure state" $|2\rangle$ orthogonal to both $|0\rangle$ and $|1\rangle$. We also address FT using a phenomenological error model where measurement errors happen independently with probability q. Such an error affects the syndrome bits but not the qubit states. Our thresholds are as follows:

Theorem 1. *Any sequence of long quantum codes (1) with stabilizer generators of weights w or less can be decoded with a vanishing error probability if channel probabilities (y, p) of erasures and depolarizing errors satisfy the restriction $2(w-1)\, \Upsilon(y, p) < e^{-1/D}$, where parameter D is defined in Eq. (1) and*

$$\Upsilon(y, p) \equiv y + (1 - y) \left\{ \frac{2p}{3} + 2 \left[\frac{p}{3}(1 - p) \right]^{1/2} \right\}. \qquad (4)$$

Theorem 2. *Any sequence of long CSS codes (1) with generator weights not exceeding w_X, w_Z can be decoded with a vanishing error probability if channel probabilities (y, p_X, p_Z) of erasures and independent X/Z errors*

satisfy the restrictions $(w_X - 1)\, \Upsilon_{\mathrm{CSS}}(y, p_Z) < e^{-1/D}$, $(w_Z - 1)\, \Upsilon_{\mathrm{CSS}}(y, p_X) < e^{-1/D}$, where

$$\Upsilon_{\mathrm{CSS}}(y, p) \equiv y + 2(1 - y)\, [p(1 - p)]^{1/2}. \qquad (5)$$

FT case gives weaker versions of Theorems 1 and 2:

Theorem 3. *If phenomenological syndrome measurement errors occur with probability q, vanishing error rates are achieved by (**a**) stabilizer codes of Theorem 1 if*

$$4\, [q(1-q)]^{1/2} + 2w\Upsilon(y, p) < e^{-1/D}, \qquad (6)$$

*(**b**) CSS codes of Theorem 2 if*

$$4\, [q(1-q)]^{1/2} + w_X \Upsilon_{\mathrm{CSS}}(y, p_Z) < e^{-1/D},$$
$$4\, [q(1-q)]^{1/2} + w_Z \Upsilon_{\mathrm{CSS}}(y, p_X) < e^{-1/D}. \qquad (7)$$

Our analysis is based on counting irreducible undetectable operators:

Definition 1. *For a given stabilizer code $\mathcal{Q}$, an undetectable operator is called irreducible if it cannot be decomposed as a product of two undetectable Pauli operators with support on non-empty disjoint sets of qubits.*

This definition implies:

Lemma 4. *Any undetectable operator $E \in \mathscr{P}_n$ can be written as $E = \prod_i J_i$, where undetectable operators $J_i \in \mathscr{P}_n$, $\mathrm{wgt}\, J_i \neq 0$, are irreducible and pairwise disjoint.*

For a given code, let $\mathscr{U} \subset \mathscr{P}_n \setminus \mathscr{S}$ denote the set of all non-trivial irreducible undetectable Pauli operators.

Given some error probability function $P(E)$, consider a syndrome-based decoder which returns the Pauli operator $E \in \mathscr{P}_n$ that maximizes $P(E)$ for a given syndrome. Notice that this is not a maximum-likelihood (ML) decoder since we ignore contributions of errors degenerate with E. Using a statistical-mechanical analogy[5, 7, 26], ML decoding corresponds to minimizing the free energy; here we ignore entropy contribution resulting from degenerate errors and just minimize the *energy* $\varepsilon(E) \equiv -\ln P(E)$. Such a procedure is intrinsically sub-optimal; thus a lower bound for decoding threshold is also a lower bound for the syndrome-based ML decoding.

Now, let $E \in \mathscr{P}_n$ be an actual error, and E' be the same-syndrome Pauli operator which minimizes the energy $\varepsilon(E')$. The product $E'E^\dagger$ is undetectable, it satisfies Lemma 4, which gives a decomposition $E'E^\dagger = \prod_i J_i$ into irreducible undetectable operators, $J_i \in \mathscr{S} \cup \mathscr{U}$. Since the operators J_i are mutually disjoint, none of them can decrease the energy of E', $\varepsilon(J_i E') \geq \varepsilon(E')$. Otherwise E' would not be the smallest-energy error with the same syndrome. The minimal-energy error E' is correct iff $E'E^\dagger$ is trivial, which implies that every irreducible component needs to be in the stabilizer, $J_j \in \mathscr{S}$ (up to a phase).

Otherwise, there is an irreducible operator $U \in \mathscr{U}$ which does not increase the energy of the original error E, $\varepsilon(UE) \leq \varepsilon(E)$. Let $\mathcal{B}(U) \equiv \{E \in \mathscr{P}_n : \varepsilon(UE) \leq \varepsilon(E)\}$ be the full set of such "bad" errors for a given $U \in \mathscr{U}$. Minimum-energy decoding gives vanishing error rate if

$$\mathrm{Prob}\left[E : E \in \bigcup_{U \in \mathscr{U}} \mathcal{B}(U)\right] \to 0, \quad n \to \infty. \quad (8)$$

Then the union bound for all $\mathcal{B}(U)$ gives the following sufficient condition for error-free decoding:

$$\sum_{U \in \mathscr{U}} \mathrm{Prob}\left[E : E \in \mathcal{B}(U)\right] \to 0, \quad n \to \infty. \quad (9)$$

For uncorrelated errors only the qubits in the support of U affect the probabilities in Eq. (9). With uniform error distributions, these probabilities depend only on the weights $f \equiv \mathrm{wgt}\, U$ of the operators $U \in \mathscr{U}$. For example, if erasures occur with a single-qubit probability y, a bad error must cover the entire support of U, which gives simply $\mathrm{Prob}[E : E \in \mathcal{B}(U)] = y^{\mathrm{wgt}(U)}$. Let N_f denote the number of operators $U \in \mathscr{U}$ of weight $f \equiv \mathrm{wgt}\, U$. Since members of the stabilizer group are excluded from $\mathscr{U}$, $N_f = 0$ for $f < d$. Thus, in the case of the erasure channel, the condition (9) is equivalent to

$$\sum_{f \geq d} N_m y^m \to 0, \quad n \to \infty. \quad (10)$$

To construct an upper bound for N_f, we use a simplified version of the cluster-enumeration algorithm originally designed for finding the distance of a quantum LDPC code[27, 28]. First, fix an arbitrary order of the r stabilizer generators G_i, $1 \leq i < r$. Start by placing any of $\{X, Y, Z\}$ at a position $j \in \{0, \ldots, n-1\}$ and place the corresponding Pauli operator as the only element of the list of the components of the operator being constructed. At every subsequent step, take the generator G_i corresponding to a non-zero syndrome bit with the smallest index i, and choose any position j in the support of G_i that is not yet selected; there are up to $\mathrm{wgt}\, G_i - 1$ choices. Choose a single-qubit Pauli operator different from the term present at the position j in the expansion (2) of G_i, and add it to the list. This sets the syndrome bit s_i to zero without modifying any of the existing entries in the list. At every step of the recursion, zero syndrome means a completed undetectable cluster; no position available to correct a chosen syndrome bit means recursion got stuck. In either case we need to go back one step by removing the element last added to the list. The procedure stops when we exhaust all choices.

If the recursion has depth f, we only construct operators of weight up to f. There are $3n$ possible choices for the first step, and up to $2(\mathrm{wgt}\, G_i - 1)$ for each subsequent step. Then a w-limited LDPC code yields at most

$$\overline{N}_f = 3n[2(w-1)]^{f-1} \quad (11)$$

recursion paths to construct operators of weight up to f. This algorithm returns only undetectable operators. While not all of them are irreducible, *all irreducible operators* of weight f are constructed with depth-f recursion, see Sec. I in the Online Supplement. These arguments give the upper bound $\overline{N}_f \geq N_f$ for the number N_f of the irreducible operators $U \in \mathscr{U}$ of weight $\mathrm{wgt}\, U = f$.
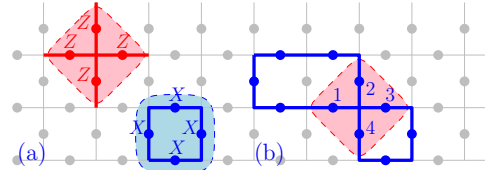

FIG. 1. (color online) (a) Toric code generators: Plaquette $A_\square$ (shaded rounded square) and vertex B_+ (shaded diamonds) operators constructed as products of four Pauli X and Pauli Z operators respectively. (b) A reducible cluster is counted as one or two clusters depending on the order in which the numbered qubits are chosen.

For CSS codes, let $\mathscr{U}_X \subset \mathscr{U}$ and $\mathscr{U}_Z \subset \mathscr{U}$ be the sets of non-trivial irreducible undetectable operators composed of only X and Z operators respectively, and $N_f^{(\mu)}$ be the number of weight-f operators in $\mathscr{U}_\mu$, $\mu \in \{X, Z\}$. For codes in Theorem 2, this gives improved bounds, e.g.,

$$N_f^{(X)} \leq \overline{N}_f^{(X)} \equiv n(w_Z - 1)^{f-1}. \quad (12)$$

We illustrate the cluster enumeration on the toric code $[[2L^2, 2, L]]$, a CSS code with $w_X = w_Z = 4$ generators local in two dimensions. The qubits are on the bonds of an $L \times L$ square lattice with periodic boundary conditions along both bond directions. The stabilizer generators are the plaquette and vertex operators, $A_\square = \prod_{j \in \square} X_j$ and $B_+ = \prod_{j \in +} Z_j$ [Fig. 1(a)]. A type-X cluster can be started by placing an X operator anywhere, which makes the two operators B_+ on the neighboring vertices unhappy (the corresponding syndrome bits are non-zero). Either can be corrected by placing an additional X operator on one of the remaining three open bonds adjoining the corresponding vertex. This produces an additional unhappy operator B_+ at the other end of the bond, etc. An undetectable cluster corresponds to a closed walk (cycle). Any cycle can be constructed this way. A topologically trivial cycle gives a member of the stabilizer group, while a cycle winding an odd number of times over one or both periodicity directions corresponds to a logical operator. Further, a cycle with self intersections gives an operator which can be decomposed into a product of two or more disjoint irreducible operators [Fig. 1(b)].

Combining Eq. (10) and the bound $N_f \leq \overline{N}_f$, see Eq. (11), we can prove a simplified version of Theorem 1 for erasures only. Namely, consider the sum

$$Q_d(y) \equiv \sum_{m \geq d} \overline{N}_f y^f = \frac{3ny\,[2y(w-1)]^{d-1}}{1 - 2y(w-1)}; \quad (13)$$

it converges absolutely for $2y(w-1) < 1$. Asymptotically, $Q_d(y)$ converges to zero as long as $n[2y(w-1)]^d \to 0$. This is true for any $y < e^{-1/D}/2(w-1)$ for codes in Eq. (1). The sum (13) majors Eq. (10) term by term, which gives a lower bound for erasure threshold, $y_c \geq e^{-1/D}/2(w-1)$, cf. Theorem 1. With the distance scaling super-logarithmically (e.g., as a power law), the sum (13) vanishes anywhere within the convergence radius, $y < [2(w-1)]^{-1}$, and we may just set $e^{-1/D} \to 1$.

Theorems 1 and 2, which combine erasures and errors, can be proved similarly if we notice that the probabilities in Eq. (9) can be bounded as in Eq. (10), with some effective erasure rate $\Upsilon \geq y$ [Online Supplement, Sec. II.]

Arguments used so far require ideal syndrome measurements. For quantum codes, it is more important to consider the FT case where errors can occur in any quantum gate during syndrome measurements[2, 4, 29–33]. Such a complete analysis is beyond the scope of this work. Instead, we give a simplified estimate based on a phenomenological error model, which assumes that measured syndrome bits can have errors, but otherwise there is no effect on the qubits[5, 10, 19]. Error correction involves repeated syndrome measurement cycles and an auxiliary code which combines the syndromes measured in subsequent cycles. We only consider the simplest case where repetition code is used for combining the syndromes. For a CSS code, with equal uncorrelated qubit and syndrome errors $q = p_X = p_Z$, the net effect is equivalent to increasing the weights of stabilizer generators in Eq. (12) and in Theorem 2 by two, $w \to w+2$. With the surface codes, decoding corresponds to minimal-weight matching of chains in three dimensions[5]. For a more general result, we have to bound the number of weight-f clusters N_{f,f_q} which include f_q "qubit" Pauli operators, and $f - f_q$ binary syndrome errors. Theorem 3 follows from the bound $N_{f,f_q} \leq \overline{N}_{f,f_q}$,

$$\overline{N}_{f,f_q} \equiv 3nm \, 2^f \binom{f}{f_q} w^{f_q}, \qquad (14)$$

where m is the number of measurement cycles (same as the code distance d, see the Online Supplement, Sec. III.)

How tight are these bounds? The toric code ($w = 4$) has an erasure threshold $y_c = 0.5$ and the ML threshold for independent X/Z errors $p_{Zc} = p_{Xc} \approx 0.11$, compared to $y_c^* = 1/3$ and $p_{Zc}^* \approx 0.029$ of Theorem 2. Bound (12) was also verified by counting irreducible clusters numerically (see Sec. IV of the Online Supplement) and fitting with $\ln N_f = A + \zeta_w f$, where $\zeta_w \leq w - 1$ for CSS codes with row weight w was expected from (12). In particular, we got $\zeta_6 \approx 4.76$, $\zeta_7 \approx 5.74$, and $\zeta_8 \approx 5.79$, indicating that our bounds for N_f are relatively tight.

In conclusion, we constructed lower bounds on the thresholds of weight-limited quantum LDPC codes with sublinear distances scaling logarithmically or faster with the code length n. These bounds are based on estimating the number of logical operators which cannot be decomposed into a product of disjoint undetectable operators. The resulting analytical expressions combine probabilities of erasures, depolarizing errors (independent X/Z errors for CSS codes), and syndrome measurement errors using a phenomenological error model. These bounds are much stronger than those constructed previously[19], and they have a different dependence on the code parameters. In particular, we no longer require that each qubit be involved in a limited number of stabilizer generators. Qualitatively, the main difference is that the present analysis is not based on percolation theory.

This technique could carry over from LDPC codes to more general degenerate codes, where the corresponding scaling of N_f can be calculated numerically or analytically (e.g., in the case of concatenated codes). It would be interesting to see if a finite FT threshold exists for finite-rate and finite relative distance quantum LDPC codes constructed by Bravyi and Hastings[18]. A related open problem is the existence of FT threshold for subsystem codes, e.g., a subclass of those constructed in Ref. [34].

Our bounds also limit the parameters of quantum LDPC codes, in particular, their rate R. Indeed, Theorem 2 gives the erasure threshold $y_c^{(\text{CSS})} \geq 1/(w-1)$ for CSS LDPC codes with super-logarithmic distance. Along with the trivial upper bound $y_c \leq (1-R)/2$, this implies that no such codes exist if $R > 1 - 2/(w-1)$. For codes with $w = 4$, this gives $R \leq 1/3$, whereas the only known example of such codes is $R = 0$ (toric codes). These can be further improved by using the tighter upper bounds constructed for quantum LDPC codes in Ref. 21.

Also, Pastawski and Yoshida pointed to us that our erasure thresholds can be combined with their upper bound[22] for codes which include non-trivial transversal logical gates from m-th level of the Clifford hierarchy[35], $y_m \leq 1/m$. Thus, e.g., only CSS codes with generators of weight $w \geq m+1$ may include such logical gates. We note that the analysis in Refs. 22 and 35 is largely based on the cleaning lemma[11, 36] and the notion of correctable subsets, which complement our irreducible undetectable operators (Def. 1). It would be interesting to check if this relation could help extending the bounds from Refs. 11 to general LDPC codes.

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